Finitely Supported Sets Containing Infinite Uniformly Supported Subsets

Andrei Alexandru
Romanian Academy, Institute of Computer Science, Iaşi, Romania
andrei.alexandru@iit.academiaromana-is.ro

Gabriel Ciobanu
Romanian Academy and A.I.Cuza University, Iaşi, Romania
gabriel@info.uaic.ro

The theory of finitely supported algebraic structures represents a reformulation of Zermelo-Fraenkel set theory in which every construction is finitely supported according to the action of a group of permutations of some basic elements named atoms. In this paper we study the properties of finitely supported sets that contain infinite uniformly supported subsets, as well as the properties of finitely supported sets that do not contain infinite uniformly supported subsets. For classical atomic sets, we study whether they contain or not infinite uniformly supported subsets.

1 Finitely Supported Sets

Finitely supported mathematics [1] is dealing with the set theory foundations for the finitely supported structures. Finitely supported structures are related to the recent development of the Fraenkel-Mostowski axiomatic set theory working with 'nominal sets' and dealing with binding and fresh names in computer science [9], but also to the theory of admissible sets of Barwise [4], in particular to the theory of hereditary finite sets. Fraenkel-Mostowski set theory (FM) represents an axiomatization of the Fraenkel Basic Model for the Zermelo-Fraenkel set theory with atoms (ZFA), a model used originally to prove the independence of the axiom of choice and other axioms of set theory with atoms. Nominal sets are actually a Zermelo-Fraenkel set theory (ZF) alternative to the non-standard Fraenkel-Mostowski set theory whose axioms are the ZFA axioms together with a new axiom of finite support claiming that any set-theoretical construction has to be finitely supported modulo a canonical hierarchically defined permutation action), since nominal sets are defined by involving group actions over standard ZF sets, without being necessary to modify the ZF axioms of extensionality or foundation. A nominal set is defined as a usual Zermelo-Fraenkel set endowed with a group action of the group of (finitary) permutations over a certain fixed countable ZF set A of basic elements whose internal structure is ignored (called *atoms*), satisfying also a finite support requirement. This finite support requirement states that for any element in a nominal set there should exist a finite set of atoms such that any permutation fixing pointwise this set of atoms also leaves the element invariant under the related group action. By now, nominal sets were used to study the binding, scope, freshness and renaming in programming languages and related formal systems. The inductively defined finitely supported sets (that are finitely supported elements in the powerset of a nominal set) involving the name-abstraction together with Cartesian product and disjoint union can encode formal syntax modulo renaming of bound variables. In this way, the standard theory of algebraic data types can be extended to include signatures involving binding operators. In particular, there exists an associated notion of structural recursion for defining syntax-manipulating functions and a notion of

M. Marin, A. Crăciun (Eds.):
Working Formal Methods Symposium 2019 (FROM 2019)
EPTCS 303, 2019, pp. 120–134, doi:10.4204/EPTCS.303.9

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proof by structural induction. Certain generalizations of nominal sets are involved in the study of automata, programming languages or Turing machines over infinite alphabets; for this, a relaxed notion of finiteness called 'orbit finiteness' was defined; it means 'having a finite number of orbits (equivalence classes) under a certain group action' [6]. Fraenkel-Mostowski generalized set theory (FMG) was introduced in [7] and generalizes both the size of atoms and the size of support from the FM set theory. More exactly, it is presented a generalization of the FM sets by replacing 'finite support' with 'well-orderable (at least countable) support' and by considering an uncountable set of atoms. Notions such as abstraction and freshness quantifier И in the FM set theory have also been extended into the new framework. In this sense, in FMG $Иa.p(a)$ for a predicate p means that p holds for all atoms except a well-orderable subset of atoms, while in FM $Иa.p(a)$ means that p holds for all atoms except a finite subset of atoms. This approach allows binding of infinitely many names in syntax instead of only finitely many names. A very recent work describing a general framework for reasoning about syntax with bindings is [5]; it overlaps the nominal sets framework, but also provides significant distinctions. In this paper, the authors employed functors for modelling the presence of variables instead of sets with atoms. Furthermore, the authors are able to remove the finite support restriction and to accept terms that are infinitely branching, terms having infinite depth, or both. Unlike nominal sets theory where atoms can only be manipulated via bijections, the functors described in [5] distinguish between binding variables (managed via bijections) and free variables (managed via possibly non-bijective functions); these functors allow the authors to apply not only swappings or permutations, but also arbitrary substitutions.

Finitely supported mathematics (shortly, FSM) is focused on the foundations of set theory (rather than on applications in computer science). In order to describe FSM as a theory of finitely supported algebraic structures, we refer to the theory of nominal sets (with the mention that the requirement regarding the countability of A is irrelevant). We call these sets *invariant sets*, using the motivation of Tarski regarding logicality (more precisely, a logical notion is defined by Tarski as one that is invariant under the permutations of the universe of discourse). FSM is actually represented by finitely supported subsets of invariant sets together with finitely supported internal algebraic operations or with finitely supported relations (that should be finitely supported as subsets in the Cartesian product of two invariant sets). There is no major technical difference between 'FSM' and 'nominal' (related to basic definitions), but conceptually the nominal approach is related to computer science, while FSM deals with the foundations of mathematics (and experimental sciences) by studying the consistency and inconsistency of various results within the framework of the atomic sets. Our goal is not to re-brand the nominal framework (whose value we certainly recognize), but to provide a collection of set theoretical results regarding foundations of finitely supported structures.

FSM contains both the family of 'non-atomic' (i.e., ordinary) ZF sets which are proved to be trivial FSM sets (i.e., their elements are left unchanged under the effect of the canonical permutation action) and the family of 'atomic' sets (i.e., sets that contain at least an atom somewhere in their structure) with finite supports (hierarchically constructed from the empty set and the fixed ZF set A of atoms). One task is to analyze whether a classical ZF result (obtained in the framework of non-atomic sets) can be adequately reformulated by replacing 'non-atomic ZF element/set/structure' with 'atomic and finitely supported element/set/structure' in order to be valid also for atomic sets with finite supports.

Note that the FSM sets is not closed under ZF subsets constructions, meaning that there exist subsets of FSM sets that fail to be finitely supported (for example the simultaneously ZF infinite and coinfinite subsets of the set A). Thus, for proving results in FSM we cannot use related results from the ZF framework without reformulating them with respect to the finite support requirement. Furthermore, not even the translation of the results from a non-atomic framework into an atomic framework (such as Zermelo Fraenkel set theory with atoms obtained by weakening ZF axiom of extensionality) is an easy task. Results from ZF may lose their validity when reformulating them in Zermelo Fraenkel set theory with atoms. For example, it is known that multiple choice principle and Kurepa's maximal antichain principle are both equivalent to the axiom of choice in ZF. However, Jech proved in [8] that multiple choice principle is valid in the Fraenkel Second Model, while the axiom of choice fails in this model. Furthermore, Kurepa's maximal antichain principle is valid in the Fraenkel Basic Model, while the axiom of choice fails in this model. This means that the following two statements that are valid in ZF, namely *'Kurepa's principle implies axiom of choice'* and *'Multiple choice principle implies axiom of choice'* fail in Zermelo Fraenkel set theory with atoms.

A proof of an FSM result should be internally consistent in FSM and not retrieved from ZF, that is it should involve only finitely supported constructions (even in the intermediate steps). The meta-theoretical techniques for the translation of a result from non-atomic structures to atomic structures are based on a refinement of the finite support principle from [9], a refinement called 'S-finite supports principle' claiming that *for any finite set S of atoms, anything that is definable in higher-order logic from S-supported structures by using S-supported constructions is also S-supported*. The formal involvement of the S-finite support principles actually implies a hierarchical constructive method for defining the support of a structure by employing, step-by-step, the supports of the substructures of a related structure.

2 Preliminary Results

A finite set is a set of the form $\{x_1,\dots,x_n\}$. Consider a fixed ZF infinite set A of elements that can be checked only for equality. The elements of A are called 'atoms' by analogy with the models of the classic ZFA set theory given by Fraenkel and Mostowski. A *transposition* is a function $(a\,b): A \to A$ that interchanges only a and b. A *(finitary) permutation* of A in FSM is a bijection of A generated by composing finitely many transpositions. We denote by S_A the group of all (finitary) permutations of A. According to Proposition 2.6 in [1], a bijection on A is finitely supported if and only if it is a (finitary) permutation of A. Thus, (finitary) permutations are simply called permutations.

