*-Continuous Kleene ω-Algebras for Energy Problems*

Zoltán Ésik
University of Szeged, Hungary

Uli Fahrenberg   Axel Legay
Inria Rennes, France

Energy problems are important in the formal analysis of embedded or autonomous systems. Using recent results on *-continuous Kleene ω-algebras, we show here that energy problems can be solved by algebraic manipulations on the transition matrix of energy automata. To this end, we prove general results about certain classes of finitely additive functions on complete lattices which should be of a more general interest.

1 Introduction

Energy problems are concerned with the question whether a given system admits infinite schedules during which (1) certain tasks can be repeatedly accomplished and (2) the system never runs out of energy (or other specified resources). These are important in areas such as embedded systems or autonomous systems and, starting with [4], have attracted some attention in recent years, for example in [3, 5–8, 16, 19, 23, 24].

With the purpose of generalizing some of the above approaches, we have in [12,17] introduced *energy automata*. These are finite automata whose transitions are labeled with *energy functions* which specify how energy values change from one system state to another. Using the theory of semiring-weighted automata [9], we have shown in [12] that energy problems in such automata can be solved in a simple static way which only involves manipulations of energy functions.

In order to put the work of [12] on a more solid theoretical footing and with an eye to future generalizations, we have recently introduced a new algebraic structure of **-continuous Kleene* ω*-algebras* [10] (see also [11] for the long version). We show here that energy functions form such a *-continuous Kleene ω-algebra. Using the fact, proven in [10], that for automata with transition weights in *-continuous Kleene ω-algebras, reachability and Büchi acceptance can be computed by algebraic manipulations on the transition matrix of the automaton, the results from [12] follow.

2 Energy Automata

The transition labels on the energy automata which we consider in the paper, will be functions which model transformations of energy levels between system states. Such transformations have the (natural) properties that below a certain energy level, the transition might be disabled (not enough energy is available to perform the transition), and an increase in input energy always yields at least the same increase in output energy. Thus the following definition:

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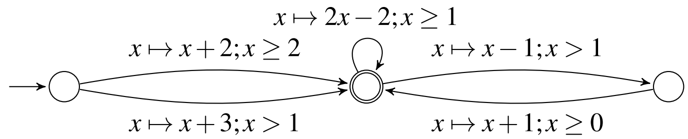

Figure 1: A simple energy automaton.

Definition 1 An *energy function* is a partial function $f : \mathbb{R}_{\geq 0} \rightharpoonup \mathbb{R}_{\geq 0}$ which is defined on a closed interval $[l_f, \infty[$ or on an open interval $]l_f, \infty[$, for some lower bound $l_f \geq 0$, and such that for all $x \leq y$ for which f is defined,

$$yf \geq xf + y - x. \tag{1}$$

The class of all energy functions is denoted by $\mathscr{F}$.

Note that we write function composition and application in diagrammatical order, *from left to right*, in this paper. Hence we write $f;g$, or simply fg, for the composition $g \circ f$ and $x;f$ or xf for function application $f(x)$. This is because we will be concerned with *algebras* of functions, in which function composition is multiplication, and where it is customary to write multiplication in diagrammatical order.

Thus energy functions are strictly increasing, and in points where they are differentiable, the derivative is at least 1. The inverse functions to energy functions exist, but are generally not energy functions. Energy functions can be *composed*, where it is understood that for a composition fg, the interval of definition is $\{x \in \mathbb{R}_{\geq 0} \mid xf \text{ and } xfg \text{ defined}\}$.

Lemma 1 *Let $f \in \mathscr{F}$ and $x \in \mathbb{R}_{\geq 0}$. If $xf < x$, then there is $N \in \mathbb{N}$ for which xf^N is not defined. If $xf > x$, then for all $P \in \mathbb{R}$ there is $N \in \mathbb{N}$ for which $xf^N \geq P$.*

Proof In the first case, we have $x - xf = M > 0$. Using (1), we see that $xf^{n+1} \leq xf^n - M$ for all $n \in \mathbb{N}$ for which xf^{n+1} is defined. Hence $(xf^n)_{n \in \mathbb{N}}$ decreases without bound, so that there must be $N \in \mathbb{N}$ such that xf^N is undefined.

In the second case, we have $xf - x = M > 0$. Again using (1), we see that $xf^{n+1} > xf^n + M$ for all $n \in \mathbb{N}$. Hence $(xf^n)_{n \in \mathbb{N}}$ increases without bound, so that for any $P \in \mathbb{R}$ there must be $N \in \mathbb{N}$ for which $xf^N \geq P$. □

Note that property (1) is not only sufficient for Lemma 1, but in a sense also necessary: if $0 < \alpha < 1$ and $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the function $xf = 1 + \alpha x$, then $xf^n = \sum_{i=0}^{n-1} \alpha^i + \alpha^n x$ for all $n \in \mathbb{N}$, hence $\lim_{n \to \infty} xf^n = \frac{1}{1-\alpha}$, so Lemma 1 does not hold for f. On the other hand, $yf = xf + \alpha(y - x)$ for all $x \leq y$, so (1) "almost" holds.

Definition 2 An *energy automaton* (S, s_0, T, F) consists of a finite set S of states, with initial state $s_0 \in S$, a finite set $T \subseteq S \times \mathscr{F} \times S$ of transitions labeled with energy functions, and a subset $F \subseteq S$ of acceptance states.

