The Triple-Pair Construction for Weighted ω-Pushdown Automata

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Let S be a complete star-omega semiring and Σ be an alphabet. For a weighted ω-pushdown automaton $\mathscr{P}$ with stateset $\{1,\ldots,n\}$, $n \geq 1$, we show that there exists a mixed algebraic system over a complete semiring-semimodule pair $((S \ll \Sigma^* \gg)^{n\times n}, (S \ll \Sigma^\omega \gg)^n)$ such that the behavior $\|\mathscr{P}\|$ of $\mathscr{P}$ is a component of a solution of this system. In case the basic semiring is $\mathbb{B}$ or $\mathbb{N}^\infty$ we show that there exists a mixed context-free grammar that generates $\|\mathscr{P}\|$. The construction of the mixed context-free grammar from $\mathscr{P}$ is a generalization of the well known triple construction and is called now triple-pair construction for ω-pushdown automata.

1 Introduction and preliminaries

Weighted pushdown automata were introduced by Kuich, Salomaa [14]. Many results on classical pushdown automata and context-free grammars can be generalized to weighted pushdown automata and algebraic systems. Classic pushdown automata can also be used to accept infinite words (see Cohen, Gold [3]) and it is this aspect we generalize in our paper. We consider weighted ω-pushdown automata and their relation to algebraic systems over a complete semiring-semimodule pair $(S^{n\times n}, V^n)$. It turns out that the well known triple construction for pushdown automata can be generalized to a triple-pair construction for weighted ω-pushdown automata. Our paper generalizes results of Droste, Kuich [5].

The paper consists of this and three more sections. In Section 2, pushdown transition matrices are introduced and their properties are studied. The main result of this section is that, for such a matrix M, the p-blocks, p a pushdown symbol, of the infinite column vector $M^{\omega,l}$ satisfy a special equality. In Section 3, weighted ω-pushdown automata are introduced. We show that for a weighted ω-pushdown automaton $\mathscr{P}$ there exists a mixed algebraic system such that the behavior $\|\mathscr{P}\|$ of $\mathscr{P}$ is a component of a solution of this system. In Section 4 we consider the case that the complete star-omega semiring S is equal to $\mathbb{B}$ or $\mathbb{N}^\infty$. Then for a given weighted ω-pushdown automaton $\mathscr{P}$ a mixed context-free grammar is constructed that generates $\|\mathscr{P}\|$. The construction is a generalization of the well known triple construction and is called *triple-pair construction for ω-pushdown automata*.

For the convenience of the reader, we quote definitions and results of Ésik, Kuich [8, 9, 10, 11] from Ésik, Kuich [7]. The reader should be familiar with Sections 5.1-5.6 of Ésik, Kuich [7].

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A semiring S is called *complete starsemiring* if sums for all families $(s_i \mid i \in I)$ of elements of S are defined, where I is an arbitrary index set, and if S is equipped with an additional unary star operation $^* : S \to S$ defined by $s^* = \sum_{j \geq 0} s^j$ for all $s \in S$. Moreover, certain conditions have to be satisfied making sure that computations with "infinite" sums can be performed analogous to those with finite sums.

A pair (S,V), where S is a complete starsemiring and V is a complete S-semimodule is called a *complete semiring-semimodule pair* if products for all sequences $(s_i \mid i \in \mathbb{N})$ of elements of S are defined and if S and V are equipped with an omega operation $^\omega : S \to V$ defined by $s^\omega = \prod_{j \geq 1} s$ for all $s \in S$. Moreover, certain conditions (e.g. "infinite" distributive laws) have to be satisfied making sure that computations with "infinite" sums and "infinite" products can be performed analogous to those with finite sums and finite products. (For details see Conway [4], Eilenberg [6], Bloom, Ésik [1], Ésik, Kuich [7], pages 30 and 105-107.)

A semiring S is called *complete star-omega semiring* if (S,S) is a complete semiring-semimodule pair.

For the theory of infinite words and finite automata accepting infinite words by the Büchi condition consult Perrin, Pin [15].

2 Pushdown transition matrices

In this section we introduce pushdown transition matrices and study their properties. Our first theorem generalizes Theorem 10.5 of Kuich, Salomaa [14]. Then we show in Theorems 3 and 6 that, for a pushdown transition matrix M, $(M^\omega)_p$ and $(M^{\omega,l})_p$, $0 \leq l \leq n$, $p \in \Gamma$, introduced below satisfy the same specific equality. In Theorem 1, S denotes a complete starsemiring; afterwards in this section, (S,V) denotes a complete semiring-semimodule pair.

Following Kuich, Salomaa [14] and Kuich [13], we introduce pushdown transitions matrices. Let Γ be an alphabet, called *pushdown alphabet* and let $n \geq 1$. A matrix $M \in (S^{n\times n})^{\Gamma^* \times \Gamma^*}$ is termed a *pushdown transition matrix* (with *pushdown alphabet* Γ and *stateset* $\{1,\dots,n\}$) if

(i) for each $p \in \Gamma$ there exist only finitely many blocks $M_{p,\pi}$, $\pi \in \Gamma^*$, that are unequal to 0;

(ii) for all $\pi_1, \pi_2 \in \Gamma^*$,

$$M_{\pi_1,\pi_2} = \begin{cases} M_{p,\pi} & \text{if there exist } p \in \Gamma, \pi, \pi' \in \Gamma^* \text{ with } \pi_1 = p\pi' \text{ and } \pi_2 = \pi\pi', \\ 0 & \text{otherwise.} \end{cases}$$

For the remaining of this paper, $M \in (S^{n\times n})^{\Gamma^* \times \Gamma^*}$ will denote a pushdown transition matrix with pushdown alphabet Γ and stateset $\{1,\dots,n\}$.

Our first theorem generalizes Theorem 10.5 of Kuich, Salomaa [14] and Theorem 6.2 of Kuich [13] to complete starsemirings. First observe that for all $\rho_1 \in \Gamma^+$, $\rho_2, \pi \in \Gamma^*$, we have $M_{\rho_1\pi,\rho_2\pi} = M_{\rho_1,\rho_2}$.

Intuitively, our next theorem states that, emptying the pushdown tape with contents $p\pi$ by finite computations has the same effect (i.e., $(M^*)_{p\pi,\varepsilon}$) as emptying first the pushdown tape with contents p (i.e., $(M^*)_{p,\varepsilon}$) by finite computations and afterwards (i.e., multiplying) emptying the pushdown tape with contents π (i.e., $(M^*)_{\pi,\varepsilon}$) by finite computations.