Definition 2.1

1. *Let X be a ZF set. An S_A-*action *on X is a group action $\cdot$ of S_A on X. An S_A-*set *is a pair $(X,\cdot)$, where X is a ZF set, and $\cdot$ is an S_A-action on X.*
2. *Let $(X,\cdot)$ be an S_A-set. We say that $S \subset A$* supports *x whenever for each $\pi \in Fix(S)$ we have $\pi \cdot x = x$, where $Fix(S) = \{\pi \,|\, \pi(a) = a, \forall a \in S\}$. The least finite set (w.r.t. the inclusion relation) supporting x (which exists according to [1]) is called* the support of *x and is denoted by $supp(x)$. An empty supported element is called* equivariant.
3. *Let $(X,\cdot)$ be an S_A-set. We say that X is an* invariant set *if for each $x \in X$ there exists a finite set $S_x \subset A$ which supports x.*

Proposition 2.2 *[1, 9] Let $(X,\cdot)$ and $(Y,\diamond)$ be S_A-sets.*

1. *The set A of atoms is an invariant set with the S_A-action $\cdot : S_A \times A \to A$ defined by $\pi \cdot a := \pi(a)$ for all $\pi \in S_A$ and $a \in A$. Furthermore, $supp(a) = \{a\}$ for each $a \in A$.*
2. *Let $\pi \in S_A$. If $x \in X$ is finitely supported, then $\pi \cdot x$ is finitely supported and $supp(\pi \cdot x) = \{\pi(u) \,|\, u \in supp(x)\} := \pi(supp(x))$.*
3. *The Cartesian product $X \times Y$ is also an S_A-set with the S_A-action $\otimes : S_A \times (X \times Y) \to (X \times Y)$ defined by $\pi \otimes (x,y) = (\pi \cdot x, \pi \diamond y)$ for all $\pi \in S_A$ and all $x \in X$, $y \in Y$. If $(X,\cdot)$ and $(Y,\diamond)$ are invariant sets, then $(X \times Y, \otimes)$ is also an invariant set.*

4. *The powerset $\wp(X) = \{Z \,|\, Z \subseteq X\}$ is also an S_A-set with the S_A-action $\star : S_A \times \wp(X) \to \wp(X)$ defined by $\pi \star Z := \{\pi \cdot z \,|\, z \in Z\}$ for all $\pi \in S_A$, and all $Z \subseteq X$. For each invariant set $(X, \cdot)$, we denote by $\wp_{fs}(X)$ the set of elements in $\wp(X)$ which are finitely supported according to the action $\star$. $(\wp_{fs}(X), \star|_{\wp_{fs}(X)})$ is an invariant set.*
5. *The finite powerset of X denoted by $\wp_{fin}(X) = \{Y \subseteq X \,|\, Y \text{finite}\}$ and the cofinite powerset of X denoted by $\wp_{cofin}(X) = \{Y \subseteq X \,|\, X \setminus Y \text{finite}\}$ are both S_A-sets with the S_A-action $\star$ defined as in the previous item (2). If X is an invariant set, then both $\wp_{fin}(X)$ and $\wp_{cofin}(X)$ are invariant sets.*
6. *We have $\wp_{fs}(A) = \wp_{fin}(A) \cup \wp_{cofin}(A)$. If $X \in \wp_{fin}(A)$, then $supp(X) = X$.
If $X \in \wp_{cofin}(A)$, then $supp(X) = A \setminus X$.*
7. *The disjoint union of X and Y defined by $X+Y = \{(0,x) \,|\, x \in X\} \cup \{(1,y) \,|\, y \in Y\}$ is an S_A-set with the S_A-action $\star : S_A \times (X+Y) \to (X+Y)$ defined by $\pi \star z = (0, \pi \cdot x)$ if $z = (0,x)$ and $\pi \star z = (1, \pi \diamond y)$ if $z = (1,y)$. If $(X, \cdot)$ and $(Y, \diamond)$ are invariant sets, then $(X+Y, \star)$ is also an invariant set.*
8. *Any ordinary (non-atomic) ZF-set X (such as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$ for example) is an invariant set with the single possible S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in X$.*

Definition 2.3

1. *Let $(X, \cdot)$ be an S_A-set. A subset Z of X is called* finitely supported *if and only if $Z \in \wp_{fs}(X)$. A subset Z of X is* uniformly supported *if all the elements of Z are supported by the same set S (and so Z is itself supported by S).*
2. *Let $(X, \cdot)$ be a finitely supported subset of an S_A- set $(Y, \cdot)$. A subset Z of Y is called* finitely supported subset of X *(and we denote this by $Z \in \wp_{fs}(X)$) if and only if $Z \in \wp_{fs}(Y)$ and $Z \subseteq X$. Similarly, we say that a uniformly supported subset of Y contained in X is a* uniformly supported subset of X.

From Definition 2.1, a subset Z of an invariant set $(X, \cdot)$ is finitely supported by a set $S \subseteq A$ if and only if $\pi \star Z \subseteq Z$ for all $\pi \in Fix(S)$, i.e. if and only if $\pi \cdot z \in Z$ for all $\pi \in S_A$ and all $z \in Z$. This is because any permutation of atoms should have finite order, and so the relation $\pi \star Z \subseteq Z$ is equivalent to $\pi \star Z = Z$.

Due to Proposition 2.2(2), whenever X is a finitely supported subset of an invariant set Y, the uniform powerset of X denoted by $\wp_{us}(X) = \{Z \subseteq X \,|\, Z \text{uniformly supported}\}$ is a subset of $\wp_{fs}(Y)$ supported by $supp(X)$. This is because, whenever $Z \subseteq X$ is uniformly supported by S and $\pi \in Fix(supp(X))$, we have $\pi \star Z \subseteq \pi \star X = X$ and $\pi \star Z$ is uniformly supported by $\pi(S)$. Similarly, $\wp_{fin}(X)$ and $\wp_{cofin}(X)$ are subsets of $\wp_{fs}(Y)$ supported by $supp(X)$. We consider that $\emptyset$, being a finite subset of X, belongs to $\wp_{us}(X)$.

Definition 2.4 *Let X and Y be invariant sets.*

1. *A function $f : X \to Y$ is* finitely supported *if $f \in \wp_{fs}(X \times Y)$.*
2. *Let Z be a finitely supported subset of X and T a finitely supported subset of Y. A function $f : Z \to T$ is* finitely supported *if $f \in \wp_{fs}(X \times Y)$. The set of all finitely supported functions from Z to T is denoted by T^Z_{fs}.*

Proposition 2.5 *[1, 9] Let $(X, \cdot)$ and $(Y, \diamond)$ be two invariant sets.*

1. *Y^X (i.e. the set of all functions from X to Y) is an S_A-set with the S_A-action $\widetilde{\star} : S_A \times Y^X \to Y^X$ defined by $(\pi \widetilde{\star} f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. A function $f : X \to Y$ is finitely supported (in the sense of Definition 2.4) if and only if it is finitely supported with respect the permutation action $\widetilde{\star}$.*

2. *Let Z be a finitely supported subset of X and T a finitely supported subset of Y. A function $f : Z \to T$ is supported by a finite set $S \subseteq A$ if and only if for all $x \in Z$ and all $\pi \in Fix(S)$ we have $\pi \cdot x \in Z$, $\pi \diamond f(x) \in T$ and $f(\pi \cdot x) = \pi \diamond f(x)$.*

3 FSM Uniformly Infinite Sets

Definition 3.1 *Let X be a finitely supported subset of an invariant set Y. X is called* FSM uniformly infinite *if there exists an infinite, uniformly supported subset of X. Otherwise, we call X* FSM non-uniformly infinite.

Theorem 3.2 *Let X be a finitely supported subset of an invariant set $(Y,\cdot)$ such that X is not FSM uniformly infinite. Then the set $\wp_{us}(X)$ is not FSM uniformly infinite.*

Proof. Suppose, by contradiction, that the set $\wp_{us}(X)$ contains an infinite subset $\mathscr{F}$ such that all the elements of $\mathscr{F}$ are different and supported by the same finite set S. By convention, without assuming that $i \mapsto X_i$ is finitely supported, we understand $\mathscr{F}$ as $\mathscr{F} = (X_i)_{i\in I}$ with the properties that $X_i \neq X_j$ whenever $i \neq j$ and $supp(X_i) \subseteq S$ for all $i \in I$. Let us fix an arbitrary $j \in I$. We prove that $supp(X_j) = \underset{x\in X_j}{\cup} supp(x)$. Indeed, since X_j is uniformly supported, there exists a finite subset of atoms T such that T supports every $x \in X_j$, i.e. $supp(x) \subseteq T$ for all $x \in X_j$. Thus, $\cup\{supp(x)\,|\,x \in X_j\} \subseteq T$. Clearly, $supp(X_j) \subseteq \cup\{supp(x)\,|\,x \in X_j\}$. Conversely, let $a \in \cup\{supp(x)\,|\,x \in X_j\}$. Thus, there exists $x_0 \in X_j$ such that $a \in supp(x_0)$. Let b be an atom such that $b \notin supp(X_j)$ and $b \notin T$. Such an atom exists because A is infinite, while $supp(X_j)$ and T are both finite. We prove by contradiction that $(b\ a) \cdot x_0 \notin X_j$. Indeed, suppose that $(b\ a) \cdot x_0 = y \in X_j$. Since $a \in supp(x_0)$, by Proposition 2.2(2), we have $b = (b\ a)(a) \in (b\ a)(supp(x_0)) = supp((b\ a) \cdot x_0) = supp(y)$. Since $supp(y) \subseteq T$, we get $b \in T$: a contradiction! Therefore, $(b\,a) \star X_j \neq X_j$, where $\star$ is the canonical S_A-action on $\wp(Y)$. Since $b \notin supp(X_j)$, we prove by contradiction that $a \in supp(X_j)$. Indeed, suppose that $a \notin supp(X_j)$. We have that $(b\,a) \in Fix(supp(X_j))$. Since $supp(X_j)$ supports X_j, it follows that $(b\,a) \star X_j = X_j$ which is a contradiction. Thus, $a \in supp(X_j)$ and so $supp(X_j) = \underset{x\in X_j}{\cup} supp(x)$.