We show an example of a simple energy automaton in Fig. 1. Here we use inequalities to give the definition intervals of energy functions.

A finite *path* in an energy automaton is a finite sequence of transitions $\pi = (s_0, f_1, s_1), (s_1, f_2, s_2), \ldots, (s_{n-1}, f_n, s_n)$. We use f_π to denote the combined energy function $f_1 f_2 \cdots f_n$ of such a finite path. We will also use infinite paths, but note that these generally do not allow for combined energy functions.

A *global state* of an energy automaton is a pair $q = (s, x)$ with $s \in S$ and $x \in \mathbb{R}_{\geq 0}$. A transition between global states is of the form $((s,x), f, (s', x'))$ such that $(s, f, s') \in T$ and $x' = f(x)$. A (finite or infinite) *run* of (S, T) is a path in the graph of global states and transitions.

We are ready to state the decision problems with which our main concern will lie. As the input to a decision problem must be in some way finitely representable, we will state them for subclasses $\mathscr{F}' \subseteq \mathscr{F}$ of *computable* energy functions; an $\mathscr{F}'$-automaton is an energy automaton (S,T) with $T \subseteq S \times \mathscr{F}' \times S$.

Problem 1 (Reachability) Given a subset $\mathscr{F}' \subseteq \mathscr{F}$ of computable functions, an $\mathscr{F}'$-automaton $A = (S, s_0, T, F)$ and a computable initial energy $x_0 \in \mathbb{R}_{\geq 0}$: does there exist a finite run of A from (s_0, x_0) which ends in a state in F?

Problem 2 (Büchi acceptance) Given a subset $\mathscr{F}' \subseteq \mathscr{F}$ of computable functions, an $\mathscr{F}'$-automaton $A = (S, s_0, T, F)$ and a computable initial energy $x_0 \in \mathbb{R}_{\geq 0}$: does there exist an infinite run of A from (s_0, x_0) which visits F infinitely often?

As customary, a run such as in the statements above is said to be accepting.

3 Algebraic Preliminaries

We now turn our attention to the algebraic setting of *-continuous Kleene algebras and related structures, before revisiting energy automata in Section 6. In this section we review some results on *-continuous Kleene algebras and *-continuous Kleene ω-algebras.

3.1 *-Continuous Kleene ω-Algebras

A *semiring* [1, 18] $S = (S, +, \cdot, 0, 1)$ consists of a commutative monoid $(S, +, 0)$ and a monoid $(S, \cdot, 1)$ such that the distributive laws

$$x(y+z) = xy + xz$$
$$(y+z)x = yx + zx$$

and the zero laws

$$0 \cdot x = 0 = x \cdot 0$$

hold for all $x, y, z \in S$. It follows that the product operation distributes over all finite sums.

An *idempotent semiring* is a semiring S whose sum operation is idempotent, so that $x + x = x$ for all $x \in S$. Each idempotent semiring S is partially ordered by the relation $x \leq y$ iff $x + y = y$, and then sum and product preserve the partial order and 0 is the least element. Moreover, for all $x, y \in S$, $x + y$ is the least upper bound of the set $\{x, y\}$. Accordingly, in an idempotent semiring S, we will usually denote the sum operation by $\vee$ and 0 by $\perp$.

A *Kleene algebra* [22] is an idempotent semiring $S = (S, \vee, \cdot, \perp, 1)$ equipped with a star operation $^* : S \to S$ such that for all $x, y \in S$, yx^* is the least solution of the fixed point equation $z = zx \vee y$ and x^*y is the least solution of the fixed point equation $z = xz \vee y$ with respect to the natural order.

A *-*continuous Kleene algebra* [22] is a Kleene algebra $S = (S, \vee, \cdot, ^*, \perp, 1)$ in which the infinite suprema $\bigvee\{x^n \mid n \geq 0\}$ exist for all $x \in S$, $x^* = \bigvee\{x^n \mid n \geq 0\}$ for every $x \in S$, and product preserves such suprema:

$$y\Big(\bigvee_{n\geq 0} x^n\Big) = \bigvee_{n\geq 0} yx^n \quad \text{and} \quad \Big(\bigvee_{n\geq 0} x^n\Big)y = \bigvee_{n\geq 0} x^n y$$

for all $x, y \in S$.

A *continuous Kleene algebra* is a Kleene algebra $S = (S, \vee, \cdot, ^*, \perp, 1)$ in which *all* suprema $\bigvee X$, $X \subseteq S$, exist and are preserved by products, *i.e.,* $y(\bigvee X) = \bigvee yX$ and $(\bigvee X)y = \bigvee Xy$ for all $X \subseteq S$, $y \in$

S. *-continuous Kleene algebras are hence a generalization of continuous Kleene algebras. There are interesting Kleene algebras which are *-continuous but not continuous, for example the Kleene algebra of all regular languages over some alphabet.