Theorem 1. *Let S be a complete starsemiring and $M \in (S^{n\times n})^{\Gamma^* \times \Gamma^*}$ be a pushdown transition matrix. Then, for all $p \in \Gamma$ and $\pi \in \Gamma^*$,*

$$(M^*)_{p\pi,\varepsilon} = (M^*)_{p,\varepsilon}(M^*)_{\pi,\varepsilon}\,.$$

Proof. Since the case $\pi = \varepsilon$ is trivial, we assume $\pi \in \Gamma^+$. We obtain

$$\begin{aligned}
(M^*)_{p\pi,\varepsilon} &= \sum_{m\geq 0} (M^{m+1})_{p\pi,\varepsilon} \\
&= \sum_{m\geq 0} \sum_{\pi_1,\ldots,\pi_m \in \Gamma^+} M_{p\pi,\pi_1} M_{\pi_1,\pi_2} \ldots M_{\pi_{m-1},\pi_m} M_{\pi_m,\varepsilon} \\
&= \Big(\sum_{m_1\geq 0} \sum_{\rho_1,\ldots,\rho_{m_1} \in \Gamma^+} M_{p\pi,\rho_1\pi} \ldots M_{\rho_{m_1}\pi,\pi} \Big) \cdot \Big(\sum_{m_2\geq 0} \sum_{\pi_1,\ldots,\pi_{m_2} \in \Gamma^+} M_{\pi,\pi_1} \ldots M_{\pi_{m_2},\varepsilon} \Big) \\
&= \Big(\sum_{m_1\geq 0} \sum_{\rho_1,\ldots,\rho_{m_1} \in \Gamma^+} M_{p,\rho_1} \ldots M_{\rho_{m_1},\varepsilon} \Big) (M^*)_{\pi,\varepsilon} = (M^*)_{p,\varepsilon} (M^*)_{\pi,\varepsilon} \,.
\end{aligned}$$

The summand for $m = 0$ is $M_{p\pi,\varepsilon}$; the summand for $m_1 = 0$ is $M_{p\pi,\pi}$ or $M_{p,\varepsilon}$; the summand for $m_2 = 0$ is $M_{\pi,\varepsilon}$. In the third line in the first factor the pushdown contents are always of the form $\rho\pi$, $\rho \in \Gamma^+$, except for the last move. Hence, in the second factor the first move has to start with pushdown contents π and it is the first time that the leftmost symbol of π is read. □

Intuitively, the next lemma states that the infinite computations starting with $p_1 \ldots p_k$ on the pushdown tape yield the same matrix $(M^\omega)_{p_1\ldots p_k}$ as summing up, for all $1 \leq j \leq k$ the product of $(M^*)_{p_1\ldots p_{j-1},\varepsilon}$ (i.e., emptying the pushdown tape with contents $p_1 \ldots p_{j-1}$ by finite computations) with the matrix $(M^\omega)_{p_j}$ (i.e., the infinite computations starting with p_j on the pushdown tape).

This means that in $p_1 \ldots p_k$ the pushdown symbols $p_1, \ldots, p_{j-1}$ are emptied by finite computations and p_j is chosen for starting the infinite computations. Clearly, $p_{j+1}, \ldots, p_k$ are not read.

Lemma 2. *Let (S,V) be a complete semiring-semimodule pair and let $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ be a pushdown transition matrix. Then for all $p_1, \ldots, p_k \in \Gamma$,*

$$(M^\omega)_{p_1\ldots p_k} = \sum_{1\leq j\leq k} (M^*)_{p_1,\ldots,p_{j-1},\varepsilon} (M^\omega)_{p_j} \,.$$

Proof.

$$(M^\omega)_{p_1,\ldots,p_k} = \sum_{\rho_1,\rho_2,\ldots \in \Gamma^+} M_{p_1\ldots p_k,\rho_1} M_{\rho_1,\rho_2} M_{\rho_2,\rho_3} \ldots \,.$$

We partition the "runs" $(p_1 \ldots p_k, \rho_1, \rho_2, \rho_3, \ldots)$ into classes:

- class (1): there exist $\rho_i' \in \Gamma^+$, $i \geq 1$, such that $\rho_i = \rho_i' p_2 \ldots p_k$.
- class (j).(t), $k \geq 3$, $2 \leq j \leq k-1$, $t \geq 1$: $\rho_t = p_j \ldots p_k$ and there exist $\rho_i' \in \Gamma^+$, for $1 \leq i \leq t-1$ and $i \geq t+1$, such that $\rho_i = \rho_i' p_j \ldots p_k$ for $1 \leq i \leq t-1$, and $\rho_i = \rho_i' p_{j+1} \ldots p_k$ for $i \geq t+1$.
- class (k).(t), $k \geq 2$, $t \geq 1$: $\rho_t = p_k$ and there exist $\rho_i' \in \Gamma^+$ for $1 \leq i \leq t-1$, such that $\rho_i = \rho_i' p_k$.

Clearly, class (1) and class (j).(t), $2 \leq j \leq k$, $t \geq 1$ are pairwise disjoint.

Intuitively, in the runs of
class (1): p_2 is never read;
class (j).(t), $2 \leq j \leq k-1$, $t \geq 1$: p_{j+1} is never read and p_j is read in the t-th step;
class (k).(t), $t \geq 1$: p_k is read in the t-th step.

We now compute for each class the value of

$$S(1) = \sum_{(1)} M_{p_1\ldots p_k,\rho_1} M_{\rho_1,\rho_2} M_{\rho_2,\rho_3} \ldots \,.$$

and

$$S(j).(t) = \sum_{(j).(t)} M_{p_1\dots p_k,\rho_1} M_{\rho_1\rho_2} M_{\rho_2,\rho_3}\dots, 2 \le j \le k, t \ge 1,$$

where $\sum_{(1)}$ and $\sum_{(j).(t)}$ means summation over all runs in the classes (1) and (j).(t), respectively. We obtain

$$S(1) = \sum_{\rho'_1,\rho'_2,\dots\in\Gamma^+} M_{p_1,\rho'_1} M_{\rho'_1,\rho'_2} M_{\rho'_2,\rho'_3}\dots = (M^\omega)_{p_1}\,.$$