Therefore, because $supp(X_j) \subseteq S$, X_j has the property that $supp(x) \subseteq S$ for all $x \in X_j$. Since j has been arbitrarily chosen from I, it follows that $\underset{i\in I}{\cup} X_i$ is an uniformly supported subset of X (all its elements being supported by S). Furthermore, $\underset{i\in I}{\cup} X_i$ is infinite since the family $(X_i)_{i\in I}$ is infinite and $X_i \neq X_j$ whenever $i \neq j$. This contradicts the hypothesis. ■

Theorem 3.3 *Let X be a finitely supported subset of an invariant set $(Y,\cdot)$ such that X is not FSM uniformly infinite. Then the set $\wp_{fin}(X)$ is not FSM uniformly infinite.*

Proof. We always have that $\wp_{fin}(X) \subseteq \wp_{us}(X)$ because any finite subset of X of form $\{x_1,\dots,x_n\}$ is uniformly supported by $supp(x_1) \cup \ldots \cup supp(x_n)$. Since $\wp_{us}(X)$ does not contain an infinite uniformly supported subset, it follows that neither $\wp_{fin}(X)$ contains an infinite uniformly supported subset. ■

Theorem 3.4 *Let X be a finitely supported subset of an invariant set $(Y,\cdot)$.*

1. *If X is not FSM uniformly infinite, then any finitely supported order-preserving (with respect to the inclusion relation) function $f : \wp_{us}(X) \to \wp_{us}(X)$ has a least fixed point supported by $supp(f) \cup supp(X)$.*
2. *If X is not FSM uniformly infinite, then any finitely supported order-preserving (with respect to the inclusion relation) function $f : \wp_{fin}(X) \to \wp_{fin}(X)$ has a least fixed point supported by $supp(f) \cup supp(X)$.*

Proof. Let $f : \wp_{us}(X) \to \wp_{us}(X)$ be a finitely supported order-preserving function. Firstly, since $\wp_{us}(X)$ is a subset of $\wp_{fs}(Y)$ supported by $supp(X)$, we have $\pi \star \emptyset, \pi^{-1} \star \emptyset \in \wp_{us}(X)$ for any permutation $\pi \in Fix(supp(X))$. Thus, $\emptyset \subseteq \pi \star \emptyset$ and $\emptyset \subseteq \pi^{-1} \star \emptyset$. Since the relation $\subseteq$ on $\wp_{us}(X)$ is supported by $supp(X)$, we get $\pi \star \emptyset \subseteq \pi \star (\pi^{-1} \star \emptyset) = (\pi \circ \pi^{-1}) \star \emptyset = \emptyset$, and so $\emptyset = \pi \cdot \emptyset$ which means that $\emptyset$ is an element in $\wp_{us}(X)$ supported by $supp(X)$. Actually, $\emptyset$ belongs to $\wp_{fin}(X)$ that is a subset of $\wp_{us}(X)$.

Since $\emptyset \subseteq f(\emptyset)$ and f is order-preserving, we can define the ascending sequence $\emptyset \subseteq f(\emptyset) \subseteq f^2(\emptyset) \subseteq \ldots \subseteq f^n(\emptyset) \subseteq \ldots$, where $f^n(\emptyset) = f(f^{n-1}(\emptyset))$ and $f^0(\emptyset) = \emptyset$. We prove by induction that $(f^n(\emptyset))_{n \in \mathbb{N}}$ is uniformly supported by $supp(f) \cup supp(X)$, namely $supp(f^n(\emptyset)) \subseteq supp(f) \cup supp(X)$ for each $n \in \mathbb{N}$. We have $supp(f^0(\emptyset)) = supp(\emptyset) \subseteq supp(X) \subseteq supp(f) \cup supp(X)$. Let us assume that $supp(f^n(\emptyset)) \subseteq supp(f) \cup supp(X)$ for some $n \in \mathbb{N}$. We have to prove that $supp(f^{n+1}(\emptyset)) \subseteq supp(f) \cup supp(X)$. Let $\pi \in Fix(supp(f) \cup supp(X))$. From the inductive hypothesis, we have $\pi \in Fix(supp(f^n(\emptyset)))$ and so $\pi \star f^n(\emptyset) = f^n(\emptyset)$. Since π fixes $supp(f)$ pointwise, according to Proposition 2.5, we have $\pi \star f^{n+1}(\emptyset) = \pi \star f(f^n(\emptyset)) = f(\pi \star f^n(\emptyset)) = f(f^n(\emptyset)) = f^{n+1}(\emptyset)$. Therefore, $(f^n(\emptyset))_{n \in \mathbb{N}} \subseteq \wp_{us}(X)$ is uniformly supported by $supp(f) \cup supp(X)$. Thus, according to Theorem 3.2, $(f^n(\emptyset))_{n \in \mathbb{N}}$ should be finite, and so there exists $n_0 \in \mathbb{N}$ such that $f^n(\emptyset) = f^{n_0}(\emptyset)$ for all $n \geq n_0$. Thus, $f(f^{n_0}(\emptyset)) = f^{n_0+1}(\emptyset) = f^{n_0}(\emptyset)$, and so $f^{n_0}(\emptyset)$ is a fixed point of f. It is supported by $supp(f) \cup supp(X)$, and obviously it is the least one.

2. A similar argument allows us to prove the second item of the proposition. This time Theorem 3.3 is used to prove that the uniformly supported ascending family $(f^n(\emptyset))_{n \in \mathbb{N}} \subseteq \wp_{fin}(X)$ is finite, and so it is stationary. ■

Theorem 3.5 *Let X be a finitely supported subset of an invariant set $(Y, \cdot)$.*

1. *If X is not FSM uniformly infinite and $f : \wp_{us}(X) \to \wp_{us}(X)$ is finitely supported with the property that $Z \subseteq f(Z)$ for all $Z \in \wp_{us}(X)$, then for each $Z \in \wp_{us}(X)$ there exists some $m \in \mathbb{N}$ such that $f^m(Z)$ is a fixed point of f.*
2. *If X is not FSM uniformly infinite and $f : \wp_{fin}(X) \to \wp_{fin}(X)$ is finitely supported with the property that $Z \subseteq f(Z)$ for all $Z \in \wp_{fin}(X)$, then for each $Z \in \wp_{fin}(X)$ there exists some $m \in \mathbb{N}$ such that $f^m(Z)$ is a fixed point of f.*

Proof. 1. Let us fix an arbitrary element $Z \in \wp_{us}(X)$. We consider the ascending (via sets inclusion) sequence $(Z_n)_{n \in \mathbb{N}}$ which has the first term $Z_0 = Z$ and the general term $Z_{n+1} = f(Z_n)$ for all $n \in \mathbb{N}$. We prove by induction that $supp(Z_n) \subseteq supp(f) \cup supp(Z) \cup supp(X)$ for all $n \in \mathbb{N}$. Clearly, $supp(Z_0) = supp(Z) \subseteq supp(f) \cup supp(Z) \cup supp(X)$. Assume that $supp(Z_k) \subseteq supp(f) \cup supp(Z) \cup supp(X)$. Let $\pi \in Fix(supp(f) \cup supp(Z) \cup supp(X))$. Thus, $\pi \cdot Z_k = Z_k$ according to the inductive hypothesis. According to Proposition 2.5, because π fixes $supp(f)$ pointwise, $supp(f)$ supports f and $\wp_{us}(X)$ is supported by $supp(X)$, we get $\pi \star Z_{k+1} = \pi \star f(Z_k) = f(\pi \star Z_k) = f(Z_k) = Z_{k+1}$. Since $supp(Z_{k+1})$ is the least set supporting Z_{k+1}, we obtain $supp(Z_{k+1}) \subseteq supp(f) \cup supp(Z) \cup supp(X)$. Thus, $(Z_n)_{n \in \mathbb{N}} \subseteq \wp_{us}(X)$ is uniformly supported by $supp(f) \cup supp(Z) \cup supp(X)$, and so $(Z_n)_{n \in \mathbb{N}}$ must be finite according to Theorem 3.2. Since by hypothesis we have $Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_n \subseteq \ldots$, there should exist $m \in \mathbb{N}$ such that $Z_m = Z_{m+1}$, i.e. $f^m(Z) = f^{m+1}(Z) = f(f^m(Z))$, and so the result follows.

2. A similar argument allows us to prove the second item of this theorem. Theorem 3.3 is used to prove that the uniformly supported ascending family $(f^n(Z))_{n \in \mathbb{N}} \subseteq \wp_{fin}(X)$ is finite, and so it is stationary for every $Z \in \wp_{fin}(X)$. ■

For self-mappings on $\wp_{fin}(A)$ we have the following stronger property.