A *semiring-semimodule pair* [2,14] (S,V) consists of a semiring $S=(S,+,\cdot,0,1)$ and a commutative monoid $V=(V,+,0)$ which is equipped with a left S-action $S\times V\to V$, $(s,v)\mapsto sv$, satisfying

$$(s+s')v = sv+s'v \qquad s(v+v') = sv+sv'$$
$$(ss')v = s(s'v) \qquad 0s = 0$$
$$s0 = 0 \qquad 1v = v$$

for all $s,s'\in S$ and $v\in V$. In that case, we also call V a *(left) S-semimodule*. If S is idempotent, then also V is idempotent, so that we then write $V=(V,\vee,\bot)$.

A *generalized *-continuous Kleene algebra* [10] is a semiring-semimodule pair (S,V) where $S=(S,\vee,\cdot,^*,\bot,1)$ is a *-continuous Kleene algebra such that

$$xy^*v = \bigvee_{n\ge 0} xy^n v$$

for all $x,y\in S$ and $v\in V$.

A **-continuous Kleene ω-algebra* [10] consists of a generalized *-continuous Kleene algebra (S,V) together with an infinite product operation $S^\omega\to V$ which maps every infinite sequence $x_0,x_1,\dots$ in S to an element $\prod_{n\ge 0}x_n$ of V. The infinite product is subject to the following conditions:

(C1) For all $x_0,x_1,\dots\in S$, $\displaystyle\prod_{n\ge 0}x_n = x_0\prod_{n\ge 0}x_{n+1}$.

(C2) Let $x_0,x_1,\dots\in S$ and $0=n_0\le n_1\le\cdots$ a sequence which increases without a bound. Let $y_k = x_{n_k}\cdots x_{n_{k+1}-1}$ for all $k\ge 0$. Then $\displaystyle\prod_{n\ge 0}x_n = \prod_{k\ge 0}y_k$.

(C3) For all $x_0,x_1,\dots,y,z\in S$, $\displaystyle\prod_{n\ge 0}(x_n(y\vee z)) = \bigvee_{x_0',x_1',\dots\in\{y,z\}}\ \prod_{n\ge 0}x_nx_n'$.

(C4) For all $x,y_0,y_1,\dots\in S$, $\displaystyle\prod_{n\ge 0}x^*y_n = \bigvee_{k_0,k_1,\dots\ge 0}\ \prod_{n\ge 0}x^{k_n}y_n$.

A *continuous Kleene ω-algebra* [14] is a semiring-semimodule pair (S,V) in which S is a continuous Kleene algebra, V is a complete lattice, and the S-action on V preserves all suprema in either argument, together with an infinite product as above which satisfies conditions (C1) and (C2) above and preserves all suprema: $\prod_{n\ge 0}(\bigvee X_n) = \bigvee\{\prod_{n\ge 0}x_n \mid x_n\in X_n, n\ge 0\}$ for all $X_0,X_1,\dots\subseteq S$ (this property implies (C3) and (C4) above). *-continuous Kleene ω-algebras are hence a generalization of continuous Kleene ω-algebras. We have in [10] given an example, based on regular languages of finite and infinite words, of a *-continuous Kleene ω-algebra which is not a continuous Kleene ω-algebra. In Section 6 we will show that energy functions give raise to another such example.

3.2 Matrix Semiring-Semimodule Pairs

For any semiring S and $n\ge 1$, we can form the matrix semiring $S^{n\times n}$ whose elements are $n\times n$-matrices of elements of S and whose sum and product are given as the usual matrix sum and product. It is known [21] that when S is a *-continuous Kleene algebra, then $S^{n\times n}$ is also a *-continuous Kleene algebra, with the *-operation defined by

$$M^*_{i,j} = \bigvee_{m\geq 0} \bigvee_{1\leq k_1,\dots,k_m\leq n} M_{i,k_1}M_{k_1,k_2}\cdots M_{k_m,j}$$

for all $M \in S^{n\times n}$ and $1 \leq i, j \leq n$. The above infinite supremum exists, as it is taken over a regular set, see [13, Thm. 9] and [10, Lemma 4]. Also, if $n \geq 2$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a and d are square matrices of dimension less than n, then

$$M^* = \begin{pmatrix} (a\vee bd^*c)^* & (a\vee bd^*c)^*bd^* \\ (d\vee ca^*b)^*ca^* & (d\vee ca^*b)^* \end{pmatrix}. \tag{2}$$

For any semiring-semimodule pair (S,V) and $n \geq 1$, we can form the matrix semiring-semimodule pair $(S^{n\times n}, V^n)$ whose elements are $n\times n$-matrices of elements of S and n-dimensional (column) vectors of elements of V, with the action of $S^{n\times n}$ on V^n given by the usual matrix-vector product.

When (S,V) is a *-continuous Kleene ω-algebra, then $(S^{n\times n}, V^n)$ is a generalized *-continuous Kleene algebra [10]. By [10, Lemma 17], there is an ω-operation on $S^{n\times n}$ defined by

$$M^\omega_i = \bigvee_{1\leq k_1,k_2,\dots\leq n} M_{i,k_1}M_{k_1,k_2}\cdots$$

for all $M \in S^{n\times n}$ and $1 \leq i \leq n$. Also, if $n \geq 2$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a and d are square matrices of dimension less than n, then

$$M^\omega = \begin{pmatrix} (a\vee bd^*c)^\omega \vee (a\vee bd^*c)^*bd^\omega \\ (d\vee ca^*b)^\omega \vee (d\vee ca^*b)^*ca^\omega \end{pmatrix}.$$

3.3 Weighted automata

Let (S,V) be a *-continuous Kleene ω-algebra and $A \subseteq S$ a subset. We write $\langle A\rangle$ for the set of all finite suprema $a_1 \vee \cdots \vee a_m$ with $a_i \in A$ for each $i = 1,\dots,m$.