For $2 \le j \le k-1$, $t \ge 1$, we obtain

$$\begin{aligned} S(j).(t) &= \Big(\sum_{\rho'_1,\rho'_2,\dots,\rho'_{t-1}\in\Gamma^+} M_{p_1\dots p_{j-1},\rho'_1}\dots M_{\rho'_{t-2},\rho'_{t-1}} M_{\rho'_{t-1},\varepsilon}\Big)\cdot\Big(\sum_{\rho'_{t+1},\rho'_{t+2},\dots\in\Gamma^+} M_{p_j,\rho'_{t+1}} M_{\rho'_{t+1},\rho'_{t+2}}\dots\Big) \\ &= (M^t)_{p_1\dots p_{j-1},\varepsilon}(M^\omega)_{p_j}\,. \end{aligned}$$

For $t \ge 1$,

$$\begin{aligned} S(k).(t) &= \Big(\sum_{\rho'_1,\rho'_2,\dots,\rho'_{t-1}\in\Gamma^+} M_{p_1\dots p_{k-1},\rho'_1}\dots M_{\rho'_{t-2},\rho'_{t-1}} M_{\rho'_{t-1},\varepsilon}\Big)\cdot\Big(\sum_{\rho_{t+1},\rho_{t+2},\dots\in\Gamma^+} M_{p_k,\rho_{t+1}} M_{\rho_{t+1},\rho_{t+2}}\dots\Big) \\ &= (M^t)_{p_1\dots p_{k-1},\varepsilon}(M^\omega)_{p_k}\,. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} (M^\omega)_{p_1\dots p_k} &= S(1) + \sum_{2\le j\le k}\sum_{t\ge 1} S(j).(t) = (M^\omega)_{p_1} + \sum_{2\le j\le k}(M^*)_{p_1\dots p_{j-1},\varepsilon}(M^\omega)_{p_j} \\ &= \sum_{1\le j\le k}(M^\star)_{p_1\dots p_{j-1},\varepsilon}(M^\omega)_{p_j}\,. \end{aligned}$$

□

Intuitively, our next theorem states that the infinite computations starting with p on the pushdown tape yield the same matrix $(M^\omega)_p$ as summing up, for all $\pi = p_1\dots p_k$ and all $1 \le j \le k$ the product of $M_{p,\pi}$ (i.e., changing the contents of the pushdown tape from p to π) with the matrix $(M^*)_{p_1\dots p_{j-1},\varepsilon}$ (i.e., emptying the pushdown tape with contents $p_1\dots p_{j-1}$ by finite computations) and eventually with the matrix $(M^\omega)_{p_j}$ (i.e., the infinite computations starting with p_j on the pushdown tape).

This means that in π the pushdown symbols $p_1,\dots,p_{j-1}$ are emptied by finite computations and p_j is chosen for starting the infinite computations. Clearly, $p_{j+1},\dots,p_k$ are not read.

Theorem 3. *Let (S,V) be a complete semiring-semimodule pair and let $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ be a pushdown transition matrix. Then, for all $p \in \Gamma$,*

$$(M^\omega)_p = \sum_{p_1\dots p_k\in\Gamma^+} M_{p,p_1\dots p_k}\sum_{1\le j\le k}(M^*)_{p_1\dots p_{j-1},\varepsilon}(M^\omega)_{p_j}\,.$$

Proof. We obtain, by Lemma 2

$$\begin{aligned} \sum_{p_1\dots p_k\in\Gamma^+} M_{p,p_1\dots p_k}\sum_{1\le j\le k}(M^*)_{p_1\dots p_{j-1},\varepsilon}(M^\omega)_{p_j} &= \sum_{p_1\dots p_k\in\Gamma^+} M_{p,p_1\dots p_k}(M^\omega)_{p_1\dots p_k} \\ &= \sum_{\pi\in\Gamma^*} M_{p,\pi}(M^\omega)_\pi = (MM^\omega) = M^\omega\,. \end{aligned}$$

□

We define the matrices $(A_M)_{p,p'} \in S^{n\times n}$, $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ a pushdown transition matrix, $p,p' \in \Gamma$, by

$$(A_M)_{p,p'} = \sum_{\substack{\pi=p_1\dots p_k\in\Gamma^+\\ p_j=p'}} M_{p,\pi}(M^*)_{p_1,\varepsilon}\dots(M^*)_{p_{j-1},\varepsilon}\,,$$

and $A_M \in (S^{n\times n})^{\Gamma\times\Gamma}$ by $A_M = ((A_M)_{p,p'})_{p,p'\in\Gamma}$. Whenever we use the notation A_M we mean the matrix just defined.

Theorem 4. *Let (S,V) be a complete semiring-semimodule pair and let $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ be a pushdown transition matrix. Then, for all $p \in \Gamma$,*

$$(M^\omega)_p = \sum_{p'\in\Gamma}(A_M)_{p,p'}(M^\omega)_{p'}\,.$$

Proof. We obtain by Theorem 3

$$\begin{aligned}\sum_{p'\in\Gamma}(A_M)_{p,p'}(M^\omega)_{p'} &= \sum_{p'\in\Gamma}\sum_{\pi=p_1\dots p_k\in\Gamma^+}\sum_{1\le j\le k}\delta_{p_j,p'}M_{p,\pi}(M^*)_{p_1\dots p_{j-1},\varepsilon}(M^\omega)_{p'}\\ &= \sum_{p_1\dots p_k\in\Gamma^+}M_{p,p_1\dots p_k}\sum_{1\le j\le k}\sum_{p'\in\Gamma}\delta_{p_j,p'}(M^*)_{p_1\dots p_{j-1},\varepsilon}(M^\omega)_{p'}\\ &= \sum_{p_1\dots p_k\in\Gamma^+}M_{p,p_1\dots p_k}\sum_{1\le j\le k}(M^*)_{p_1\dots p_{j-1},\varepsilon}(M^\omega)_{p_j} = (M^\omega)_p\,.\end{aligned}$$

□

When we say "G is the graph with adjacency matrix $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$" then it means that G is the graph with adjacency matrix $M' \in S^{(\Gamma^*\times n)\times(\Gamma^*\times n)}$, where M corresponds to M' with respect to the canonical isomorphism between $(S^{n\times n})^{\Gamma^*\times\Gamma^*}$ and $S^{(\Gamma^*\times n)\times(\Gamma^*\times n)}$.

Let now M be a pushdown transition matrix and $0 \le l \le n$. Then $M^{\omega,l}$ is the column vector in $(V^n)^{\Gamma^*}$ defined as follows: For $\pi \in \Gamma^*$ and $1 \le i \le n$, let $((M^{\omega,l})_\pi)_i$ be the sum of all weights of paths in the graph with adjacency matrix M that have initial vertex (π,i) and visit vertices (π',i'), $\pi' \in \Gamma^*$, $1 \le i' \le l$, infinitely often. Observe that $M^{\omega,0} = 0$ and $M^{\omega,n} = M^\omega$.