Proposition 3.6 *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be a finitely supported function with the property that $Z \subseteq f(Z)$ for all $Z \in \wp_{fin}(A)$. There are infinitely many fixed points of f, namely the finite subsets of A containing all the elements of $supp(f)$.*

Proof. Let $Z \in \wp_{fin}(A)$. Since the support of a finite subset of atoms coincides with the related subset, we have $supp(Z) = Z$ and $supp(f(Z)) = f(Z)$. According to Proposition 2.5, for any permutation $\pi \in Fix(supp(f) \cup supp(Z)) = Fix(supp(f) \cup Z)$, we have $\pi \star f(Z) = f(\pi \star Z) = f(Z)$ which means $supp(f) \cup Z$ supports $f(Z)$, that is, $f(Z) = supp(f(Z)) \subseteq supp(f) \cup Z$ (claim 1). Since we also have $Z \subseteq f(Z)$, we get $Z \setminus supp(f) \subseteq f(Z) \setminus supp(f) \subseteq Z \setminus supp(f)$, that is, $Z \setminus supp(f) = f(Z) \setminus supp(f)$ (claim 2). If $supp(f) = \emptyset$, the result follows obviously. Let $supp(f) = \{a_1, \dots, a_n\}$. According to (claim 1), we have $supp(f) \subseteq f(supp(f)) \subseteq supp(f)$, and so $f(supp(f)) = supp(f)$. If Z has the form $Z = \{a_1, \dots, a_n, b_1, \dots, b_m\}$ with $b_1, \dots, b_m \in A \setminus supp(f)$, $m \geq 1$, we should have by hypothesis that $a_1, \dots, a_n \in f(Z)$, and by (claim 2) $f(Z) \setminus supp(f) = \{b_1, \dots, b_m\}$. Since no other elements different from $a_1, \dots, a_n$ are in $supp(f)$, from (claim 1) we get $f(Z) = \{a_1, \dots, a_n, b_1, \dots, b_m\}$. ■

Theorem 3.7 *The following properties of FSM uniformly infinite sets hold.*

1. *Let X be an infinite, finitely supported subset of an invariant set Y. Then the sets $\wp_{fs}(\wp_{fin}(X))$ and $\wp_{fs}(T_{fin}(X))$ are FSM uniformly infinite.*
2. *Let X be an infinite, finitely supported subset of an invariant set Y. Then the set $\wp_{fs}(\wp_{fs}(X))$ is FSM uniformly infinite.*
3. *Let X and Y be two finitely supported subsets of an invariant set Z. If neither X nor Y is FSM uniformly infinite, then $X \times Y$ is not FSM uniformly infinite.*
4. *Let X and Y be two finitely supported subsets of an invariant set Z. If neither X nor Y is FSM uniformly infinite, then $X + Y$ is not FSM uniformly infinite.*

Proof. 1. Obviously, $\wp_{fin}(X)$ is a finitely supported subset of the invariant set $\wp_{fs}(Y)$, supported by $supp(X)$. This is because whenever Z is an element of $\wp_{fin}(X)$ (i.e. whenever Z is a finite subset of X) and π fixes $supp(X)$ pointwise, we have that $\pi \star Z$ is also a finite subset of X. The family $\wp_{fs}(\wp_{fin}(X))$ represents the family of those subsets of $\wp_{fin}(X)$ which are finitely supported as subsets of the invariant set $\wp_{fs}(Y)$ in the sense of Definition 2.3. As above, according to Proposition 2.2, we have that $\wp_{fs}(\wp_{fin}(X))$ is a finitely supported subset of the invariant set $\wp_{fs}(\wp_{fs}(Y))$, supported by $supp(\wp_{fin}(X)) \subseteq supp(X)$.

Let X_i be the set of all i-sized subsets from X, i.e. $X_i = \{Z \subseteq X \,|\, |Z| = i\}$. Since X is infinite, it follows that each $X_i, i \geq 1$ is non-empty. Obviously, we have that any i-sized subset $\{x_1, \dots, x_i\}$ of X is finitely supported (as a subset of Y) by $supp(x_1) \cup \ldots \cup supp(x_i)$. Therefore, $X_i \subseteq \wp_{fin}(X)$ and $X_i \subseteq \wp_{fs}(Y)$ for all $i \in \mathbb{N}$. Since $\cdot$ is a group action, the image of an i-sized subset of X under an arbitrary permutation is an i-sized subset of Y. However, any permutation of atoms that fixes $supp(X)$ pointwise also leaves X invariant, and so for any permutation $\pi \in Fix(supp(X))$ we have that $\pi \star Z$ is an i-sized subset of X whenever Z is an i-sized subset of X. Thus, each X_i is a subset of $\wp_{fin}(X)$ finitely supported by $supp(X)$, and so $X_i \in \wp_{fs}(\wp_{fin}(X))$. The family $(X_i)_{i \in \mathbb{N}}$ is infinite and uniformly supported.

If we consider Y_i the set of all i-sized injective tuples formed by elements of X, we have that each Y_i is a subset of $T_{fin}(X)$ supported by $supp(X)$, and the family $(Y_i)_{i \in \mathbb{N}}$ is an infinite, uniformly supported, subset of $\wp_{fs}(T_{fin}(X))$.

2. The proof is actually the same as in the above item since every $X_i \in \wp_{fs}(\wp_{fs}(X))$.

3. Suppose, by contradiction, that $X \times Y$ is FSM uniformly infinite. Thus, there exists an infinite injective family $((x_i, y_i))_{i \in I} \subseteq X \times Y$ and a finite $S \subseteq A$ with the property that $supp((x_i, y_i)) \subseteq S$ for all $i \in I$ (1). Fix some $j \in I$. We claim that $supp((x_j, y_j)) = supp(x_j) \cup supp(y_j)$. Let $U = (x_j, y_j)$, and $S = supp(x_j) \cup supp(y_j)$. Obviously, S supports U. Indeed, let us consider $\pi \in Fix(S)$. We have that $\pi \in Fix(supp(x_j))$ and also $\pi \in Fix(supp(y_j))$ Therefore, $\pi \cdot x_j = x_j$ and $\pi \cdot y_j = y_j$, and so $\pi \otimes (x_j, y_j) =$

$(\pi \cdot x_j, \pi \cdot y_j) = (x_j, y_j)$, where $\otimes$ represent the S_A action on $X \times Y$ described in Proposition 2.2. Thus, $supp(U) \subseteq S$. It remains to prove that $S \subseteq supp(U)$. Fix $\pi \in Fix(supp(U))$. Since $supp(U)$ supports U, we have $\pi \otimes (x_j, y_j) = (x_j, y_j)$, and so $(\pi \cdot x_j, \pi \cdot y_j) = (x_j, y_j)$, from which we get $\pi \cdot x_j = x_j$ and $\pi \cdot y_j = y_j$. Thus, $supp(x_j) \subseteq supp(U)$ and $supp(y_j) \subseteq supp(U)$. Hence $S = supp(x_j) \cup supp(y_j) \subseteq supp(U)$.

According to relation (1) we obtain, $supp(x_i) \cup supp(y_i) \subseteq S$ for all $i \in I$. Thus, $supp(x_i) \subseteq S$ for all $i \in I$ and $supp(y_i) \subseteq S$ for all $i \in I$ (2). Since the family $((x_i, y_i))_{i\in I}$ is infinite and injective, then at least one of the uniformly supported families $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ is infinite, a contradiction.

4. Suppose, by contradiction, that $X+Y$ is FSM uniformly infinite. Thus, there exists an infinite injective family $(z_i)_{i\in I} \subseteq X \times Y$ and a finite $S \subseteq A$ such that $supp(z_i) \subseteq S$ for all $i \in I$. According to the construction of the disjoint union of two S_A-sets (see Proposition 2.2), there should exist an infinite family of $(z_i)_i$ of form $((0, x_j))_{x_j \in X}$ which is uniformly supported by S, or an infinite family of form $((1, y_k))_{y_k \in Y}$ which is uniformly supported by S. Since 0 and 1 are constants, this means there should exist at least an infinite uniformly supported family of elements from X, or an infinite uniformly supported family of elements from Y, a contradiction. ■

The following result represents a significant extension of Theorem 2 in [3] since we are able to prove that $\wp_{fs}(A)^A_{fs}$ does not contain an infinite uniformly supported subset (an so, neither one of its subsets such as S_A or A^A_{fs} does not contain an infinite uniformly supported subset).

Theorem 3.8 *All the sets presented below are FSM non-uniformly infinite (i.e. none of them contains infinite uniformly supported subsets).*

1. *The invariant set A of atoms.*
2. *The powerset $\wp_{fs}(A)$ of the set of atoms.*
3. *The set $T_{fin}(A)$ of all finite injective tuples of atoms.*
4. *The invariant set of all finitely supported functions $f : A \to \wp_{fs}(A)$.*
5. *The invariant set A^A_{fs} of all finitely supported functions from A to A.*
6. *The invariant set of all finitely supported functions $f : A \to A^n$, where $n \in \mathbb{N}$ and A^n is the n-times Cartesian product of A.*
7. *The invariant set of all finitely supported functions $f : A \to T_{fin}(A)$.*
8. *The sets $\wp_{fin}(A)$, $\wp_{cofin}(A)$, $\wp_{fin}(\wp_{fs}(A))$, or $\wp_{fin}(A^A_{fs})$.*
9. *Any construction of finite powersets of the following forms $\wp_{fin}(\dots \wp_{fin}(A))$, $\wp_{fin}(\dots \wp_{fin}(A^A_{fs}))$, or $\wp_{fin}(\dots \wp_{fin}(\wp_{fs}(A)))$.*
10. *Every finite Cartesian combination between the set A, $\wp_{fin}(A)$, $\wp_{cofin}(A)$, $\wp_{fs}(A)$ and A^A_{fs}.*
11. *The disjoint unions $A + A^A_{fs}$, $A + \wp_{fs}(A)$, $\wp_{fs}(A) + A^A_{fs}$ and $A + \wp_{fs}(A) + A^A_{fs}$ and all finite disjoint unions between A, A^A_{fs} and $\wp_{fs}(A)$.*

Proof. 1. A does not contain an infinite uniformly supported subset since for any finite set $S \subseteq A$ there are at most $|S|$ atoms supported by S, namely the elements of S.