A *weighted automaton* [15] over A of dimension $n \geq 1$ is a tuple (α, M, k), where $\alpha \in \{\bot, 1\}^n$ is the initial vector, $M \in \langle A\rangle^{n\times n}$ is the transition matrix, and k is an integer $0 \leq k \leq n$. Combinatorially, this may be represented as a transition system whose set of states is $\{1,\dots,n\}$. For any pair of states i, j, the transitions from i to j are determined by the entry $M_{i,j}$ of the transition matrix: if $M_{i,j} = a_1 \vee \cdots \vee a_m$, then there are m transitions from i to j, respectively labeled $a_1,\dots,a_n$. The states i with $\alpha_i = 1$ are *initial*, and the states $\{1,\dots,k\}$ are *accepting*.

The *finite behavior* of a weighted automaton $A = (\alpha, M, k)$ is defined to be

$$|A| = \alpha M^*\kappa,$$

where $\kappa \in \{\bot, 1\}^n$ is the vector given by $\kappa_i = 1$ for $i \leq k$ and $\kappa_i = \bot$ for $i > k$. (Note that α has to be used as a *row* vector for this multiplication to make sense.) It is clear by (2) that $|A|$ is the supremum of the products of the transition labels along all paths in A from any initial to any accepting state.

The *Büchi behavior* of a weighted automaton $A = (\alpha, M, k)$ is defined to be

$$\|A\| = \alpha \begin{pmatrix} (a+bd^*c)^\omega \\ d^*c(a+bd^*c)^\omega \end{pmatrix},$$

where $a \in \langle A\rangle^{k\times k}$, $b \in \langle A\rangle^{k\times(n-k)}$, $c \in \langle A\rangle^{(n-k)\times n}$ and $d \in \langle A\rangle^{(n-k)\times(n-k)}$ are such that $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. By [10, Thm. 20], $\|A\|$ is the supremum of the products of the transition labels along all infinite paths in A from any initial state which infinitely often visit an accepting state.

4 Generalized *-continuous Kleene Algebras of Functions

In the following two sections our aim is to establish properties which ensure that semiring-semimodule pairs of functions form *-continuous Kleene ω-algebras. We will use these properties in Section 6 to show that energy functions form a *-continuous Kleene ω-algebra.

Let L and L' be complete lattices with bottom and top elements $\bot$ and $\top$. Then a function $f : L \to L'$ is said to be *finitely additive* if $\bot f = \bot$ and $(x \vee y)f = xf \vee yf$ for all $x, y \in L$. (Recall that we write function application and composition in the diagrammatic order, from left to right.) When $f : L \to L'$ is finitely additive, then $(\bigvee X)f = \bigvee Xf$ for all finite sets $X \subseteq L$.

Consider the collection $\mathrm{FinAdd}_{L,L'}$ of all finitely additive functions $f : L \to L'$, ordered pointwise. Since the (pointwise) supremum of any set of finitely additive functions is finitely additive, $\mathrm{FinAdd}_{L,L'}$ is also a complete lattice, in which the supremum of any set of functions can be constructed pointwise. The least and greatest elements are the constant functions with value $\bot$ and $\top$, respectively. By an abuse of notation, we will denote these functions by $\bot$ and $\top$ as well.

Definition 3 A function $f \in \mathrm{FinAdd}_{L,L'}$ is said to be $\top$-*continuous* if $f = \bot$ or for all $X \subseteq L$ with $\bigvee X = \top$, also $\bigvee Xf = \top$.

Note that if $f \neq \bot$ is $\top$-continuous, then $\top f = \top$. The functions id and $\bot$ are $\top$-continuous. Also, the (pointwise) supremum of any set of $\top$-continuous functions is again $\top$-continuous.

We will first be concerned with functions in $\mathrm{FinAdd}_{L,L}$, which we just denote FinAdd_L. Since the composition of finitely additive functions is finitely additive and the identity function id over L is finitely additive, and since composition of finitely additive functions distributes over finite suprema, FinAdd_L, equipped with the operation $\vee$ (binary supremum), ; (composition), and the constant function $\bot$ and the identity function id as 1, is an idempotent semiring. It follows that when f is finitely additive, then so is $f^* = \bigvee_{n\geq 0} f^n$. Moreover, $f \leq f^*$ and $f^* \leq g^*$ whenever $f \leq g$. Below we will usually write just fg for the composition $f;g$.

Lemma 2 *Let S be any subsemiring of* FinAdd_L *closed under the *-operation. Then S is a *-continuous Kleene algebra iff for all $g, h \in S$, $g^*h = \bigvee_{n\geq 0} g^n h$.*

Proof Suppose that the above condition holds. We need to show that $f(\bigvee_{n\geq 0} g^n)h = \bigvee_{n\geq 0} fg^n h$ for all $f, g, h \in S$. But $f(\bigvee_{n\geq 0} g^n)h = f(\bigvee_{n\geq 0} g^n h)$ by assumption, and we conclude that $f(\bigvee_{n\geq 0} g^n h) = \bigvee_{n\geq 0} fg^n h$ since the supremum is pointwise. □

Compositions of $\top$-continuous functions in FinAdd_L are again $\top$-continuous, so that the collection of all $\top$-continuous functions in FinAdd_L is itself an idempotent semiring.

