An Optimal Decision Procedure for MPNL over the Integers

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Interval temporal logics provide a natural framework for qualitative and quantitative temporal reasoning over interval structures, where the truth of formulae is defined over intervals rather than points. In this paper, we study the complexity of the satisfiability problem for Metric Propositional Neighborhood Logic (MPNL). MPNL features two modalities to access intervals "to the left" and "to the right" of the current one, respectively, plus an infinite set of length constraints. MPNL, interpreted over the naturals, has been recently shown to be decidable by a doubly exponential procedure. We improve such a result by proving that MPNL is actually EXPSPACE-complete (even when length constraints are encoded in binary), when interpreted over finite structures, the naturals, and the integers, by developing an EXPSPACE decision procedure for MPNL over the integers, which can be easily tailored to finite linear orders and the naturals (EXPSPACE-hardness was already known).

1   Introduction

Interval temporal logics provide a natural framework for temporal representation and reasoning on interval structures over linearly (or partially) ordered domains. They take time intervals as the primitive ontological entities and define truth of formulae with respect to them instead of to time instants. Modal operators of interval temporal logics correspond to binary relations between pairs of intervals (in fact, an interval temporal logic of ternary interval relations was developed by Venema in [15]). In the realm of interval temporal logics, a prominent role is accorded to Halpern and Shoham's modal logic of time intervals (HS), whose modalities make it possible to express all Allen's binary interval relations [1].

Interval-based temporal formalisms have been extensively used in various areas of computer science and artificial intelligence, including hardware specification and verification, constraint processing, planning and plan validation, theories of action and change, and natural language understanding. However, many applications impose severe syntactic and semantic restrictions that considerably weaken their expressive power. Interval temporal logics relax these restrictions, thus allowing one to express much more complex temporal properties. Unfortunately, most of them, including HS and the majority of its fragments, turn out to be undecidable (a comprehensive survey on interval logics can be found in [11]; an up-to-date picture of decidability and undecidability results about them can be obtained from [9, 13]).

One of the few cases of a decidable temporal logic with genuine interval semantics, that is, not reducible to point-based semantics, is the propositional logic of temporal neighborhood (Propositional Neighborhood Logic, PNL for short), interpreted over various classes of temporal structures, including all, dense, discrete, and finite linear orders, as well as rational, integer, and natural numbers [10]. PNL is the fragment of HS featuring two modalities corresponding to Allen's relations *meets* and *met by* (the

one is the inverse of the other). Decidability of PNL with respect to various classes of linear orders has been proved in [3] via a reduction to the satisfiability problem for the two-variable fragment of first-order logic for binary relational structures over ordered domains [12]. Decidability of PNL with respect to other classes of linear orders via a direct model-theoretic argument has been recently shown in [7], where tableau-based optimal decision procedures for PNL, interpreted in the considered classes of linear orders, have also been developed.

Despite its seeming simplicity, PNL is well-suited for a number of concrete application domains. One of them is that of transaction-time databases (also called append-only databases), that keep track of the sequence of timestamped versions of the database, where information is never removed and new information is appended to existing information, respecting the temporal ordering. However, in such an application domain as well in various other ones, a metric dimension turns out to be a very useful ingredient. A metric extension of PNL has been developed by Bresolin et al. in [2]. The resulting interval temporal logic, called Metric PNL (MPNL for short), pairs PNL modalities with a family of special atomic propositions expressing integer constraints (equalities and inequalities) on the length of the intervals over which they are evaluated. The authors show that the satisfiability problem for MPNL, interpreted over natural numbers, is decidable. However, they leave the precise characterization of its complexity as an open problem. Metric constraints in MPNL are expressed in terms of some $k \in \mathbb{N}$. When k is a constant of the formula or it is expressed in unary, MPNL is NEXPTIME-complete, but when k is expressed in binary, then the satisfiability problem for MPNL has been shown to be somewhere in between EXPSPACE and 2NEXPTIME only.

In this paper, we focus our attention on MPNL with a binary encoding of metric constraints. We first provide an original model-theoretic proof of the decidability of its satisfiability problem over finite linear orders, natural numbers, and integer numbers. As a matter of fact, the proof gives us a doubly-exponential upper bound to the size of the (pseudo-)model for the input MPNL formula (if any), when interpreted in the linear orders under consideration. Then, we devise an EXPSPACE decision procedure for MPNL, interpreted over the integer numbers, and we show how to adapt it to the cases of finite linear orders and natural numbers. EXPSPACE-completeness immediately follows from the already known EXPSPACE-hardness of the problem. As a by-product, we solve the issue about the exact complexity of MPNL, with a binary encoding of metric constraints, interpreted over the natural numbers, which was left open in [2]. Moreover, since MPNL is expressively complete for a fragment of first-order logic with two variables and one successor function, interpreted over the same classes of linear orders [2], the proposed decision procedure can be used to check the satisfiability of formulae of such a logic as well.