2. $\wp_{fs}(A)$ does not contain an infinite uniformly supported subset since for any finite set $S \subseteq A$ there are at most $2^{|S|+1}$ subsets of A supported by a certain finite set $S \subseteq A$, namely the subsets of S and the supersets of $A \setminus S$.

3. $T_{fin}(A)$ does not contain an infinite uniformly supported subset because the finite injective tuples of atoms supported by a finite set S are only those injective tuples formed by elements of S, being at most $1 + A^1_{|S|} + A^2_{|S|} + \dots + A^{|S|}_{|S|}$ such tuples, where $A^k_n = n(n-1)\dots(n-k+1)$.

4. We prove that $\wp_{fs}(A)^A_{fs}$ does not contain infinite uniformly supported subsets.

We remark that if $S=\{s_1,\dots,s_n\}$ is a finite subset of an invariant set $(X,\cdot)$ containing no infinite uniformly supported subset, then X^S_{fs} does not contain an infinite uniformly supported subset. For this we claim that there is an injection φ from X^S_{fs} into $X^{|S|}$ defined by: if $f\in X^S_{fs}$, then $\varphi(f)=(f(s_1),\dots,f(s_n))$; if π fixes $supp(s_1)\cup\ldots\cup supp(s_n)$ pointwise, then $\varphi(\pi\widetilde{\star}f)=((\pi\widetilde{\star}f)(s_1),\dots,(\pi\widetilde{\star}f)(s_n))=(\pi\cdot f(\pi^{-1}\cdot s_1),\dots,\pi\cdot f(\pi^{-1}\cdot s_n))=(\pi\cdot f(s_1),\dots,\pi\cdot f(s_n))=\pi\otimes\varphi(f)$ for all $f\in X^S_{fs}$, where $\otimes$ is the S_A-action on $X^{|S|}$, and $\widetilde{\star}$ is the canonical action on X^S_{fs}. Therefore φ is finitely supported. Obviously, $X^{|S|}$ does not contain an infinite uniformly supported subset; otherwise X should contain itself an infinite uniformly supported subset.

Let us fix $n\in\mathbb{N}$. Assume, by contradiction, that there exist infinitely many functions $g:A\to\wp_n(A)$ (where $\wp_n(A)$ is the invariant set of all n-sized subsets of A) supported by the same finite set $S'\subseteq A$. Each S'-supported function $g:A\to\wp_n(A)$ can be uniquely decomposed into two S'-supported functions $g|_{S'}$ and $g|_{A\setminus S'}$ (this follows since both S' and $A\setminus S'$ are supported by S'). Since there exist only finitely many functions from S' to $\wp_n(A)$ supported by S', there should exist infinitely many functions $g:(A\setminus S')\to\wp_n(A)$ supported by S'. For such a function g, let us fix an element $a\in A\setminus S'$. For each π fixing $S'\cup\{a\}$ pointwise we have $\pi\star g(a)=g(\pi(a))=g(a)$ which means that $g(a)$ is supported by $S'\cup\{a\}$. Since $g(a)$ is an n-sized (i.e. finite) subset of atoms, we have $g(a)=supp(g(a))\subseteq S'\cup\{a\}$. We distinguish two cases. In the first case, $g(a)=\{a,x_2,\dots,x_n\}$ with $x_2,\dots,x_n\in S'$. Let b be an arbitrary element from $A\setminus S'$, and so (ab) fixes S' pointwise, which means $g(b)=g((ab)(a))=(ab)\star g(a)=(ab)\star\{a,x_2,\dots,x_n\}=\{(ab)(a),(ab)(x_2),\dots,(ab)(x_n)\}=\{b,x_2,\dots,x_n\}$. Thus, only the choice of $x_2,\dots,x_n$ provides the distinction between g's. Since S' is finite, $\{x_2,\dots,x_n\}$ can be selected in $C^{n-1}_{|S'|}$ ways if $|S'|\geq n-1$, or in 0 ways otherwise. In the second case we have $g(a)=\{x_1,\dots,x_n\}$ with $x_1,\dots,x_n\in S'$. For all $b\in A\setminus S$ we have that (ab) fixes S' pointwise, and so $g(b)=g((ab)(a))=(ab)\star g(a)=(ab)\star\{x_1,\dots,x_n\}=\{x_1,\dots,x_n\}$. Since S' is finite, $\{x_1,\dots,x_n\}$ can be selected in $C^n_{|S'|}$ ways if $|S'|\geq n$, or in 0 ways otherwise. In both cases, g's can be defined only in finitely many ways.

We proved that there exist at most finitely many functions from A to $\wp_n(A)$ supported by the same set of atoms. Let us assume by contradiction that $\wp_{fin}(A)^A$ contains an infinite S-uniformly supported subset. If $f:A\to\wp_{fin}(A)$ is a function supported by S, then we have $|f(a)|=|(ab)\star f(a)|=|f((ab)(a))|=|f(b)|$ for all $a,b\notin S$. As above, each S-supported function $f:A\to\wp_{fin}(A)$ is uniquely decomposed into two S-supported functions $f|_S$ and $f|_{A\setminus S}$. However $f(A\setminus S)\subseteq\wp_n(A)$ for some $n\in\mathbb{N}$. We also know that there are at most finitely many S-supported functions from S to $\wp_{fin}(A)$. Furthermore, there exist at most finitely many S-supported functions from $A\setminus S$ to $\wp_n(A)$ for each fixed $n\in\mathbb{N}$. Therefore, it should exist an infinite subset $M\subseteq\mathbb{N}$ such that we have at least one S-supported function $f:A\setminus S\to\wp_k(A)$ for any $k\in M$. Fix $a\in A\setminus S$. For each of the above f's (that form an S-uniformly supported family $\mathscr{F}$) we have that $f(a)$'s form an uniformly supported family (by $S\cup\{a\}$) of $\wp_{fin}(A)$. If $S\cup\{a\}$ has l elements, there exists a fixed $m\in M$ with $m>l$. However, $f(a)$ for a function $f:A\setminus S\to\wp_m(A)$ from $\mathscr{F}$, which is an m-sized subset of atoms cannot be supported by $S\cup\{a\}$ whose cardinality is less than m. Therefore, the set of all $f(a)$'s cannot be infinite and uniformly supported.

Since there exists the empty supported bijection $X\mapsto A\setminus X$ from $\wp_{fin}(A)$ onto $\wp_{cofin}(A)$, we also have that there exist at most finitely many S-supported functions from A to $\wp_{cofin}(A)$. Assume, by contradiction, that $\wp_{fs}(A)^A$ contains an infinite S-uniformly supported subset. If $h:A\to\wp_{fs}(A)$ is a function supported by S, then consider $h(a)=X$ for some $a\in A\setminus S$. For $b\in A\setminus S$ we have $h(b)=(ab)\star X$, which means $h(A\setminus S)$ is formed only by finite subsets of atoms if X is finite, and $h(A\setminus S)$ is formed only by cofinite subsets of atoms if X is cofinite. However, we have at most finitely many S-supported functions from S to $\wp_{fs}(A)$. Furthermore, we have at most finitely many S-supported functions from

$A \setminus S$ to $\wp_{fin}(A)$, and at most finitely many S-supported functions from $A \setminus S$ to $\wp_{cofin}(A)$. We get a contradiction, and we conclude that $\wp_{fs}(A)^A_{fs}$ does not contain an infinite uniformly supported subset.

5. There is an equivariant injection from A^A_{fs} into $\wp_{fs}(A)^A_{fs}$, and the result is immediate.

6. There is an equivariant bijection between $(A^n)^A_{fs}$ and $(A^A_{fs})^n$ defined as below. If $f : A \to A^n$ is a finitely supported function with $f(a) = (a_1, \dots, a_n)$, we associate to f the Cartesian pair $(f_1, \dots, f_n)$ where for each $i \in \mathbb{N}$, $f_i : A \to A$ is defined by $f_i(a) = a_i$ for all $a \in A$. Since A^A_{fs} does not contain an infinite uniformly supported subset, neither $(A^A_{fs})^n$ contains an infinite uniformly supported subset.

7. Assume by contradiction that $T_{fin}(A)^A$ contains an infinite S-uniformly supported subset. If $f : A \to T_{fin}(A)$ is a function supported by S, then consider $f(a) = x$ for some $a \notin S$. For $b \notin S$ we have that $(a\,b)$ fixes S pointwise, and so $f(b) = f((a\,b)(a)) = (a\,b) \otimes f(a) = (a\,b) \otimes x$ which means $|f(a)| = |f(b)|$ for all $a, b \notin S$. Each S-supported function $f : A \to T_{fin}(A)$ can be uniquely decomposed into two S-supported functions $f|_S$ and $f|_{A \setminus S}$. However $f(A \setminus S) \subseteq A'^n$ for some $n \in \mathbb{N}$, where A'^n is the set of all injective n-tuples of A. We have at most finitely many S-supported functions from S to $T_{fin}(A)$ (since $T_{fin}(A)^S$ cannot contain an infinite uniformly supported subset; otherwise $T_{fin}(A)$ would itself contain an infinite uniformly supported subset). Since A'^n is a subset of A^n and $A \setminus S$ is a subset of A, we have at most finitely many S-supported functions from $A \setminus S$ to A'^n for each fixed $n \in \mathbb{N}$. Therefore, there should exist an infinite subset $M \subseteq \mathbb{N}$ such that we have at least one S-supported function $g : A \setminus S \to A'^k$ for any $k \in M$. Fix $a \in A \setminus S$. For each of the above g's (that form an S-supported family $\mathscr{F}$) we have that $g(a)$'s form an uniformly supported family (by $S \cup \{a\}$) of $T_{fin}(A)$, which is also infinite because tuples having different cardinalities are different and M is infinite. We thus obtained a contradiction.