Definition 4 A function $f \in \mathrm{FinAdd}_L$ is said to be *locally *-closed* if for each $x \in L$, either $xf^* = \top$ or there exists $N \geq 0$ such that $xf^* = x \vee \cdots \vee xf^N$.

The functions id and $\bot$ are locally *-closed. As the next example demonstrates, compositions of locally *-closed (and $\top$-continuous) functions are not necessarily locally *-closed.

Example 1 Let L be the following complete lattice (the linear sum of three infinite chains):

$$\bot < x_0 < x_1 < \cdots < y_0 < y_1 < \cdots < z_0 < z_1 < \cdots < \top$$

Since L is a chain, a function $L \to L$ is finitely additive iff it is monotone and preserves $\bot$.

Let $f, g : L \to L$ be the following functions. First, $\bot f = \bot g = \bot$ and $\top f = \top g = \top$. Moreover, $x_i f = y_i$, $y_i f = z_i g = \top$ and $x_i g = \bot$, $y_i g = x_{i+1}$, and $z_i g = \top$ for all i. Then f, g are monotone, $uf^* =$

$u \vee uf \vee uf^2$ and $ug^* = u \vee ug$ for all $u \in L$. Also, f and g are $\top$-continuous, since if $\bigvee X = \top$ then either $\top \in X$ or $X \cap \{z_0, z_1, \dots\}$ is infinite, but then $\bigvee Xf = \bigvee Xg = \top$. However, fg is not locally *-closed, since $x_0(fg)^* = x_0 \vee x_0(fg) \vee x_0(fg)^2 \cdots = x_0 \vee x_1 \vee \cdots = y_0$. □

Lemma 3 *Let $f \in \mathrm{FinAdd}_L$ be locally *-closed. Then also f^* is locally *-closed. If f is additionally $\top$-continuous, then so is f^*.*

Proof We prove that $xf^{**} = x \vee xf^* = xf^*$ for all $x \in L$. Indeed, this is clear when $xf^* = \top$, since $f^* \le f^{**}$. Otherwise $xf^* = \bigvee_{k \le n} xf^k$ for some $n \ge 0$.

By finite additivity, it follows that $xf^*f^* = \bigvee_{k \le n} xf^k f^*$. But for each k, $xf^k f^* = xf^k \vee xf^{k+1} \vee \cdots \le xf^*$, thus $xf^* = xf^*f^*$ and $xf^* = xf^{**}$. It follows that f^* is locally *-closed.

Suppose now that f is additionally $\top$-continuous. We need to show that f^* is also $\top$-continuous. To this end, let $X \subseteq L$ with $\bigvee X = \top$. Since $x \le xf^*$ for all $x \in X$, it holds that $\bigvee Xf^* \ge \bigvee X = \top$. Thus $\bigvee Xf^* = \top$. □

Proposition 4 *Let S be any subsemiring of FinAdd_L closed under the *-operation. If each $f \in S$ is locally *-closed and $\top$-continuous, then S is a *-continuous Kleene algebra.*

Proof Suppose that $g, h \in S$. By Lemma 2, it suffices to show that $g^*h = \bigvee_{n \ge 0} g^n h$. Since this is clear when $h = \perp$, assume that $h \ne \perp$. As $g^n h \le g^* h$ for all $n \ge 0$, it holds that $\bigvee_{n \ge 0} g^n h \le g^* h$. To prove the opposite inequality, suppose that $x \in L$. If $xg^* = \top$, then $\bigvee_{n \ge 0} xg^n = \top$, so $\bigvee_{n \ge 0} xg^n h = \top$ by $\top$-continuity. Thus, $xg^*h = \top = \bigvee_{n \ge 0} xg^n h$.

Suppose that $xg^* \ne \top$. Then there is $m \ge 0$ with

$$xg^*h = (x \vee \cdots \vee xg^m)h = xh \vee \cdots \vee xg^m h \le \bigvee_{n \ge 0} xg^n h = x(\bigvee_{n \ge 0} g^n h)\,. \qquad \square$$

Now define a left action of FinAdd_L on $\mathrm{FinAdd}_{L,L'}$ by $fv = f;v$, for all $f \in \mathrm{FinAdd}_L$ and $v \in \mathrm{FinAdd}_{L,L'}$. It is a routine matter to check that $\mathrm{FinAdd}_{L,L'}$, equipped with the above action, the binary supremum operation $\vee$ and the constant $\perp$ is an (idempotent) left FinAdd_L-semimodule, that is, $(\mathrm{FinAdd}_L, \mathrm{FinAdd}_{L,L'})$ is a semiring-semimodule pair.