Let $P_l = \{(j_1,j_2,\dots) \in \{1,\dots,n\}^\omega \mid j_t \le l \text{ for infinitely many } t \ge 1\}$.

Then for $\pi \in \Gamma^+$, $1 \le j \le n$, we obtain

$$((M^{\omega,l})_\pi)_j = \sum_{\pi_1,\pi_2,\dots\in\Gamma^+}\sum_{(j_1,j_2,\dots)\in P_l}(M_{\pi,\pi_1})_{j,j_1}(M_{\pi_1,\pi_2})_{j_1,j_2}(M_{\pi_2,\pi_3})_{j_2,j_3}\dots\,.$$

Lemma 5. *Let (S,V) be a complete semiring-semimodule pair and let $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ be a pushdown transition matrix. Then, for all $p_1,\dots,p_k \in \Gamma$, $0 \le l \le n$,*

$$(M^{\omega,l})_{p_1\dots p_k} = \sum_{1\le j\le k}(M^*)_{p_1\dots p_{j-1},\varepsilon}(M^{\omega,l})_{p_j}.$$

Proof. By the proof of Lemma 2 and the following summation identity: Assume that $A_1,A_2,\dots$ are matrices in $S^{n\times n}$. Then, for $0 \le l \le n$, $1 \le j \le n$, and $m \ge 1$,

$$\sum_{(j_1,j_2,\dots)\in P_l}(A_1)_{j,j_1}(A_2)_{j_1,j_2}\dots = \sum_{1\le j_1,\dots,j_m\le n}(A_1)_{j,j_1}\dots(A_m)_{j_{m-1},j_m}\sum_{(j_{m+1},j_{m+2},\dots)\in P_l}(A_{m+1})_{j_m,j_{m+1}}\dots\,.$$

□

Theorem 6 generalizes Theorem 4 from $M^{\omega,n}$ to $M^{\omega,l}$, $0 \le l \le n$.

Theorem 6. *Let (S,V) be a complete semiring-semimodule pair and let $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ be a pushdown transition matrix. Then, for all $p \in \Gamma$, $0 \le l \le n$,*

$$(M^{\omega,l})_p = \sum_{p'\in\Gamma} (A_M)_{p,p'}(M^{\omega,l})_{p'} .$$

3 Algebraic systems and ω-pushdown automata

In this section, we define ω-pushdown automata and show that for an ω-pushdown automaton $\mathscr{P}$ there exists an algebraic system over a quemiring such that the behavior $\|\mathscr{P}\|$ of $\mathscr{P}$ is a component of a solution of this system.

For the definition of an S'-algebraic system over a quemiring $S\times V$ we refer the reader to [7], page 136, and for the definition of quemirings to [7], page 110. Here we note that a quemiring T is isomorphic to a quemiring $S\times V$ determined by the semiring-semimodule pair (S,V), cf. [7], page 110.

In the sequel, (S,V) is a complete semiring-semimodule pair and S' is a subset of S containing 0 and 1. Let $M \in (S'^{n\times n})^{\Gamma^*\times\Gamma^*}$ be a pushdown matrix. Consider the $S'^{n\times n}$-algebraic system over the complete semiring-semimodule pair $(S^{n\times n}, V^n)$, i.e., over the quemiring $S^{n\times n}\times V^n$,

$$y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi} y_\pi , \; p \in \Gamma . \tag{1}$$

(See Section 5.6 of Ésik, Kuich [7].) The variables of this system (1) are $y_p, p \in \Gamma$, and $y_\pi, \pi \in \Gamma^*$, is defined by $y_{p\pi} = y_p y_\pi$ for $p \in \Gamma$, $\pi \in \Gamma^*$ and $y_\varepsilon = \varepsilon$. Hence, for $\pi = p_1 \dots p_k$, $y_\pi = y_{p_1} \dots y_{p_k}$. The variables y_p are variables for $(S^{n\times n}, V^n)$.

Let $x = (x_p)_{p\in\Gamma}$, where x_p, $p \in \Gamma$, are variables for $S^{n\times n}$. Then, for $p \in \Gamma$, $\pi = p_1 p_2 \dots p_k$, $(M_{p,\pi} y_\pi)_x$ is defined to be

$$(M_{p,\pi} y_\pi)_x = (M_{p,\pi} y_{p_1} \dots y_{p_k})_x = M_{p,\pi} z_{p_1} + M_{p,\pi} x_{p_1} z_{p_2} + \dots + M_{p,\pi} x_{p_1} \dots x_{p_{k-1}} z_{p_k} .$$

Here z_p, $p \in \Gamma$, are variables for V^n.

We obtain, for $p \in \Gamma$, $\pi = p_1 \dots p_k$,

$$\begin{aligned}(M_{p,\pi} y_\pi)_x &= \sum_{p'\in\Gamma} \sum_{\substack{\pi = p_1\dots p_k \in \Gamma^+ \\ p_j = p'}} M_{p,\pi} x_{p_1} \dots x_{p_{j-1}} z_{p'} \\ &= \sum_{\pi = p_1\dots p_k \in \Gamma^+} M_{p,\pi} \sum_{1\le j\le k} x_{p_1} \dots x_{p_{j-1}} z_{p_j} .\end{aligned}$$

The system (1) induces the following mixed ω-algebraic system:

$$x_p = \sum_{\pi\in\Gamma^*} M_{p\pi} x_\pi , \; p \in \Gamma , \tag{2}$$

$$z_p = \sum_{\pi\in\Gamma^*} (M_{p,\pi} y_\pi)_{(x_p)_{p\in\Gamma}} = \sum_{p'\in\Gamma} \sum_{\substack{\pi = p_1\dots p_k \in \Gamma^+ \\ p_j = p'}} M_{p,\pi} x_{p_1} \dots x_{p_{j-1}} z_{p'} . \tag{3}$$

Here (2) is an $S'^{n\times n}$-algebraic system over the semiring $S^{n\times n}$ (see Section 2.3 of Ésik, Kuich [7]) and (3) is an $S^{n\times n}$-linear system over the semimodule V^n (see Section 5.5 of Ésik, Kuich [7]).

In the classical theory of automata and formal languages, equation (2) plays a crucial role in the transition from pushdown automata to context-free grammars. It is, in the form of matrix notation, the well-known triple construction. (See Harrison [12], Theorem 5.4.3; Bucher, Maurer [2], Sätze 2.3.10, 2.3.30; Kuich, Salomaa [14], pages 178, 306; Kuich [13], page 642; Ésik, Kuich [7], pages 77, 78.)