Items 8,9,10,11 follow from the above items involving Theorem 3.7 ■

Remark 3.9 *Despite of Theorem 3.8(3), it is worth noting that the set* $T^{\delta}_{fin}(A) = \bigcup_{n \in \mathbb{N}} A^n$ *of all finite tuples of atoms (not necessarily injective) is FSM uniformly infinite. This follows as below. Fix* $a \in A$ *and* $i \in \mathbb{N}$*. We consider the tuple* $x_i = (a, \dots, a) \in A^i$*. Clearly,* x_i *is supported by* $\{a\}$ *for each* $i \in \mathbb{N}$*, and so* $(x_n)_{n \in \mathbb{N}}$ *is a uniformly supported subset of* $T^{\delta}_{fin}(A)$*.*

Theorem 3.10

1. *Let* X *be a finitely supported subset of an invariant set. If* X *is not FSM uniformly infinite, then each finitely supported injective mapping* $f : X \to X$ *should be surjective.*

2. *Let* X *be a finitely supported subset of an invariant set. If* $\wp_{fs}(X)$ *is not FSM uniformly infinite, then each finitely supported surjective mapping* $f : X \to X$ *should be injective. The converse does not hold since every finitely supported surjective mapping* $f : \wp_{fin}(A) \to \wp_{fin}(A)$ *is also injective, while* $\wp_{fs}(\wp_{fin}(A))$ *is FSM uniformly infinite.*

Proof. 1. Assume, by contradiction, that $f : X \to X$ is a finitely supported injection with the property that $Im(f) \subsetneq X$. This means that there exists $x_0 \in X$ such that $x_0 \notin Im(f)$. We can form a sequence of elements from X which has the first term x_0 and the general term $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$. Since $x_0 \notin Im(f)$ it follows that $x_0 \neq f(x_0)$. Since f is injective and $x_0 \notin Im(f)$, by induction we obtain that $f^n(x_0) \neq f^m(x_0)$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Furthermore, x_{n+1} is supported by $supp(f) \cup supp(x_n)$ for all $n \in \mathbb{N}$. Indeed, let $\pi \in Fix(supp(f) \cup supp(x_n))$. According to Proposition 2.5, $\pi \cdot x_{n+1} = \pi \cdot f(x_n) = f(\pi \cdot x_n) = f(x_n) = x_{n+1}$. Since $supp(x_{n+1})$ is the least set supporting x_{n+1}, we obtain $supp(x_{n+1}) \subseteq supp(f) \cup supp(x_n)$ for all $n \in \mathbb{N}$. By induction, we have $supp(x_n) \subseteq supp(f) \cup supp(x_0)$ for all $n \in \mathbb{N}$. Thus, all x_n are supported by the same set of atoms $supp(f) \cup supp(x_0)$, which means the family $(x_n)_{n \in \mathbb{N}}$ is infinite and uniformly supported, contradicting the hypothesis.

2. Let $f : X \to X$ be a finitely supported surjection. Since f is surjective, we can define the function $g : \wp_{fs}(X) \to \wp_{fs}(X)$ by $g(Y) = f^{-1}(Y)$ for all $Y \in \wp_{fs}(X)$ which is finitely supported by $supp(f) \cup supp(X)$ (according to the S-finite support principle) and injective. Alternatively, we can provide a direct proof that g is finitely supported. Let Y be an arbitrary element from $\wp_{fs}(X)$. We claim that $f^{-1}(Y) \in \wp_{fs}(X)$. Let π fix $supp(f) \cup supp(Y) \cup supp(X)$ pointwise, and $y \in f^{-1}(Y)$. This means $f(y) \in Y$. Since π fixes $supp(f)$ pointwise and $supp(f)$ supports f, we have $f(\pi \cdot y) = \pi \cdot f(y) \in \pi \star Y = Y$, and so $\pi \cdot y \in f^{-1}(Y)$. Therefore, $f^{-1}(Y)$ is finitely supported, and so the function g is well defined. We claim that g is supported by $supp(f) \cup supp(X)$. Let π fix $supp(f) \cup supp(X)$ pointwise. For any arbitrary $Y \in \wp_{fs}(X)$ we get $\pi \star Y \in \wp_{fs}(X)$ and $\pi \star g(Y) \in \wp_{fs}(X)$. Furthermore, π^{-1} fixes $supp(f)$ pointwise, and so $f(\pi^{-1} \cdot x) = \pi^{-1} \cdot f(x)$ for all $x \in X$. For any arbitrary $Y \in \wp_{fs}(X)$, we have that $z \in g(\pi \star Y) = f^{-1}(\pi \star Y) \Leftrightarrow f(z) \in \pi \star Y \Leftrightarrow \pi^{-1} \cdot f(z) \in Y \Leftrightarrow f(\pi^{-1} \cdot z) \in Y \Leftrightarrow \pi^{-1} \cdot z \in f^{-1}(Y) \Leftrightarrow z \in \pi \star f^{-1}(Y) = \pi \star g(Y)$. If follows that $g(\pi \star Y) = \pi \star g(Y)$ for all $Y \in \wp_{fs}(X)$, and so g is finitely supported. Now, since $\wp_{fs}(X)$ is not FSM uniformly infinite, it follows from item 1 that g is surjective.

Now let us consider two elements $a, b \in X$ such that $f(a) = f(b)$. We prove by contradiction that $a = b$. Suppose that $a \neq b$. Let us consider $Y = \{a\}$ and $Z = \{b\}$. Obviously, $Y, Z \in \wp_{fs}(X)$. Since g is surjective, for Y and Z there is $Y_1, Z_1 \in \wp_{fs}(X)$ such that $f^{-1}(Y_1) = g(Y_1) = Y$ and $f^{-1}(Z_1) = g(Z_1) = Z$. We know that $f(Y) \cap f(Z) = \{f(a)\}$. Thus, $f(a) \in f(Y) = f(f^{-1}(Y_1)) \subseteq Y_1$. Similarly, $f(a) = f(b) \in f(Z) = f(f^{-1}(Z_1)) \subseteq Z_1$, and so $f(a) \in Y_1 \cap Z_1$. Thus, $a \in f^{-1}(Y_1 \cap Z_1) = f^{-1}(Y_1) \cap f^{-1}(Z_1) = Y \cap Z$. However, since we assumed that $a \neq b$, we have that $Y \cap Z = \emptyset$, which represents a contradiction. It follows that $a = b$, and so f is injective.

In order to prove the invalidity of the reverse implication, we prove that any finitely supported surjective mapping $f : \wp_{fin}(A) \to \wp_{fin}(A)$ is also injective, while $\wp_{fs}(\wp_{fin}(A))$ is FSM uniformly infinite (since it contains an infinite uniformly supported countable subset $(X_n)_{n \in \mathbb{N}}$ where, for any $n \in \mathbb{N}$, X_n is defined as the equivariant set of all n-sized subsets of atoms). Let us consider a finitely supported surjection $f : \wp_{fin}(A) \to \wp_{fin}(A)$. Let $X \in \wp_{fin}(A)$. Then $supp(X) = X$ and $supp(f(X)) = f(X)$. Since $supp(f)$ supports f and $supp(X)$ supports X, for any π fixing pointwise $supp(f) \cup supp(X) = supp(f) \cup X$ we have $\pi \star f(X) = f(\pi \star X) = f(X)$ which means $supp(f) \cup X$ supports $f(X)$, that is $f(X) = supp(f(X)) \subseteq supp(f) \cup X$ (claim 1).