Lemma 5 *Let $S \subseteq \mathrm{FinAdd}_L$ be a *-continuous Kleene algebra and $V \subseteq \mathrm{FinAdd}_{L,L'}$ an S-semimodule. Then (S,V) is a generalized *-continuous Kleene algebra iff for all $f \in S$ and $v \in V$, $f^*v = \bigvee_{n \ge 0} f^n v$.*

Proof Similar to the proof of Lemma 2 □

Proposition 6 *Let $S \subseteq \mathrm{FinAdd}_L$ be a *-continuous Kleene algebra and $V \subseteq \mathrm{FinAdd}_{L,L'}$ an S-semimodule. If each $f \in S$ is locally *-closed and $\top$-continuous and each $v \in V$ is $\top$-continuous, then (S,V) is a generalized *-continuous Kleene algebra.*

Proof Similar to the proof of Proposition 4. □

5 *-continuous Kleene ω-Algebras of Functions

In this section, let L be an arbitrary complete lattice and $L' = \mathbf{2}$, the 2-element lattice $\{\perp, \top\}$. We define an infinite product $\mathrm{FinAdd}_L^\omega \to \mathrm{FinAdd}_{L,\mathbf{2}}$. Let $f_0, f_1, \dots \in \mathrm{FinAdd}_L$ be an infinite sequence and define $v = \prod_{n \ge 0} f_n : L \to \mathbf{2}$ by

$$xv = \begin{cases} \perp & \text{if there is } n \ge 0 \text{ such that } xf_0 \cdots f_n = \perp, \\ \top & \text{otherwise} \end{cases}$$

for all $x \in L$. We will write $\prod_{n\geq k} f_n$, for $k \geq 0$, as a shorthand for $\prod_{n\geq 0} f_{n+k}$.

It is easy to see that $\prod_{n\geq 0} f_n$ is finitely additive. Indeed, $\bot \prod_{n\geq 0} f_n = \bot$ clearly holds, and for all $x \leq y \in L$, $x \prod_{n\geq 0} f_n \leq y \prod_{n\geq 0} f_n$. Thus, to prove that $(x \vee y) \prod_{n\geq 0} f_n = x \prod_{n\geq 0} f_n \vee y \prod_{n\geq 0} f_n$ for all $x, y \in L$, it suffices to show that if $x \prod_{n\geq 0} f_n = y \prod_{n\geq 0} f_n = \bot$, then $(x \vee y) \prod_{n\geq 0} f_n = \bot$. But if $x \prod_{n\geq 0} f_n = y \prod_{n\geq 0} f_n = \bot$, then there exist $m, k \geq 0$ such that $x f_0 \cdots f_m = y f_0 \cdots f_k = \bot$. Let $n = \max\{m, k\}$. We have $(x \vee y) f_0 \cdots f_n = x f_0 \cdots f_n \vee y f_0 \cdots f_n = \bot$, and thus $(x \vee y) \prod_{n\geq 0} f_n = \bot$.

It is clear that this infinite product satisfies conditions (C1) and (C2) in the definition of *-continuous Kleene ω-algebra. Below we show that also (C3) and (C4) hold.

Lemma 7 *For all* $f_0, f_1, \ldots, g_0, g_1, \ldots \in \mathrm{FinAdd}_L$,

$$\prod_{n\geq 0} (f_n \vee g_n) = \bigvee_{h_n \in \{f_n, g_n\}} \prod_{n\geq 0} h_n \,.$$

Proof Since infinite product is monotone, the term on the right-hand side of the equation is less than or equal to the term on the left-hand side. To prove that equality holds, let $x \in L$ and suppose that $x \prod_{n\geq 0} (f_n \vee g_n) = \top$. It suffices to show that there is a choice of the functions $h_n \in \{f_n, g_n\}$ such that $x \prod_{n\geq 0} h_n = \top$.

Consider the infinite ordered binary tree where each node at level $n \geq 0$ is the source of an edge labeled f_n and an edge labeled g_n, ordered as indicated. We can assign to each node u the composition h_u of the functions that occur as the labels of the edges along the unique path from the root to that node.

Let us mark a node u if $x h_u \neq \bot$. As $x \prod_{n\geq 0} (f_n \vee g_n) = \top$, each level contains a marked node. Moreover, whenever a node is marked and has a predecessor, its predecessor is also marked. By König's lemma [20] there is an infinite path going through marked nodes. This infinite path gives rise to the sequence $h_0, h_1, \ldots$ with $x \prod_{n\geq 0} h_n = \top$. □

Lemma 8 *Let* $f \in \mathrm{FinAdd}_L$ *and* $v \in \mathrm{FinAdd}_{L,\mathbf{2}}$ *such that* f *is locally* **-closed and* v *is* $\top$*-continuous. If* $x f^* v = \top$*, then there exists* $k \geq 0$ *such that* $x f^k v = \top$.

Proof If $x f^* = \bigvee_{n=0}^{N} x f^n$ for some $N \geq 0$, then $x f^* v = \bigvee_{n=0}^{N} x f^n v = \top$ implies the claim of the lemma. If $x f^* = \top$, then $\top$-continuity of v implies that $\bigvee_{n\geq 0} x f^n v = \top$, which again implies the claim. □

Lemma 9 *Let* $f, g_0, g_1, \ldots \in \mathrm{FinAdd}_L$ *be locally* **-closed and* $\top$*-continuous such that for each* $m \geq 0$, $g_m \prod_{n\geq m+1} f^* g_n \in \mathrm{FinAdd}_{L,\mathbf{2}}$ *is* $\top$*-continuous. Then*

$$\prod_{n\geq 0} f^* g_n = \bigvee_{k_0, k_1, \ldots \geq 0} \prod_{n\geq 0} f^{k_n} g_n \,.$$

Proof As infinite product is monotone, the term on the right-hand side of the equation is less than or equal to the term on the left-hand side. To prove that equality holds, let $x \in L$ and suppose that $x \prod_{n\geq 0} f^* g_n = \top$. We want to show that there exist integers $k_0, k_1, \ldots \geq 0$ such that $x \prod_{n\geq 0} f^{k_n} g_n = \top$.