By Theorem 5.6.1 of Ésik, Kuich [7], $(A,U) \in ((S^{n\times n})^{\Gamma}, (V^n)^{\Gamma})$ is a solution of (1) iff A is a solution of (2) and (A,U) is a solution of (3). We now compute such solutions (A,U).

Theorem 7. *Let S be a complete starsemiring and $M \in (S'^{n\times n})^{\Gamma^*\times\Gamma^*}$ be a pushdown transition matrix. Then $((M^*)_{p,\varepsilon})_{p\in\Gamma}$ is a solution of* (2).

Proof. By Theorem 1. □

We now substitute in (3) for $(x_p)_{p\in\Gamma}$ the solution $((M^*)_{p,\varepsilon})_{p\in\Gamma}$ of (1) and obtain the $S'^{n\times n}$-linear system (4) over the semimodule V^n

$$z_p = \sum_{p'\in\Gamma} (A_M)_{p,p'} z_{p'},\ p \in \Gamma. \tag{4}$$

Theorem 8. *Let (S,V) be a complete semiring-semimodule pair and $M \in (S'^{n\times n})^{\Gamma^*\times\Gamma^*}$ be a pushdown transition matrix. Then, for all $0 \le l \le n$, $((M^{\omega,l})_p)_{p\in\Gamma}$ is a solution of* (4).

Proof. By Theorem 6. □

Corollary 9. *Let (S,V) be a complete semiring-semimodule pair and $M \in (S'^{n\times n})^{\Gamma^*\times\Gamma^*}$ be a pushdown transition matrix. Then, for all $0 \le l \le n$,*

$$(((M^*)_{p,\varepsilon})_{p\in\Gamma}, ((M^{\omega,l})_p)_{p\in\Gamma})$$

is a solution of (1).

We can write the system (4) in matrix notation in the form

$$z = A_M z \tag{5}$$

with column vector $z = (z_p)_{p\in\Gamma}$.

Corollary 10. *Let (S,V) be a complete semiring-semimodule pair and $M \in (S'^{n\times n})^{\Gamma^*\times\Gamma^*}$ be a pushdown transition matrix. Then for all $0 \le l \le n$, $((M^{\omega,l})_p)_{p\in\Gamma}$ is a solution of* (5).

We now introduce pushdown automata and ω-pushdown automata (see Kuich, Salomaa [14], Kuich [13], Cohen, Gold [3]).

Let S be a complete semiring and $S' \subseteq S$ with $0,1 \in S'$. An *S'-pushdown automaton over S*

$$\mathscr{P} = (n, \Gamma, I, M, P, p_0)$$

is given by

(*i*) a finite set of *states* $\{1,\ldots,n\}$, $n \ge 1$,

(*ii*) an alphabet Γ of *pushdown symbols*,

(*iii*) a *pushdown transition matrix* $M \in (S'^{n\times n})^{\Gamma^*\times\Gamma^*}$,

(*iv*) an *initial state vector* $I \in S'^{1\times n}$, (*v*) a *final state vector* $P \in S'^{n\times 1}$,

(*vi*) an *initial pushdown symbol* $p_0 \in \Gamma$,

The *behavior* $\|\mathscr{P}\|$ *of* $\mathscr{P}$ is an element of S and is defined by $\|\mathscr{P}\| = I(M^*)_{p_0,\varepsilon}P$.

For a complete semiring-semimodule pair (S,V), an S'-ω-pushdown automaton (over (S,V))

$$\mathscr{P} = (n,\Gamma,I,M,P,p_0,l)$$

is given by an S'-pushdown automaton (n,Γ,I,M,P,p_0) and an $l \in \{0,\ldots,n\}$ indicating that the states $1,\ldots,l$ are *repeated states*.

The *behavior* $\|\mathscr{P}\|$ of the S'-ω-pushdown automaton $\mathscr{P}$ is defined by

$$\|\mathscr{P}\| = I(M^*)_{p_0,\varepsilon}P + I(M^{\omega,l})_{p_0}\,.$$

Here $I(M^*)_{p_0,\varepsilon}P$ is the behavior of the S'-ω-pushdown automaton $\mathscr{P}_1 = (n,\Gamma,I,M,P,p_0,0)$ and $I(M^{\omega,l})_{p_0}$ is the behavior of the S'-ω-pushdown automaton $\mathscr{P}_2 = (n,\Gamma,I,M,0,p_0,l)$. Observe that $\mathscr{P}_2$ is an automaton with the Büchi acceptance condition: if G is the graph with adjacency matrix M, then only paths that visit the repeated states $1,\ldots,l$ infinitely often contribute to $\|\mathscr{P}_2\|$. Furthermore, $\mathscr{P}_1$ contains no repeated states and behaves like an ordinary S'-pushdown automaton.

Theorem 11. *Let (S,V) be a complete semiring-semimodule pair and let $\mathscr{P} = (n,\Gamma,I,M,P,p_0,l)$ be an S'-ω-pushdown automaton over (S,V). Then $(\|\mathscr{P}\|,(((M^*)_{p,\varepsilon})_{p\in\Gamma},((M^{\omega,l})_p)_{p\in\Gamma}))$ is a solution of the $S'^{n\times n}$-algebraic system*

$$y_0 = Iy_{p_0}P, y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi}y_\pi,\ p \in \Gamma$$

over the complete semiring-semimodule pair $(S^{n\times n},V^n)$.

Proof. By Corollary 9, $(((M^*)_{p,\varepsilon})_{p\in\Gamma},((M^{\omega,l})_p)_{p\in\Gamma})$ is a solution of the second equation. Since

$$I(((M^*)_{p_0,\varepsilon}),((M^{\omega,l})_{p_0}))P = (I(M^*)_{p_0,\varepsilon}P, I(M^{\omega,l})_{p_0}) = \|\mathscr{P}\|\,,$$

$(\|\mathscr{P}\|,(((M^*)_{p,\varepsilon})_{p\in\Gamma},((M^{\omega,l})_p)_{p\in\Gamma}))$ is a solution of the given $S'^{n\times n}$-algebraic system. □