For a fixed $m \geq 1$, let us fix m (arbitrarily chosen) atoms $b_1, \ldots, b_m \in A \setminus supp(f)$. Let us consider $\mathscr{U} = \{\{a_1, \ldots, a_n, b_1, \ldots, b_m\} \mid a_1, \ldots, a_n \in supp(f), n \geq 1\} \cup \{\{b_1, \ldots, b_m\}\}$. The set $\mathscr{U}$ is finite since $supp(f)$ is finite and $b_1, \ldots, b_m \in A \setminus supp(f)$ are fixed. Let us consider $Y \in \mathscr{U}$, that is $Y \setminus supp(f) = \{b_1, \ldots, b_m\}$. There exists $Z \in \wp_{fin}(A)$ such that $f(Z) = Y$. According to (claim 1), Z must be either of form $Z = \{c_1, \ldots, c_k, b_{i_1}, \ldots, b_{i_l}\}$ with $c_1, \ldots, c_k \in supp(f)$ and $b_{i_1}, \ldots, b_{i_l} \in A \setminus supp(f)$ or of form $Z = \{b_{i_1}, \ldots, b_{i_l}\}$ with $b_{i_1}, \ldots, b_{i_l} \in A \setminus supp(f)$. In both cases we have $\{b_1, \ldots, b_m\} \subseteq \{b_{i_1}, \ldots, b_{i_l}\}$. We should prove that $l = m$. Assume, by contradiction, that there exists b_{i_j} with $j \in \{1, \ldots, l\}$ such that $b_{i_j} \notin \{b_1, \ldots, b_m\}$. Then $(b_{i_j}\ b_1) \star Z = Z$ since both $b_{i_j}, b_1 \in Z$ and Z is a finite subset of A (b_{i_j} and b_1 are interchanged in Z under the effect of the transposition $(b_{i_j}\ b_1)$, while the other atoms belonging to Z are left unchanged, meaning that the whole Z is left invariant under $\star$). Furthermore, since $b_{i_j}, b_1 \notin supp(f)$ we have that $(b_{i_j}\ b_1)$ fixes $supp(f)$ pointwise, and, because $supp(f)$ supports f, we get $f(Z) = f((b_{i_j}\ b_1) \star Z) = (b_{i_j}\ b_1) \star f(Z)$ which is a contradiction because $b_1 \in f(Z)$ while $b_{i_j} \notin f(Z)$. Thus, $\{b_{i_1}, \ldots, b_{i_l}\} = \{b_1, \ldots, b_m\}$, and so $Z \in \mathscr{U}$. Therefore, $\mathscr{U} \subseteq f(\mathscr{U})$ which means $|\mathscr{U}| \leq |f(\mathscr{U})|$. However, since f is a function and $\mathscr{U}$ is finite, we get $|f(\mathscr{U})| \leq |\mathscr{U}|$. We obtain $|\mathscr{U}| = |f(\mathscr{U})|$ and, because $\mathscr{U}$ is finite with $\mathscr{U} \subseteq f(\mathscr{U})$, we get $\mathscr{U} = f(\mathscr{U})$ (claim 2) which means that $f|_{\mathscr{U}} : \mathscr{U} \to \mathscr{U}$ is surjective. Since $\mathscr{U}$ is finite, $f|_{\mathscr{U}}$ should be injective, i.e. $f(U_1) \neq f(U_2)$ whenever $U_1, U_2 \in \mathscr{U}$ with $U_1 \neq U_2$ (claim 3).

Whenever $d_1, \ldots, d_v \in A \setminus supp(f)$ with$\{d_1, \ldots, d_v\} \neq \{b_1, \ldots, b_m\}$, $v \geq 1$, and considering $\mathscr{V} =$

$\{\{a_1,\ldots,a_n,d_1,\ldots,d_v\} \,|\, a_1,\ldots,a_n \in supp(f), n \geq 1\} \cup \{\{d_1,\ldots,d_v\}\}$, we conclude that $\mathscr{U}$ and $\mathscr{V}$ are disjoint. Whenever $U_1 \in \mathscr{U}$ and $V_1 \in \mathscr{V}$ we have $f(U_1) \in \mathscr{U}$ and $f(V_1) \in \mathscr{V}$ by using the same arguments used to prove (claim 2), and so $f(U_1) \neq f(V_1)$ (claim 4). If $\mathscr{T} = \{\{a_1,\ldots,a_n\} \,|\, a_1,\ldots,a_n \in supp(f)\}$ and $Y \in \mathscr{T}$, then there is $T' \in \wp_{fin}(A)$ such that $Y = f(T')$. Similarly as in (claim 2), we should have $T' \in \mathscr{T}$. Otherwise, if T' belongs to some $\mathscr{V}$ considered above, i.e. if T' contains an element outside $supp(f)$, we get the contradiction $Y = f(T') \in \mathscr{V})$ and so $\mathscr{T} \subseteq f(\mathscr{T})$ from which $\mathscr{T} = f(\mathscr{T})$ since $\mathscr{T}$ is finite (using similar arguments as those involved to prove (claim 3) from $\mathscr{U} \subseteq f(\mathscr{U})$). Thus, $f|_{\mathscr{T}} : \mathscr{T} \to \mathscr{T}$ is surjective. Since $\mathscr{T}$ is finite, $f|_{\mathscr{T}}$ should be also injective, namely $f(T_1) \neq f(T_2)$ whenever $T_1, T_2 \in \mathscr{T}$ with $T_1 \neq T_2$ (claim 5). The case $supp(f) = \emptyset$ is contained in the above analysis; it leads to $f(\emptyset) = \emptyset$ and $f(X) = X$ for all $X \in \wp_{fin}(A)$. We also have $f(T_1) \neq f(V_1)$ whenever $T_1 \in \mathscr{T}$ and $V_1 \in \mathscr{V}$ since $f(T_1) \in \mathscr{T}$, $f(V_1) \in \mathscr{V}$ and $\mathscr{T}$ and $\mathscr{V}$ are disjoint (claim 6). Since $b_1,\ldots,b_m$ and $d_1,\ldots,d_v$ were arbitrarily chosen from $A \setminus supp(f)$, the injectivity of f leads from the claims (3), (4), (5) and (6) covering all the possible cases for two different finite subsets of atoms and comparison of the values of f over the related subsets of atoms. ■

Theorem 3.10 (related to Theorem 2 in [3]) allows us to establish a strong result generalizing the approach in [3] by claiming that a finitely supported mapping $f : \wp_{fin}(A) \to \wp_{fin}(A)$ is injective *if and only if* it is surjective.

Theorem 3.11 *Let X be a finitely supported subset of an invariant set $(Z,\cdot)$. If X contains an infinite, finitely supported, totally ordered subset, then it is FSM uniformly infinite.*

Proof. Assume that X contains an infinite, finitely supported, totally ordered subset $(Y,\leq)$. We claim that Y is uniformly supported by $supp(\leq) \cup supp(Y)$. Let π be a permutation fixing $supp(\leq) \cup supp(Y)$ pointwise and let $y \in Y$ an arbitrary element. Since π fixes $supp(Y)$ pointwise and $supp(Y)$ supports Y, we obtain that $\pi \cdot y \in Y$, and so we should have either $y < \pi \cdot y$, or $y = \pi \cdot y$, or $\pi \cdot y < y$. If $y < \pi \cdot y$, then, because π fixes $supp(\leq)$ pointwise and because the mapping $z \mapsto \pi \cdot z$ is bijective from Y to $\pi \star Y$, we get $y < \pi \cdot y < \pi^2 \cdot y < \ldots < \pi^n \cdot y$ for all $n \in \mathbb{N}$. However, since any permutation of atoms interchanges only finitely many atoms, it has a finite order in the group S_A, and so there is $m \in \mathbb{N}$ such that $\pi^m = Id$ (where Id is the identity on A). This means $\pi^m \cdot y = y$, and so we get $y < y$ which is a contradiction. Similarly, the assumption $\pi \cdot y < y$, leads to the relation $\pi^n \cdot y < \ldots < \pi \cdot y < y$ for all $n \in \mathbb{N}$ which is also a contradiction since π has finite order. Therefore, $\pi \cdot y = y$, and because y was arbitrary chosen form Y, Y should be a uniformly supported infinite subset of X. ■

Definition 3.12

- *Two FSM sets X and Y are* FSM equipollent *if there exists a finitely supported bijection $f : X \to Y$.*
- *The* FSM cardinality *of X is defined as the equivalence class of all FSM sets equipollent to X, and is denoted by $|X|$.*

According to Definition 3.12 for two FSM sets X and Y, we have $|X| = |Y|$ if and only if there exists a finitely supported bijection $f : X \to Y$. On the family of cardinalities we can define the relations:

- $\leq$ by: $|X| \leq |Y|$ if and only if there is a finitely supported injective (one-to-one) mapping $f : X \to Y$.
- $\leq^*$ by: $|X| \leq^* |Y|$ if and only if there is a finitely supported surjective (onto) mapping $f : Y \to X$.

By using Theorem 4.5 and Theorem 4.6 from [2], we can present the following result.

Theorem 3.13

1. *The relation $\leq$ is equivariant, reflexive, anti-symmetric and transitive, but it is not total.*

2. *The relation $\leq^*$ is equivariant, reflexive and transitive, but it is not anti-symmetric, nor total.*

Theorem 3.14 *Let X be a finitely supported subset of an invariant set $(Y,\cdot)$*

1. *If $|X|=|X\times X|$, then $|X|=2|X|$. The converse does not hold.*
2. *If $|X|=2|X|$, then X is FSM uniformly infinite. The converse does not hold.*

Proof. 1. Fix two elements $x_1,x_2\in X$ with $x_1\neq x_2$. We can define an injection $f:X\times\{0,1\}\to X\times X$ by $f(u)=\begin{cases}(x,x_1) & \text{for}\, u=(x,0)\\ (x,x_2) & \text{for}\, u=(x,1)\end{cases}$. Clearly, by checking the condition in Proposition 2.5 and using Proposition 2.2, we have that f is supported by $supp(X)\cup supp(x_1)\cup supp(x_2)$ (since $\{0,1\}$ is necessarily a trivial invariant set), and so $|X\times\{0,1\}|\leq|X\times X|$. Thus, $|X\times\{0,1\}|\leq|X|$. Obviously, there is an injection $i:X\to X\times\{0,1\}$ defined by $i(x)=(x,0)$ for all $x\in X$ which is supported by $supp(X)$. According to Theorem 3.13, we get $2|X|=|X\times\{0,1\}|=|X|$.