The paper is organized as follows. In Section 2, we introduce the logic. Then, in Section 3, we provide some basic definitions and results to be used in the following. In Section 4, we prove the decidability of the satisfiability problem for MPNL over finite linear orders. In the following two sections, we generalize such a result to the cases of natural and integer numbers by showing that every satisfiable formula has a model that can be represented with a suitable small "generator". Finally, in Section 7, we outline an EXPSPACE decision procedure for satisfiability checking in the most general case of integer numbers, which can be easily tailored to the cases of finite linear orders and natural numbers.

2 The logic MPNL

The logic MPNL can be viewed as a natural metric extension of PNL. The language of PNL consists of a set $\mathcal{AP}$ of atomic propositions, the propositional connectives $\neg$ and $\vee$, and the modal operators $\Diamond_r$ and $\Diamond_l$ for Allen's relations *meets* and *met by*, respectively [1]. Representation theorems, axiomatic

systems, and decidability results for PNL, interpreted over various classes of linear orders, have been given in [3, 10]. An optimal tableau-based method for deciding the satisfiability problem for the future fragment of PNL (RPNL) over the natural numbers has been presented in [8], and later extended to the full PNL over the integers in [5], while an optimal tableau system for RPNL over the class of all linear orders can be found in [6]. Optimal tableau-based decision procedures for PNL, interpreted over various classes of linear orders, can be found in [7].

An extension of PNL, interpreted over the natural numbers, with (a limited set of) metric constraints has been defined and systematically studied in [2] (as a matter of fact, a metric extension of RPNL was first considered in [4]). Let δ be the *distance* function over natural numbers defined as $\delta(x, y) = |x - y|$ (the same definition applies to any finite linear order and to the integer numbers). Metric PNL (MPNL) is obtained from PNL by adding a set of (pre-interpreted) atomic propositions for length constraints. These propositions allow one to constrain the length of the current interval and can be viewed as the natural metric generalization of the modal constant π of propositional interval logics [10], which evaluates to true precisely over point-intervals. Formally, for each $\sim \in \{<, \leqslant, =, \geqslant, >\}$, MPNL features a length constraint $\mathsf{len}_{\sim k}$, whose semantics is defined as follows: $M, [x, y] \Vdash \mathsf{len}_{\sim k}$ iff $\delta(x, y) \sim k$. Hereafter, we limit ourselves to one type of metric constraints only, namely, $\mathsf{len}_{<k}$, as all the remaining ones can be expressed in terms of it. As an example, we have that $M, [x, y] \Vdash \mathsf{len}_{=k} \Leftrightarrow M, [x, y] \Vdash \mathsf{len}_{<k+1} \wedge \neg \mathsf{len}_{<k}$. Formulae of MPNL (denoted by $\varphi, \psi, \dots$) are generated by the following grammar:

$$\varphi ::= \mathsf{len}_{<k} \mid p \mid \neg \varphi \mid \varphi \vee \varphi \mid \Diamond_l \varphi \mid \Diamond_r \varphi, \text{ where } p \in \mathcal{AP} \text{ and } k \in \mathbb{N}.$$

The other propositional connectives, the logical constants $\top$ (`true`) and $\bot$ (`false`), and the dual modal operators $\Box_r$ and $\Box_l$ are defined as usual. Moreover, the modal constant π can be defined as $\mathsf{len}_{<1}$.

Given a linearly-ordered domain $\mathbb{D} = \langle D, < \rangle$, a (*non-strict*) *interval* over $\mathbb{D}$ is an ordered pair $[x, y]$, with $x \leqslant y$. We denote by $\mathbb{I}(\mathbb{D})$ the set of all intervals over $\mathbb{D}$. Moreover, we denote by y_{max} the greatest point in D (if there is not such a point, we put $y_{max} = +\infty$) and by y_{min} the least point in D (if there is not such a point, we put $y_{min} = -\infty$). The semantics of MPNL is given in terms of *models* of the form $M = \langle \mathbb{D}, V \rangle$, where $V : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{D})}$ is a valuation function assigning a set of intervals to every atomic proposition. From now on, we assume the domain D to be either $\mathbb{Z}$, $\mathbb{N}$, or a finite prefix of $\mathbb{N}$. We recursively define the truth relation $\Vdash$ as follows:

- $M, [x, y] \Vdash p$ iff $[x, y] \in V(p)$, for any $p \in \mathcal{AP}$;

- $M, [x, y] \Vdash \mathsf{len}_{<k}$ iff $\delta(x, y) < k$;

- $M, [x, y] \Vdash \neg \varphi$ iff it is not the case that $M, [x, y] \Vdash \varphi$;

- $M, [x, y] \Vdash \varphi \vee \psi$ iff $M, [x, y] \Vdash \varphi$ or $M, [x, y] \Vdash \psi$;

- $M, [x, y] \Vdash \Diamond_l \varphi$ iff there exists $z \leqslant x$ such that $M, [z, x] \Vdash \varphi$;

- $M, [x, y] \Vdash \Diamond_r \varphi$ iff there exists $z \geqslant y$ such that $M, [y, z] \Vdash \varphi$.

An MPNL-formula φ is said to be *satisfiable* if there exist a model $M = \langle \mathbb{D}, V \rangle$ and an interval $[x, y] \in \mathbb{I}(\mathbb{D})$ such that $M, [x, y] \Vdash \varphi$.

In [2], the satisfiability problem for MPNL has been shown to be decidable when interpreted over the set of natural numbers. More precisely, it has been shown that the satisfiability problem for MPNL over the set of natural numbers is NEXPTIME-complete when either the maximal k that occurs in metric constraints is a constant or the parameter k of metric constraints is represented in unary, and it is in between EXPSPACE and 2NEXPTIME when the parameter k is represented in binary. In the following, we will

show that the satisfiability problem for MPNL, with a binary encoding of metric constraints, interpreted over finite linear orders, the natural numbers, and the integer numbers, is actually EXPSPACE-complete, by developing an EXPSPACE decision procedure for it. It is worth noticing that the model-theoretic argument behaves, in a way, worse than the one in [2], as it provides a doubly-exponential upper bound on the size of (pseudo-)models, regardless of the representation of k. Nevertheless, we will show that in the search for a (pseudo-)model of a given formula, at any time it suffices to keep track of a portion of it that can be recorded in exponential space, thus obtaining an EXPSPACE decision procedure.

3   Atoms, types, dependencies, and compass structures

In this section, we introduce the basic logical machinery to be used in the following sections. Let $M = \langle \mathbb{D}, V \rangle$ be a model for an MPNL-formula φ. In the sequel, we relate every interval in M to the set of sub-formulae of φ it satisfies. To do that, we introduce the key notions of φ-*atom* and φ-*type*. First of all, we define the *closure* $\mathcal{Cl}(\varphi)$ of φ as the set of all sub-formulae of φ and of their negations (we identify $\neg\neg\alpha$ with α, $\neg\Diamond_r\alpha$ with $\Box_r\neg\alpha$, and so on), and we define $\mathcal{K}_\varphi = \{k \mid \mathsf{len}_{<k} \in \mathcal{Cl}(\varphi)\}$ as the set of all metric parameters that appear in φ.

Definition 1. *A φ-atom is any non-empty set $F \subseteq \mathcal{Cl}(\varphi)$ such that:*

1. *for every $\alpha \in \mathcal{Cl}(\varphi)$, we have $\alpha \in F$ iff $\neg\alpha \notin F$,*
2. *for every $\gamma = \alpha \vee \beta \in \mathcal{Cl}(\varphi)$, we have $\gamma \in F$ iff $\alpha \in F$ or $\beta \in F$, and*
3. *for every k, k' in $\mathcal{K}_\varphi$ such that $k < k'$, we have that $\mathsf{len}_{<k} \in A$ implies $\mathsf{len}_{<k'} \in A$.*