Let S be a complete star-omega semiring and Σ be an alphabet. Then by Theorem 5.5.5 of Ésik, Kuich [7], $(S\ll\Sigma^*\gg, S\ll\Sigma^\omega\gg)$ is a complete semiring-semimodule pair. Let $\mathscr{P} = (n,\Gamma,M,I,P,p_0,l)$ be an $S\langle\Sigma\cup\{\varepsilon\}\rangle$-$\omega$-pushdown automaton over $(S\ll\Sigma^*\gg,\ S\ll\Sigma^\omega\gg)$. Consider the algebraic system over the complete semiring-semimodule pair $((S\ll\Sigma^*\gg)^{n\times n},(S\ll\Sigma^\omega\gg)^n)$

$$y_0 = Iy_{p_0}P, y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi}y_\pi, p \in \Gamma \tag{6}$$

and the mixed algebraic system (7) over $((S\ll\Sigma^*\gg)^{n\times n},(S\ll\Sigma^\omega\gg)^n)$ induced by (6)

$$\begin{aligned} x_0 &= Ix_{p_0}P, x_p = \sum_{\pi=p_1\ldots p_k\in\Gamma^*} M_{p,\pi}x_{p_1}\ldots x_{p_k}, p\in\Gamma, \\ z_0 &= Iz_{p_0}, z_p = \sum_{\pi=p_1\ldots p_k\in\Gamma^+} M_{p,\pi}\sum_{1\le j\le k} x_{p_1}\ldots x_{p_{j-1}}z_{p_j}, p\in\Gamma. \end{aligned} \tag{7}$$

Corollary 12. *Let (S,V) be a complete semiring-semimodule pair, Σ be an alphabet and $\mathscr{P} = (n,\Gamma,M,I,P,p_0,l)$ be an $S\langle\Sigma\cup\{\varepsilon\}\rangle$-$\omega$-pushdown automaton over $(S\ll\Sigma^*\gg,S\ll\Sigma^\omega\gg)$.*

Then $(I(M^*)_{p_0,\varepsilon}P,((M^*)_{p,\varepsilon})_{p\in\Gamma},I(M^{\omega,l})_{p_0},((M^{\omega,l})_p)_{p\in\Gamma})$ is a solution of (7)*. It is called solution of order l.*

Let now in (7)

$$x = ([i,p,j])_{1\le i,j\le n}, p \in \Gamma,$$

be $n \times n$-matrices of variables and

$$z = ([i,p])_{1\le i\le n}, p \in \Gamma$$

be n-dimensional column vectors of variables. If we write the mixed algebraic system (7) component-wise, we obtain a mixed algebraic system over $((S \ll \Sigma^* \gg),(S \ll \Sigma^\omega \gg))$ with variables $[i,p,j]$ over $S \ll \Sigma^* \gg$, where $p \in \Gamma$, $1 \le i,j \le n$, and variables $[i,p]$ over $S \ll \Sigma^\omega \gg$, where $p \in \Gamma$, $1 \le i \le n$.

Writing the mixed algebraic system (7) component-wise, we obtain the system (8):

$$\begin{aligned}
x_0 &= \sum_{1\le m_1,m_2\le n} I_{m_1}[m_1,p_0,m_2]P_{m_2},\\
[i,p,j] &= \sum_{k\ge 0}\sum_{p_1,\dots,p_k\in\Gamma}\sum_{1\le m_1,\dots,m_k\le n}(Mp,p_1\dots p_k)_{i,m_1}[m_1,p_1,m_2][m_2,p_2,m_3]\dots[m_k,p_k,j],\\
&\qquad\qquad p\in\Gamma, 1\le i,j\le n,\\
z_0 &= \sum_{1\le m\le n} I_m[m,p_0]\\
[i,p] &= \sum_{k\ge 1}\sum_{p_1,\dots,p_k\in\Gamma}\sum_{1\le j\le k}\sum_{1\le m_1,\dots,m_j\le n}(Mp,p_1\dots p_k)_{i,m_1}[m_1,p_1,m_2]\dots[m_{j-1},p_{j-1},m_j][m_j,p_j],\\
&\qquad\qquad p\in\Gamma, 1\le i\le n.
\end{aligned} \tag{8}$$

Theorem 13. *Let (S,V) be a complete semiring-semimodule pair and $\mathscr{P} = (n,\Gamma,M,I,p_0,P,l)$ be a S'-ω-pushdown automaton. Then*

$$(I(M^*)_{p_0,\varepsilon}P,(((M^*)_{p,\varepsilon})_{i,j})_{p\in\Gamma,1\le i,j\le n}, I(M^{\omega,l})_{p_0},((M^{\omega,l})_p)_i)_{p\in\Gamma,1\le i\le n}$$

is a solution of the system (8) *called solution of order l with $\|\mathscr{P}\| = (I(M^*)_{p_0,\varepsilon}P, I(M^{\omega,l})_{p_0})$.*

4 Mixed algebraic systems and mixed context-free grammars

In this section we associate a mixed context-free grammar with finite and infinite derivations to the algebraic system (8). The language generated by this mixed context-free grammar is then the behavior $\|\mathscr{P}\|$ of the ω-pushdown automaton $\mathscr{P}$. The construction of the mixed context-free grammar from the ω-pushdown automaton $\mathscr{P}$ is a generalization of the well known triple construction and is called now *triple-pair construction for ω-pushdown automata*. We will consider the commutative complete star-omega semirings $\mathbb{B} = (\{0,1\},\vee,\wedge,*,0,1)$ with $0^* = 1^* = 1$ and $\mathbb{N}^\infty = (\mathbb{N}\cup\{\infty\},+,\cdot,^*,0,1)$ with $0^* = 1$ and $a^* = \infty$ for $a \ne 0$.

If $S = \mathbb{B}$ or $S = \mathbb{N}^\infty$ and $0 \le l \le n$, then we associate to the mixed algebraic system (8) over $((S \ll \Sigma^* \gg),(S \ll \Sigma^\omega \gg))$, and hence to the ω-pushdown automaton $\mathscr{P} = (n,\Gamma,I,M,P,p_0,l)$, the *mixed context-free grammar*

$$G_l \;=\; (X,Z,\Sigma,P_X,P_Z,x_0,z_0,l)\,.$$

(See also Ésik, Kuich [7, page 139].) Here

(i) $X = \{x_0\} \cup \{[i,p,j] \mid 1 \le i,j \le n,\ p \in \Gamma\}$ is a set of *variables for finite derivations*;

(ii) $Z = \{z_0\} \cup \{[i,p] \mid 1 \le i \le n,\ p \in \Gamma\}$ is a set of *variables for infinite derivations*;

(iii) Σ is an alphabet of *terminal symbols*;

(iv) P_X is a finite set of *productions for finite derivations* given below;

(v) P_Z is a finite set of *productions for infinite derivations* given below;

(vi) x_0 is the *start variable for finite derivations*;

(vii) z_0 is the *start variable for infinite derivations*;

(viii) $\{[i,p] \mid 1 \le i \le l,\, p \in \Gamma\}$ is the set of *repeated variables for infinite derivations*.