Let $x_0 = x$. By Lemma 8, $x \prod_{n\geq 0} f^* g_n = x_0 f^* g_0 \prod_{n\geq 1} f^* g_n = \top$ implies that there is $k_0 \geq 0$ for which $x_0 f^{k_0} g_0 \prod_{n\geq 1} f^* g_n = \top$. We finish the proof by induction. Assume we have $k_0, \ldots, k_m \geq 0$ such that $x f^{k_0} g_0 \cdots f^{k_m} g_m \prod_{n\geq m+1} f^* g_n = \top$ and let $x_{m+1} = x f^{k_0} g_0 \cdots f^{k_m} g_m$. Then $x_{m+1} f^* g_{m+1} \prod_{n\geq m+2} f^* g_n = \top$ implies, using Lemma 8, that there exists $k_{m+1} \geq 0$ for which $x_{m+1} f^{k_{m+1}} g_{m+1} \prod_{n\geq m+2} f^* g_n = \top$. □

Proposition 10 *Let* $S \subseteq \mathrm{FinAdd}_L$ *and* $V \subseteq \mathrm{FinAdd}_{L,\mathbf{2}}$ *such that* (S, V) *is a generalized* **-continuous Kleene algebra of locally* **-closed and* $\top$*-continuous functions* $L \to L$ *and* $\top$*-continuous functions* $L \to \mathbf{2}$*. If* $\prod_{n\geq 0} f_n \in V$ *for all sequences* $f_0, f_1, \ldots$ *of functions in* S*, then* (S, V) *is a* **-continuous Kleene* ω*-algebra.*

Proof This is clear from Lemmas 7 and 9. □

We finish the section by a lemma which exhibits a condition on the lattice L which ensures that infinite products of locally *-closed and $\top$-continuous functions are again $\top$-continuous.

Lemma 11 *Assume that L has the property that whenever $\bigvee X = \top$ for some $X \subseteq L$, then for all $x < \top$ in L there is $y \in X$ with $x \le y$. If $f_0, f_1, \ldots \in \mathrm{FinAdd}_L$ is a sequence of locally *-closed and $\top$-continuous functions, then $\prod_{n\ge 0} f_n \in \mathrm{FinAdd}_{L,\mathbf{2}}$ is $\top$-continuous.*

Proof Let $v = \prod_{n\ge 0} f_n$. We already know that v is finitely additive. We need to show that if $v \ne \bot$, then v is $\top$-continuous. But if $v \ne \bot$, then there is some $x < \top$ with $xv = \top$, *i.e.,* such that $xf_0 \cdots f_n > \bot$ for all n. By assumption, there is some $y \in X$ with $x \le y$. It follows that $yf_0 \cdots f_n \ge xf_0 \cdots f_n > \bot$ for all n and thus $\bigvee Xv = \top$. □

6 Energy Automata Revisited

We finish this paper by showing how the setting developed in the last sections can be applied to solve the energy problems of Section 2. Let $L = [0, \top]_\bot$ be the complete lattice of nonnegative real numbers together with $\top = \infty$ and an extra bottom element $\bot$, and extend the usual order and operations on real numbers to L by declaring that $\bot < x < \top$, $\bot - x = \bot$ and $\top + x = \top$ for all $x \in \mathbb{R}_{\ge 0}$. Note that L satisfies the precondition of Lemma 11.

We extend the definition of energy function:

Definition 5 An *extended energy function* is a mapping $f : L \to L$ for which $\bot f = \bot$, $\top f = \bot$ if $xf = \bot$ for all $x < \top$ and $\top f = \top$ otherwise, and $yf \ge xf + y - x$ whenever $\bot < x < y < \top$. The set of such functions is denoted $\mathscr{E}$.

Every energy function $f : \mathbb{R}_{\ge 0} \rightharpoonup \mathbb{R}_{\ge 0}$ as of Definition 1 gives rise to an extended energy function $\tilde f : L \to L$ given by $\bot \tilde f = \bot$, $x\tilde f = \bot$ if xf is undefined, $x\tilde f = xf$ otherwise for $x \in \mathbb{R}_{\ge 0}$, and $\top \tilde f = \top$. This defines an embedding $\mathscr{F} \hookrightarrow \mathscr{E}$.

The definition entails that for all $f \in \mathscr{E}$ and all $x < y \in L$, $xf = \top$ implies $yf = \top$ and $yf = \bot$ implies $xf = \bot$. Note that $\mathscr{E}$ is closed under (pointwise) binary supremum $\vee$ and composition and contains the functions $\bot$ and id.