   Intuitively, a φ-*atom* is a maximal *locally consistent* set of formulas chosen from $\mathcal{Cl}(\varphi)$. Note that the cardinality of $\mathcal{Cl}(\varphi)$ is *linear* in the length $|\varphi|$ of φ, while the number of φ-atoms is *at most exponential* in $|\varphi|$ (precisely, we have that $|\mathcal{Cl}(\varphi)|$ is at most $2|\varphi|$ and there are at most $2^{|\varphi|}$ distinct atoms). We define $\mathcal{A}_\varphi$ as the set of all possible atoms that can be built over $\mathcal{Cl}(\varphi)$. For every model M and every interval $[x,y] \in \mathbb{I}(\mathbb{D})$, we associate the set of all formulas $\psi \in \mathcal{Cl}(\varphi)$ such that $M, [x,y] \vDash \psi$ with $[x,y]$. We call such a set the φ-*type* of $[x,y]$ and we denote it by $\mathcal{Type}_M([x,y])$. We have that every φ-type is a φ-atom, but not vice versa. Hereafter, φ-atoms (resp., φ-types) will be simply called atoms (resp., types). Given an atom F, we denote by $\mathcal{Obs}_r(F)$ (resp., $\mathcal{Obs}_l(F)$) the set of all *future* (resp., *past*) *observable formulae* of F, namely, the set of formulae $\psi \in F$ such that $\Diamond_r\psi \in \mathcal{Cl}(\varphi)$ (resp., $\Diamond_l\psi \in \mathcal{Cl}(\varphi)$). Similarly, given an atom F, we denote by $\mathcal{Req}_r(F)$ (resp., $\mathcal{Req}_l(F)$) the set of all $\Diamond_r$-*requests* (resp., $\Diamond_l$-*requests*) of F, namely, the set of formulae $\psi \in \mathcal{Cl}(\varphi)$ such that $\Diamond_r\psi \in F$ (resp., $\Diamond_l\psi \in F$), and we use the shorthand $\mathcal{Req}(F)$ for $\mathcal{Req}_r(F) \cup \mathcal{Req}_l(F)$. Making use of the above notions, we can define the following relation between two atoms F and G:

$$F \xrightarrow{\ \mathsf{R}\ } G \quad \text{iff} \quad \mathcal{Obs}_r(G) \subseteq \mathcal{Req}_r(F) \text{ and } \mathcal{Obs}_l(F) \subseteq \mathcal{Req}_l(G)$$

The relation $\xrightarrow{\ \mathsf{R}\ }$ satisfies a *view-to-type dependency*, that is, for every pair of intervals $[x,y], [x',y']$ in $\mathbb{I}(\mathbb{D})$, we have that $y = x'$ implies $\mathcal{Type}_M([x,y]) \xrightarrow{\ \mathsf{R}\ } \mathcal{Type}_M([x',y'])$.

   We provide now a natural interpretation of MPNL over grid-like structures (*compass structures*) by exploiting the existence of a natural bijection between the intervals $[x,y]$ and the points (x,y) of a $D \times D$ grid with $x \leqslant y$. Such an interpretation was originally proposed by Venema in [14], and it can be given for HS and all its fragments as well. As an example, Figure 1 shows four intervals $[x_0,y_0], ..., [x_3,y_3]$ such that (i) $y_0 = x_1$, (ii) $x_0 = y_2$, (iii) the length of $[x_2,y_2]$ is less than k, and (iv) the length of $[x_3,y_3]$ is greater than k, together with the corresponding points $(x_0,y_0), ..., (x_3,y_3)$ of the grid (notice that Allen's
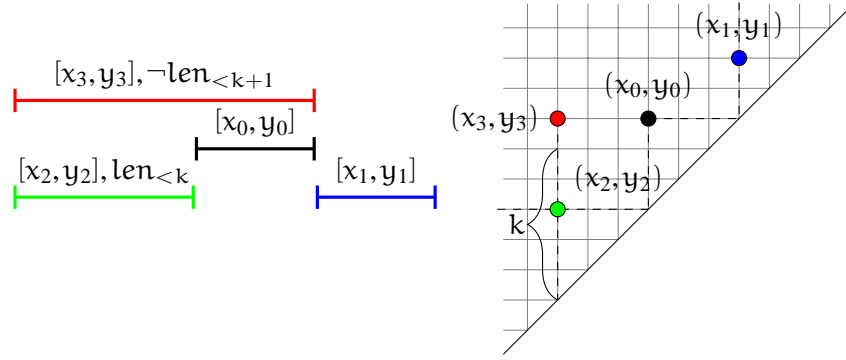
Figure 1: Correspondence between intervals and the points of the compass structure.

interval relations *meets* and *met by* are mapped into the corresponding spatial relations between pairs of points). Such an alternative interpretation of MPNL over compass structures will be exploited in the decidability proofs to make them easier to understand.