In the definition of G_l the sets P_X and P_Z are as follows:

$$\begin{aligned}
P_X = \{&x_0 \to a_1[m_1,p_0,m_2]a_2 \mid \\
&1 \le m_1,m_2 \le n, (I_{m_1},a_1) \neq 0, (P_{m_2},a_2) \neq 0, a_1,a_2 \in \Sigma \cup \{\varepsilon\}\} \cup \\
&\{[i,p,j] \to a[m_1,p_1,m_2][m_2,p_2,m_3]\dots[m_k,p_k,j] \mid p \in \Gamma, 1 \le i,j \le n, k \ge 0, \\
&p_1,\dots,p_k \in \Gamma, 1 \le m_1,\dots,m_k \le n, ((M_{p,p_1\dots p_k})_{i,m_1},a) \neq 0, a \in \Sigma \cup \{\varepsilon\}\}\,, \\
P_Z = \{&z_0 \to a[m,p_0] \mid 1 \le m \le n, (I_m,a) \neq 0, a \in \Sigma \cup \{\varepsilon\}\} \cup \\
&\{[i,p] \to a[m_1,p_1,m_2]\dots[m_{j-1},p_{j-1},m_j][m_j,p_j] \mid p,p_1,\dots,p_k \in \Gamma, 1 \le i \le n, \\
&k \ge 1, 1 \le j \le k, 1 \le m_1,\dots,m_j \le n, ((M_{p,p_1\dots p_k})_{i,m_1},a) \neq 0, a \in \Sigma \cup \{\varepsilon\}\}\,.
\end{aligned}$$

For the remainder of this section, $\mathscr{P}$ always denotes the ω-pushdown automaton $\mathscr{P} = (n,\Gamma,I,M,P,p_0,l)$. Especially this means that l is a fixed parameter. Observe that $((M_{p,p_1\dots p_k})_{i,m_1},a) \neq 0$ iff $(m_1,p_k\dots p_1) \in \delta(i,a,p)$ in the usual δ-notation for the transition function of a classical pushdown automaton. (See Harrison [12] and Kuich [13] pages 638/639.) Here we have to reverse $p_1\dots p_k$ since the pushdown tape of classical pushdown automata has its rightmost element as top element.

A *finite leftmost derivation* $\alpha_1 \Rightarrow_L^* \alpha_2$, where $\alpha_1,\alpha_2 \in (X \cup \Sigma)^*$, by productions in P_X is defined as usual. An *infinite (leftmost) derivation* $\pi : z_0 \Rightarrow_L^\omega w$, for $z_0 \in Z, w \in \Sigma^\omega$, is defined as follows:

$$\begin{aligned}
\pi : z_0 &\Rightarrow_L \alpha_0[i_0,p_0] \Rightarrow_L^* w_0[i_0,p_0] \Rightarrow_L w_0\alpha_1[i_1,p_1] \Rightarrow_L^* w_0w_1[i_1,p_1] \Rightarrow_L \dots \\
&\Rightarrow_L^* w_0w_1\dots w_m[i_m,p_m] \Rightarrow_L w_0w_1\dots w_m\alpha_{m+1}[i_{m+1},p_{m+1}] \Rightarrow_L^* \dots\,,
\end{aligned}$$

where $z_0 \to \alpha_0[i_0,p_0], [i_0,p_0] \to \alpha_1[i_1,p_1], \dots, [i_m,p_m] \to \alpha_{m+1}[i_{m+1},p_{m+1}], \dots$ are productions in P_Z and $w = w_0w_1\dots w_m\dots$.

We now define an infinite derivation $\pi_l : z_0 \Rightarrow_L^{\omega,l} w$ for $0 \le l \le n$, $z_0 \in Z$, $w \in \Sigma^\omega$: We take the above definition for $\pi : z_0 \Rightarrow_L^\omega w$ and consider the sequence of the first elements i of the triple variables $[i,p,j]$ of X that are rewritten in the finite leftmost derivation $\alpha_m \Rightarrow_L^* w_m$, $m \ge 0$. Assume this sequence is $i_m^1, i_m^2, \dots, i_m^{t_m}$ for some t_m, $m \ge 1$. Then, to obtain π_l from π, the condition $i_0, i_1^1, i_1^2 \dots, i_1^{t_1}, i_1, i_2^1, \dots, i_2^{t_2}, i_2, \dots, i_m, i_{m+1}^1, \dots, i_{m+1}^{t_{m+1}}, i_{m+1}, \dots \in P_l$ has to be satisfied.

Then we define

$$L(G_l) = \{w \in \Sigma^* \mid x_0 \Rightarrow_L^* w\} \,\cup\, \{w \in \Sigma^\omega \mid \pi : z_0 \Rightarrow_L^{\omega,l} w\}\,.$$

Observe that the construction of G_l from $\mathscr{P}$ is nothing else than a generalization of the triple construction for ω-pushdown automata, since the construction of the context-free grammar $G = (X,\Sigma,P_X,x_0)$ is the triple construction. (See Harrison [12], Theorem 5.4.3; Bucher, Maurer [2], Sätze 2.3.10, 2.3.30; Kuich, Salomaa [14], pages 178, 306; Kuich [13], page 642; Ésik, Kuich [7], pages 77, 78.)

We call the construction of the mixed context-free grammar G_l from $\mathscr{P}$ the *triple-pair construction for ω-pushdown automata*. This is justified by the definition of the sets of variables $\{[i,p,j] \mid 1 \le i,j \le n, p \in \Gamma\}$ and $\{[i,p] \mid 1 \le i \le n, p \in \Gamma\}$ of G_l and by the forthcoming Corollary 15.

In the next theorem we use the isomorphism between $\mathbb{B} \ll \Sigma^* \gg \times\, \mathbb{B} \ll \Sigma^\omega \gg$ and $2^{\Sigma^*} \times 2^{\Sigma^\omega}$.