Let us consider $Z=\mathbb{N}\times A$. We make the remark that $|\mathbb{N}\times\mathbb{N}|=|\mathbb{N}|$ by considering the equivariant injection $h:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ defined by $h(m,n)=2^m3^n$ and using Theorem 3.13. Similarly, $|\{0,1\}\times\mathbb{N}|=|\mathbb{N}|$ by considering the equivariant injection $h':\mathbb{N}\times\{0,1\}\to\mathbb{N}$ defined by $h'(n,0)=2^n$ and $h'(n,1)=3^n$ and using Theorem 3.13. We have $2|Z|=2|\mathbb{N}||A|=|\mathbb{N}||A|=|Z|$. However, we prove that $|Z\times Z|\neq|Z|$. Assume the contrary, and so we have $|\mathbb{N}\times(A\times A)|=|\mathbb{N}\times A\times\mathbb{N}\times A|=|\mathbb{N}\times A|$. Thus, there is a finitely supported injection $g:A\times A\to\mathbb{N}\times A$, and so there is a finitely supported surjection $f:\mathbb{N}\times A\to A\times A$ defined as $f(y)=\begin{cases}g^{-1}(y), & \text{if}\, y\in Im(g)\\ x_0, & \text{if}\, y\notin Im(g)\end{cases}$ where x_0 is a fixed element in $A\times A$. Let us consider three different atoms $a,b,c\notin supp(f)$. There exists $(i,x)\in\mathbb{N}\times A$ such that $f(i,x)=(a,b)$. Since $(a\,b)\in Fix(supp(f))$ and $\mathbb{N}$ is trivial invariant set, we have $f(i,(a\,b)(x))=(a\,b)f(i,x)=(a\,b)(a,b)=((a\,b)(a),(a\,b)(b))=(b,a)$. We should have $x=a$ or $x=b$, otherwise f is not a function. Assume without losing the generality that $x=a$, which means $f(i,a)=(a,b)$. Therefore $f(i,b)=f(i,(a\,b)(a))=(a\,b)f(i,a)=(a\,b)(a,b)=(b,a)$. Similarly, since $(a\,c),(b\,c)\in Fix(supp(f))$, we have $f(i,c)=f(i,(a\,c)(a))=(a\,c)f(i,a)=(a\,c)(a,b)=(c,b)$ and $f(i,b)=f(i,(b\,c)(c))=(b\,c)f(i,c)=(b\,c)(c,b)=(b,c)$. But $f(i,b)=(b,a)$ contradicting the functionality of f.

2. Let us consider an element y_1 belonging to an invariant set (whose action is also denoted by $\cdot$) with $y_1\notin X$ (such an element can be a non-empty element in $\wp_{fs}(X)\setminus X$, for instance). Fix $y_2\in X$. One can define a mapping $f:X\cup\{y_1\}\to X\times\{0,1\}$ by $f(x)=\begin{cases}(x,0) & \text{for}\, x\in X\\ (y_2,1) & \text{for}\, x=y_1\end{cases}$. Clearly, f is injective and it is supported by $S=supp(X)\cup supp(y_1)\cup supp(y_2)$ because for all π fixing S pointwise we have $f(\pi\cdot x)=\pi\cdot f(x)$ for all $x\in X\cup\{y_1\}$. Therefore, $|X\cup\{y_1\}|\leq|X\times\{0,1\}|=|X|$, and so there is a finitely supported injection $g:X\cup\{y_1\}\to X$. The mapping $h:X\to X$ defined by $h(x)=g(x)$ is injective, supported by $supp(g)\cup supp(X)$, and $g(y_1)\in X\setminus h(X)$, which means h is not surjective. According to Theorem 3.10(1), X should be FSM uniformly infinite.

Let us denote $Z=A\cup\mathbb{N}$. Since A and $\mathbb{N}$ are disjoint, we have that Z is an invariant set. Clearly, Z is FSM uniformly infinite. Assume, by contradiction, that $|Z|=2|Z|$, that is $|A\cup\mathbb{N}|=|A+A+\mathbb{N}|=|(\{0,1\}\times A)\cup\mathbb{N}|$. Thus, there is a finitely supported injection $f':(\{0,1\}\times A)\cup\mathbb{N}\to A\cup\mathbb{N}$, and so there exists a finitely supported injection $f:(\{0,1\}\times A)\to A\cup\mathbb{N}$. We prove that whenever $\varphi:A\to A\cup\mathbb{N}$ is finitely supported and injective, we have $\varphi(a)\in A$ for $a\notin supp(\varphi)$. Let us assume by contradiction that there is $a\notin supp(\varphi)$ such that $\varphi(a)\in\mathbb{N}$. Since $supp(\varphi)$ is finite, there exists $b\notin supp(\varphi)$, $b\neq a$. Thus, $(a\,b)$ fixes $supp(\varphi)$ pointwise, and so $\varphi(b)=\varphi((a\,b)(a))=(a\,b)\diamond\varphi(a)=\varphi(a)$ since $(\mathbb{N},\diamond)$ is a trivial invariant set. This contradicts the injectivity of φ. We can consider the mappings $\varphi_1,\varphi_2:A\to A\cup\mathbb{N}$ defined by $\varphi_1(a)=f(0,a)$ for all $a\in A$ and $\varphi_2(a)=f(1,a)$ for all $a\in A$, that are injective and supported by $supp(f)$. Therefore, $f(\{0\} \times A) = \varphi_1(A)$ contains at most finitely many element from $\mathbb{N}$, and $f(\{1\} \times A) = \varphi_2(A)$ also contains at most finitely many element from $\mathbb{N}$. Thus, f is an injection from $(\{0,1\} \times A)$ to $A \cup T$ where T is a finite subset of $\mathbb{N}$. It follows that $f(\{0\} \times A)$ contains an infinite finitely supported subset of atoms U, and $f(\{1\} \times A)$ contains an infinite finitely supported subset of atoms V. Since f is injective, it follows that U and V are infinite disjoint finitely supported subsets of A, which contradicts the fact that any subset of A is either finite or cofinite. ■

4 Conclusion

The newly developed theory of finitely supported sets allows the computational study of structures which are very large, possibly infinite, but containing enough symmetries such that they can be clearly/concisely represented and manipulated. Uniformly supported sets are particularly of interest because they involve boundedness properties of supports, meaning that the support of each element in an uniformly supported set is contained in the same finite set of atoms. In this way, all the individuals in an infinite uniformly supported family can be characterized by involving only finitely many characteristics.

In this paper we described FSM uniformly infinite sets that are finitely supported sets containing infinite, uniformly supported subsets. Firstly we proved that the finite powerset and the uniform powerset of a set that is FSM uniformly finite is also FSM non-uniformly infinite (Theorem 3.2 and Theorem 3.3). Finitely supported order-preserving self-mappings on the finite powerset and, respectively, on the uniform powerset of a set that is FSM non-uniformly infinite have least fixed points (Theorem 3.4). This is an important extension of Tarski's fixed point theorem for complete lattices that is specific to FSM; generally, order-preserving functions on finite powersets do not have fixed points since the finite powersets are not complete lattices. Particularly, finitely supported order-preserving mappings $f : \wp_{fin}(A) \to \wp_{fin}(A)$, finitely supported order-preserving mappings $f : \wp_{fin}(\wp_{fs}(A)) \to \wp_{fin}(\wp_{fs}(A))$ and finitely supported order-preserving mappings $f : \wp_{fin}(A^A_{fs}) \to \wp_{fin}(A^A_{fs})$ should have least fixed points that are supported by $supp(f)$ in each case. Another fixed point property is described in Theorem 3.5. Particularly, finitely supported progressive (inflationary) self-mappings defined on $\wp_{fin}(A)$ have infinitely many fixed points as proved in Proposition 3.6. We can also prove that any finitely supported, strict order-preserving, self-mapping f on $\wp_{fin}(A)$ has infinitely many fixed points (namely all the sets $X \setminus supp(f)$ with $X \in \wp_{fin}(A)$).

Operations with FSM uniformly (in)finite sets are presented in Theorem 3.7. We were able to prove that A, $\wp_{fs}(A)$, $T_{fin}(A)$, $\wp_{fin}(\wp_{fs}(A))$, A^A_{fs}, $\wp_{fin}(A^A_{fs})$, $(A^n)^A_{fs}$ (for a fixed $n \in \mathbb{N}$), $T_{fin}(A)^A_{fs}$ and $\wp_{fs}(A)^A_{fs}$ are FSM non-uniformly infinite, while $\wp_{fs}(\wp_{fin}(A))$ and $T^{\delta}_{fin}(A)$ are FSM uniformly infinite. Connections between FSM uniformly non-infinity and injectivity/surjectivity of self-mappings on FSM sets are presented in Theorem 3.10. One can easily remark that a finitely supported function $f : A \to A$ is injective if and only if it is surjective. Furthermore, any finitely supported injection $f : \wp_{fs}(A) \to \wp_{fs}(A)$ is also surjective, any finitely supported injection $f : \wp_{fin}(\wp_{fs}(A)) \to \wp_{fin}(\wp_{fs}(A))$ is also surjective, and any finitely supported injection $f : A^A_{fs} \to A^A_{fs}$ is also surjective. These results generalize/extend related results presented in Theorem 2 of [3]. In Theorem 3.11 we proved that a finitely supported subset of an invariant set containing an infinite, finitely supported, totally ordered subset is FSM uniformly infinite. Finally, we connected the concept of being FSM uniformly infinite with cardinality properties of form $|X| = |X \times X|$ and $|X| = 2|X|$, respectively (Theorem 3.14).

The case study presented in this paper can be significantly extended by presenting several other definitions of infinity (Dedekind type, Mostowski type, Tarski type and Kuratowski type), and then comparing them in the framework of atomic finitely supported sets. This is the topic of a future paper.

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