Definition 2. *Given an* MPNL *formula* φ*, a* compass φ-structure *is a pair* $\mathcal{G} = (\mathbb{P}_\mathbb{D}, \mathcal{L})$*, where* $\mathbb{P}_\mathbb{D}$ *is the set of points of the form* (x, y)*, with* $x, y \in D$ *and* $x \leqslant y$*, and* $\mathcal{L}$ *is a function that maps any point* $(x, y) \in \mathbb{P}_\mathbb{D}$ *to a* φ-atom $\mathcal{L}(x, y)$ *in such a way that:*

- *for every pair of points* $(x, y), (x', y') \in \mathbb{P}_\mathbb{D}$ *, if* $y = x'$ *then* $\mathcal{L}(x, y) \xrightarrow{\text{R}} \mathcal{L}(x', y')$ *(**temporal consistency**);*
- *for every point* $(x, y) \in \mathbb{P}_\mathbb{D}$*, and every* $\text{len}_{<k} \in \mathcal{L}(x, y)$*,* $y - x < k$ *(**length consistency**).*

We say that a compass φ-structure $\mathcal{G} = (\mathbb{P}_\mathbb{D}, \mathcal{L})$ *features* a formula ψ if there exists a point $(x, y) \in \mathbb{P}_\mathbb{D}$ such that $\psi \in \mathcal{L}(x, y)$. Fulfilling compass structures are defined as follows.

Definition 3. *Given an* MPNL *formula* φ *and compass* φ-structure $\mathcal{G} = (\mathbb{P}_\mathbb{D}, \mathcal{L})$ *for it, we say that* $\mathcal{G}$ *is **fulfilling** if and only if for every point* $(x, y) \in \mathbb{P}_\mathbb{D}$ *and every formula* $\psi \in \mathcal{R}eq_r(\mathcal{L}(x, y))$ *(resp.,* $\psi \in \mathcal{R}eq_l(\mathcal{L}(x, y))$*), there exists a point* $(x', y') \in \mathbb{P}_\mathbb{D}$ *such that* $x' = y$ *(resp.,* $y' = x$*) and* $\psi \in \mathcal{L}(x', y')$*.*

The following proposition proves that the satisfiability problem for MPNL is reducible to the problem of deciding, for any given formula φ, whether there exists a compass φ-structure featuring φ. Its easy proof is left to the reader.

Proposition 1. *An* MPNL-*formula* φ *is satisfiable if and only if there exists a fulfilling compass* φ-*structure that features* φ*.*

Without loss of generality, we will assume φ to be satisfied by the initial point-interval 0 (resp., to belong to $\mathcal{L}(0, 0)$) [13].

Given an MPNL-formula φ, we denote by k_φ the maximum k occurring in φ. If there is not any k in φ, we simply put $k_\varphi = 0$. We assume k_φ, as well as any length constraint occurring in φ, to be encoded in binary, and thus it immediately follows that $k_\varphi \leqslant 2^{|\varphi|}$.

Given a compass φ-structure $\mathcal{G} = (\mathbb{P}_\mathbb{D}, \mathcal{L})$, we define a *marking function* $\mathcal{M} : \mathbb{P}_\mathbb{D} \to \mathcal{A}_\varphi \times 2^{\mathcal{C}l(\varphi)} \times \{0, \ldots, k_\varphi\}$ such that, for every $(x, y) \in \mathbb{P}_\mathbb{D}$, $\mathcal{M}(x, y) = (F, \Psi, h)$, where (i) $F = \mathcal{L}(x, y)$, (ii) $\Psi = \{\psi \in \mathcal{C}l(\varphi) \mid \psi \in \mathcal{R}eq_r(x, x) \wedge \forall x \leqslant y' \leqslant y(\psi \notin \mathcal{L}(x, y'))\}$, and (iii) h is defined as follows:

$$h = \begin{cases} y - x & \text{if } y - x < k_\varphi; \\ k_\varphi & \text{otherwise.} \end{cases}$$

Notice that, for every point (x, y), Ψ is the set of formulae that must belong to the labeling of points (x, y'), with $y' > y$ (points "above" (x, y)), to guarantee the fulfilling of all $\Diamond_r$-requests in $\mathcal{L}(x, x)$, that is, for each $\psi \in \Psi$, there must exist at least one point (x, y') such that $\psi \in \mathcal{L}(x, y')$).

Let $\mathcal{A}_\varphi^\mathcal{M}$ be the image of $\mathcal{M}$. We call any triplet in $\mathcal{A}_\varphi^\mathcal{M}$ a *marked atom*. It can be easily shown that $|\mathcal{A}_\varphi^\mathcal{M}| \leqslant 2^{3|\varphi|}$ ($|A_\varphi| \leqslant 2^{|\varphi|}$, $|\mathcal{R}eq_r(\mathcal{L}(x,x))| \leqslant |\varphi|$, and $k_\varphi \leqslant 2^{|\varphi|}$).