Theorem 14. *Assume that* (σ,τ) *is the solution of order* l *of the mixed algebraic system* (8) *over* $(\mathbb{B} \ll \Sigma^* \gg, \mathbb{B} \ll \Sigma^\omega \gg)$ *for* $k \in \{0,\dots,n\}$. *Then*

$$L(G_l) \;=\; \sigma_{x_0} \cup \tau_{z_0}\,.$$

Proof. By Theorem IV.1.2 of Salomaa, Soittola [16] and by Theorem 13, we obtain $\sigma_{x_0} = \{w \in \Sigma^* \mid x_0 \Rightarrow_L^* w\}$. We now show that τ_{z_0} is generated by the infinite derivations $\Rightarrow_L^{\omega,l}$ from z_0. First observe that the rewriting by the typical $[i,p,j]$- and $[i,p]$- production corresponds to the situation that in the graph of the ω-pushdown automaton $\mathscr{P}$ the edge from $(p\rho,i)$ to $(p_1\dots p_j\rho,j)$, $\rho \in \Gamma^*$, is passed after the state i is visited. The first step of the infinite derivation π_l is given by $z_0 \Rightarrow_L \alpha_0[i_0,p]$ and indicates that the path in the graph of $\mathscr{P}$ corresponding to π_l starts in state i_0. Furthermore, the sequence of the first elements of variables that are rewritten in π_l, i.e., $i_0, i_1^1, \dots, i_1^{t_1}, i_1, i_2^1, \dots, i_2^{t_2}, i_2, \dots, i_m, i_{m+1}^1, \dots, i_{m+1}^{t_{m+1}}, i_{m+1}, \dots$ indicates that the path in the graph of $\mathscr{P}$ corresponding to π_l visits these states. Since this sequence is in P_l the corresponding path contributes to $\|\mathscr{P}\|$. Hence, by Theorem IV.1.2 of Salomaa, Soittola [16] and Theorem 13 for the finite leftmost derivations $\alpha_m \Rightarrow_L^* w_m$, $m \geq 1$, and by Theorem 5.5.9 of Ésik, Kuich [7] and Theorem 13 for the infinite derivation $[i_0,p_0] \Rightarrow \alpha_1[i_1,p_1] \Rightarrow \alpha_1\alpha_2[i_2,p_2] \Rightarrow \dots \Rightarrow \alpha_1\alpha_2\dots\alpha_m[i_m,p_m] \Rightarrow \dots$ we obtain

$$\tau_{z_0} = \{w \in \Sigma^\omega \mid \pi : z_0 \Rightarrow_L^{\omega,l} w\}\,.$$

□

Corollary 15. *Assume that the mixed context free grammar* G_l *associated to the mixed algebraic system* (8) *is constructed from the* $\mathbb{B}\langle\Sigma \cup \{\varepsilon\}\rangle$-$\omega$*-pushdown automaton* $\mathscr{P}$. *Then*

$$L(G_l) = \|\mathscr{P}\|\,.$$

Proof. By Theorems 13 and 14. □

For the remainder of this section our basic semiring is $\mathbb{N}^\infty$, which allows us to draw some stronger conclusions.

Theorem 16. *Assume that* (σ,τ) *is the solution of order* l *of the mixed algebraic system* (8) *over* $(\mathbb{N}^\infty \ll \Sigma^* \gg, \mathbb{N}^\infty \ll \Sigma^\omega \gg)$ *where the entries of* I,M,P *are in* $\{0,1\}\langle\Sigma \cup \{\varepsilon\}\rangle$. *Denote by* $d(w)$*, for* $w \in \Sigma^*$*, the number (possibly* ∞*) of distinct finite leftmost derivations of* w *from* x_0 *with respect to* G_l*; and by* $c(w)$*, for* $w \in \Sigma^\omega$*, the number (possibly* ∞*) of distinct infinite leftmost derivations* π *of* w *from* z_0 *with respect to* G_l. *Then*

$$\sigma_{x_0} = \sum_{w \in \Sigma^*} d(w)w \qquad \text{and} \qquad \tau_{z_0} = \sum_{w \in \Sigma^\omega} c(w)w\,.$$

Proof. The proof of Theorem 16 is identical to the proof of Theorem 14 with the exceptions that Theorem IV.1.2 of Salomaa, Soittola [16] is replaced by Theorem IV.1.5 and Theorem 5.5.9 of Ésik, Kuich [7] is replaced by Theorem 5.5.10. □

In the forthcoming Corollary 17 we consider, for a given $\{0,1\}\langle\Sigma \cup \{\varepsilon\}\rangle$-$\omega$-pushdown automaton $\mathscr{P} = (n,\Gamma,I,M,P,p_0,l)$ the number of distinct computations from an initial instantaneous description (i,w,p_0) for $w \in \Sigma^*$, $I_i \neq 0$, to an accepting instantaneous description $(j,\varepsilon,\varepsilon)$, with $P_j \neq 0$, $i,j \in \{0,\dots,n\}$.

Here (i,w,p_0) means that $\mathscr{P}$ starts in the initial state i with w on its input tape and p_0 on its pushdown tape; and $(j,\varepsilon,\varepsilon)$ means that $\mathscr{P}$ has entered the final state j with empty input tape and empty pushdown tape.

Furthermore, we consider the number of distinct infinite computations starting in an initial instantaneous description (i, w, p_0) for $w \in \Sigma^\infty$, $I_i \neq 0$.

Corollary 17. *Assume that the mixed context-free grammar G_l associated to the mixed algebraic system (8) is constructed from the $\{0,1\}\langle\Sigma \cup \{\varepsilon\}\rangle$-$\omega$-pushdown automaton $\mathscr{P}$. Then the number (possibly ∞) of distinct finite leftmost derivations of w, $w \in \Sigma^*$, from x_0 equals the number of distinct finite computations from an initial instantaneous description for w to an accepting instantaneous description; moreover, the number (possibly ∞) of distinct infinite (leftmost) derivations of w, $w \in \Sigma^\omega$, from z_0 equals the number of distinct infinite computations starting in an initial instantaneous description for w.*

Proof. By Corollary 6.11 of Kuich [13] and the definition of infinite derivations with respect to G_l. □

The context-free grammar G_l associated to (8) is called *unambiguous* if each $w \in L(G_l)$, $w \in \Sigma^*$ has a unique finite leftmost derivation and each $w \in L(G_l)$, $w \in \Sigma^\omega$, has a unique infinite (leftmost) derivation.

An $\mathbb{N}^\infty\langle\Sigma \cup \{\varepsilon\}\rangle$-$\omega$-pushdown automaton $\mathscr{P}$ is called *unambiguous* if $(\|\mathscr{P}\|, w) \in \{0,1\}$ for each $w \in \Sigma^* \cup \Sigma^\omega$.

Corollary 18. *Assume that the mixed context-free grammar G_l associated to the mixed algebraic system (8) is constructed from the $\{0,1\}\langle\Sigma \cup \{\varepsilon\}\rangle$-$\omega$-pushdown automaton $\mathscr{P}$. Then G_l is unambiguous iff $\|\mathscr{P}\|$ is unambiguous.*

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