Lemma 12 *Extended energy functions are finitely additive and $\top$-continuous, hence $\mathscr{E} \subseteq \mathrm{FinAdd}_L$ is a semiring.*

Proof Finite additivity follows from monotonicity. For $\top$-continuity, let $X \subseteq L$ such that $\bigvee X = \top$ and $f \in \mathscr{E}$, $f \ne \bot$. We have $X \ne \{\bot\}$, so let $x_0 \in X \setminus \{\bot\}$ and, for all $n \ge 0$, $x_n = x_0 + n$. Let $y_n = x_n f$. If $y_n = \bot$ for all $n \ge 0$, then also $nf = \bot$ for all $n \ge 0$ (as $x_n \ge n$), hence $f = \bot$. We must thus have an index N for which $y_N > \bot$. But then $y_{N+k} \ge y_N + k$ for all $k \ge 0$, hence $\bigvee Xf = \top$. □

Lemma 13 *For $f \in \mathscr{E}$, f^* is given by $xf^* = x$ if $xf \le x$ and $xf^* = \top$ if $xf > x$. Hence f is locally *-closed and $f^* \in \mathscr{E}$.*

Proof We have $\bot f^* = \bot$ and $\top f^* = \top$. Let $x \ne \bot, \top$. If $xf \le x$, then $xf^n \le x$ for all $n \ge 0$, so that $x \le \bigvee_{n\ge 0} xf^n \le x$, whence $xf^* = x$. If $xf > x$, then let $a = xf - x > 0$. We have $xf \ge x + a$, hence $xf^n \ge x + na$ for all $n \ge 0$, so that $xf^* = \bigvee_{n\ge 0} xf^n = \top$. □

Not all locally *-closed functions $f : L \to L$ are energy functions: the function f defined by $xf = 1$ for $x < 1$ and $xf = x$ for $x \ge 1$ is locally *-closed, but $f \notin \mathscr{E}$.

Corollary 14 *$\mathscr{E}$ is a *-continuous Kleene algebra.*

Proof This is clear by Proposition 4. □

Remark It is *not* true that $\mathscr{E}$ is a *continuous* Kleene algebra: Let $f_n, g \in \mathscr{E}$ be defined by $xf_n = x + 1 - \frac{1}{n+1}$ for $x \geq 0$, $n \geq 0$ and $xg = x$ for $x \geq 1$, $xg = \bot$ for $x < 1$. Then $0(\bigvee_{n \geq 0} f_n)g = (\bigvee_{n \geq 0} 0f_n)g = 1g = 1$, whereas $0\bigvee_{n \geq 0}(f_n g) = \bigvee_{n \geq 0}(0f_n g) = \bigvee_{n \geq 0}((1 - \frac{1}{n+1})g) = \bot$.

Let $\mathscr{V}$ denote the $\mathscr{E}$-semimodule of all $\top$-continuous functions $L \to \mathbf{2}$. For $f_0, f_1, \ldots \in \mathscr{E}$, define the infinite product $f = \prod_{n \geq 0} f_n : L \to \mathbf{2}$ by $xf = \bot$ if there is an index n for which $xf_0 \cdots f_n = \bot$ and $xf = \top$ otherwise, like in Section 5. By Lemma 11, $\prod_{n \geq 0} f_n$ is $\top$-continuous, *i.e.,* $\prod_{n \geq 0} f_n \in \mathscr{V}$.

By Proposition 6, $(\mathscr{E}, \mathscr{V})$ is a generalized *-continuous Kleene algebra.

Corollary 15 *$(\mathscr{E}, \mathscr{V})$ is a *-continuous Kleene ω-algebra.*

Proof This is clear by Proposition 10. □

Remark As $\mathscr{E}$ is not a continuous Kleene algebra, it also holds that $(\mathscr{E}, \mathscr{V})$ is not a continuous Kleene ω-algebra; in fact it is clear that there is no $\mathscr{E}$-semimodule $\mathscr{V}'$ for which $(\mathscr{E}, \mathscr{V})$ would be a continuous Kleene ω-algebra. The initial motivation for the work in [10] and the present paper was to generalize the theory of continuous Kleene ω-algebras so that it would be applicable to energy functions.

Noting that energy automata are weighted automata over $\mathscr{E}$ in the sense of Section 3.3, we can now solve the reachability and Büchi problem for energy automata:

Theorem 1 *Let $A = (\alpha, M, k)$ be an energy automaton and $x_0 \in \mathbb{R}_{\geq 0}$. There exists a finite run of A from an initial state to an accepting state with initial energy x_0 iff $x_0|A| > \bot$.*

Theorem 2 *Let $A = (\alpha, M, k)$ be an energy automaton and $x_0 \in \mathbb{R}_{\geq 0}$. There exists an infinite run of A from an initial state which infinitely often visits an accepting state iff $x_0\|A\| = \top$.*

Corollary 16 *Problems 1 and 2 are decidable.*

In [12], the complexity of the decision procedure has been established for important subclasses of energy functions.

7 Conclusion and Further Work

We have shown that energy functions form a *-continuous Kleene ω-algebra [10], hence that *-continuous Kleene ω-algebras provide a proper algebraic setting for energy problems. On our way, we have proven more general results about properties of finitely additive functions on complete lattices which should be of a more general interest.

There are interesting generalizations of our setting of energy automata which, we believe, can be attacked using techniques similar to ours. One such generalization are energy problems for *real time* or *hybrid* models, as for example treated in [3–5, 23]. Another generalization is to higher dimensions, like in [16, 19, 24] and other papers.

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