Definition 4. *Given an* MPNL *formula* φ*, a compass* φ*-structure* $\mathcal{G} = (\mathbb{P}_\mathbb{D}, \mathcal{L})$ *for* φ*, and* $y \in D$*, we define the* horizontal configuration *of* y *in* $\mathcal{G}$ *as a counting function* $\mathcal{C}_y : \mathcal{A}_\varphi^\mathcal{M} \to \mathbb{N} \cup \{\omega\}$ *such that for every* $(F, \Psi, h) \in \mathcal{A}_\varphi^\mathcal{M}$*,* $\mathcal{C}_y(F, \Psi, h) = |\{x \mid \mathcal{M}(x, y) = (F, \Psi, h)\}|$*.*

It is worth noticing that, for any given y, (i) there exists a unique marked atom of the form $(F, \Psi, 0)$, with $\mathcal{C}_y(F, \Psi, 0) = 1$, and (ii) for every $0 < h < k_\varphi$, there exists at most 1 marked atom of the form (F, Ψ, h), and if for every marked atom (F, Ψ, h), $\mathcal{C}(F, \Psi, h) = 0$, then $\mathcal{C}(F', \Psi', h') = 0$ for every marked atom (F', Ψ', h') with $h' > h$. On the contrary, there is not a bound on the number of occurrences of a marked node of the form (F, Ψ, k_φ) (it can be equal to ω).

Finally, we define the following equivalence relation on the set of horizontal configurations, where p and f are defined as $p = |\{\Diamond_l \psi \in \mathcal{C}l(\varphi)\}|$ and $f = |\{\Diamond_r \psi \in \mathcal{C}l(\varphi)\}|$, respectively.

Definition 5. *Given an* MPNL *formula* φ *and a compass* φ*-structure* $\mathcal{G} = (\mathbb{P}_\mathbb{D}, \mathcal{L})$ *for it, we say that two horizontal configurations* $\mathcal{C}_y$ *and* $\mathcal{C}_{y'}$ *are* equivalent *(written* $\mathcal{C}_y \equiv \mathcal{C}_{y'}$*) if and only if for every* $(F, \Psi, h) \in \mathcal{A}_\varphi^\mathcal{M}$*, either* $\mathcal{C}_{y'}(A, \Psi, h) = \mathcal{C}_y(F, \Psi, h)$ *or* $(h = k_\varphi$ *and) both* $\mathcal{C}_y(F, \Psi, k_\varphi) \geqslant p \cdot f + p$ *and* $\mathcal{C}_{y'}(F, \Psi, k_\varphi) \geqslant p \cdot f + p$*.*

It can be easily shown that $\equiv$ is an equivalence relation of finite index. For every marked atom $(F, \Psi, h) \in \mathcal{A}_\varphi^\mathcal{M}$, we do not distinguish between two configurations $\mathcal{C}_y$ and $\mathcal{C}_{y'}$ such that $\mathcal{C}_y(F, \Psi, h)$ and $\mathcal{C}_{y'}(F, \Psi, h)$ are different, but both greater than or equal to $p \cdot f + p$. Hence, the number of equivalence classes in $\equiv$ is bounded by

$$\left(p \cdot f + p + 1\right)^{|\mathcal{A}_\varphi^\mathcal{M}|} \leqslant \left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1\right)^{2^{3|\varphi|}},$$

since $p \cdot f + p \leqslant \frac{|\varphi|^2}{4} + \frac{|\varphi|}{2}$ and $\left|\mathcal{A}_\varphi^\mathcal{M}\right| \leqslant 2^{3|\varphi|}$.

4 Decidability of MPNL over finite linear orders

In this section, we show that if there exists a finite fulfilling compass structure $\mathcal{G}$ for an MPNL formula φ, then there exists a finite fulfilling compass structure $\mathcal{G}'$ whose size is at most doubly exponential in the length of φ. To prove this result, we will make use of the following lemma, which states that we can always shrink the size of a fulfilling compass structure, provided that there exist y, y' such that $\mathcal{C}_y \equiv \mathcal{C}_{y'}$.

Lemma 1. *Let* φ *be an* MPNL *formula and let* $\mathcal{G} = (\mathbb{P}_\mathbb{D}, \mathcal{L})$ *be a finite fulfilling compass* φ*-structure which features* φ*. If there exist* $\overline{y}, \overline{y}' \in D$*, with* $\overline{y} < \overline{y}'$*, such that* $\mathcal{C}_{\overline{y}} \equiv \mathcal{C}_{\overline{y}'}$*, then it is possible to build a finite fulfilling compass* φ*-structure* $\mathcal{G}' = (\mathbb{P}_{\mathbb{D}'}, \mathcal{L}')$ *featuring* φ *with* $|D'| = |D| - (\overline{y}' - \overline{y})$*.*

Proof. Suppose that $\mathcal{G} = (\mathbb{P}_\mathbb{D}, \mathcal{L})$ is a finite fulfilling compass φ-structure which features φ and such that there exist $\overline{y}, \overline{y}' \in D$, with $\overline{y} < \overline{y}'$, such that $\mathcal{C}_{\overline{y}} \equiv \mathcal{C}_{\overline{y}'}$. We build a compass φ-structure $\mathcal{G}' = (\mathbb{P}_{\mathbb{D}'}, \mathcal{L}')$, with $|D'| = |D| - (\overline{y}' - \overline{y})$, by executing the following procedure.

1. For every $(x,y) \in \mathbb{P}_{\mathbb{D}'}$, with $y \leqslant \overline{y}$, we put $\mathcal{L}'(x,y) = \mathcal{L}(x,y)$.

2. For every $(x,y) \in \mathbb{P}_{\mathbb{D}'}$, with $y > \overline{y}$ and $\overline{y} - k_\varphi < x \leqslant y$, we put $\mathcal{L}'(x,y) = \mathcal{L}(x + (\overline{y}' - \overline{y}), y + (\overline{y}' - \overline{y}))$.

3. For every $(A, \Psi, k_\varphi) \in \mathcal{A}_\varphi^{\mathcal{M}}$, we define a partial injective function $g : \{0, \ldots, \overline{y} - k_\varphi\} \rightarrow \{0, \ldots, \overline{y}' - k_\varphi\}$ as follows:

$$g(x) = \begin{cases} x' \text{ with } \mathcal{M}(x', \overline{y}') = (A, \Psi, k_\varphi) & \text{if } \mathcal{M}(x, \overline{y}) = (A, \Psi, k_\varphi) \text{ and} \\ & \mathcal{C}_{\overline{y}}(A, \Psi, k_\varphi) = \mathcal{C}_{\overline{y}'}(A, \Psi, k_\varphi) \\ \texttt{undefined} & \text{otherwise} \end{cases}$$

   By injectivity of g, every x (where g is defined) is associated with a distinct x'. Moreover, since $\mathcal{C}_{\overline{y}}(A, \Psi, k_\varphi) = \mathcal{C}_{\overline{y}'}(A, \Psi, k_\varphi)$, for every x' such that $\mathcal{M}(x', \overline{y}') = (A, \Psi, k_\varphi)$, there exists (a unique) x such that $g(x) = x'$. Now, for every $0 \leqslant x \leqslant \overline{y} - k_\varphi$ such that $g(x)$ is defined, we put $\mathcal{L}'(x, \overline{y} + i) = \mathcal{L}(g(x), \overline{y}' + i)$ for every $1 \leqslant i \leqslant y_{max} - \overline{y}'$.

4. For every $(A, \Psi, k_\varphi) \in \mathcal{A}_\varphi^{\mathcal{M}}$ such that $\mathcal{C}_{\overline{y}'}(A, \Psi, k_\varphi) \geqslant p \cdot f + p$, we choose a "witness" $w_{(A, \Psi)}$ such that $\mathcal{M}(w_{(A, \Psi)}, \overline{y}') = (A, \Psi, k_\varphi)$. Then, we identify a minimal set of *essential elements* $\mathcal{ES}_{(A, \Psi)}^{\overline{y}'} = \{y_1', \ldots, y_m'\}$ such that, for every $\psi \in \Psi$, there exists a point $y_j' \in \mathcal{ES}_{(A, \Psi)}^{\overline{y}'}$ with $\psi \in \mathcal{L}(w_{(A, \Psi)}, y_j')$. As $|\Psi| \leqslant f$, it immediately follows that $m \leqslant f$. Moreover, by definition of (the second component of a) marked atom, $y_i' > \overline{y}'$ for every $1 \leqslant i \leqslant m$. Now, let $\text{Blocked}_{(A, \Psi)}^{\overline{y}'} = \{x_1', \ldots, x_{m'}'\}$ be a minimal set of elements, called *blocked elements*, satisfying the following condition: for every $1 \leqslant i \leqslant m$ and every $\psi \in \mathcal{R}eq_l(y_i', y_i')$, if there exists $x' \in D$ such that $\psi \in \mathcal{L}(x', y_i')$ and $\mathcal{M}(x', \overline{y}') = (A, \Psi, k_\varphi)$, then there exists $x_j' \in \text{Blocked}_{(A, \Psi)}^{\overline{y}'}$ such that $\psi \in \mathcal{L}(x_j', y_i')$ and $\mathcal{M}(x_j', \overline{y}') = (A, \Psi, k_\varphi)$. As $m \leqslant f$ and $|\mathcal{R}eq_l(y_i', y_i')| \leqslant p$, $|\text{Blocked}_{(A, \Psi)}^{\overline{y}'}| \leqslant p \cdot f$. Since $\mathcal{C}_{\overline{y}} \equiv \mathcal{C}_{\overline{y}'}$, a set $\text{Blocked}_{(A, \Psi)}^{\overline{y}} = \{x_1, \ldots, x_{m'}\}$ exists such that, for every $1 \leqslant i \leqslant m'$, $\mathcal{M}(x_i, \overline{y}) = (A, \Psi, k_\varphi)(= \mathcal{M}(w_{(A, \Psi)}, \overline{y}'))$. For every $1 \leqslant i \leqslant m'$ and every $1 \leqslant j \leqslant y_{max} - \overline{y}'$, we put $\mathcal{L}'(x_i, \overline{y} + j) = \mathcal{L}(x_i', \overline{y}' + j)$. In such a way, all points (x_i, y) in $\mathcal{G}'$, with $1 \leqslant i \leqslant m'$, turn out to be labeled and all $\Diamond_r$-requests of points (x_i, x_i) are fulfilled.

5. Once the above steps have been executed, there may exist some $x \in D$ such that the labeling of points $(x, y) \in \mathbb{P}_{\mathbb{D}'}$, with $y > \overline{y}$, is still undefined. Let $\mathcal{M}(x, \overline{y}) = (A, \Psi, k_\varphi)$. By construction, $\mathcal{C}_{\overline{y}}(\mathcal{M}(x, \overline{y})) \geqslant p \cdot f + p$. For every unlabeled point (x, y), we put $\mathcal{L}'(x, y) = \mathcal{L}(w_{(A, \Psi)}, y + (\overline{y}' - \overline{y}))$, where $w_{(A, \Psi)}$ is the witness chosen at step 4.

Unfortunately, there is no guarantee that all $\Diamond_l$-requests are fulfilled in $\mathcal{G}'$. Let $y > \overline{y}$ such that there exists $\psi \in \mathcal{R}eq_l(\mathcal{L}'(y, y))$ which is not fulfilled in $\mathcal{G}'$. By construction, $\mathcal{L}'(y, y) = \mathcal{L}(y + (\overline{y}' - \overline{y}), y + (\overline{y}' - \overline{y}))$, and thus, since $\mathcal{G}$ is fulfilling, there exists a point $(x_\psi', y + (\overline{y}' - \overline{y}))$ such that $\psi \in \mathcal{L}(x_\psi', y + (\overline{y}' - \overline{y}))$. We must distinguish two cases:

a) for every witness $w_{(A, \Psi)}$, $y + (\overline{y}' - \overline{y}) \notin \mathcal{ES}_{(A, \Psi)}^{\overline{y}'}$. Let (A, Ψ, k_φ) be the marked atom associated with $(x_\psi', \overline{y}')$ in $\mathcal{G}$, that is, $\mathcal{M}(x_\psi', \overline{y}') = (A, \Psi, k_\varphi)$. It holds that $\mathcal{C}_{\overline{y}'}(\mathcal{M}(x_\psi', \overline{y}')) \geqslant p \cdot f + p$ (if this was not the case, x_ψ' would not belong to the range of g, thus violating the properties we impose on it at step 3), and thus $\mathcal{C}_{\overline{y}}(\mathcal{M}(x_\psi', \overline{y})) \geqslant p \cdot f + p$ as well. Since $|\text{Blocked}_{(A, \Psi)}^{\overline{y}}|(= |\text{Blocked}_{(A, \Psi)}^{\overline{y}'}|) \leqslant p \cdot f$, there exist at least p elements $x_{m'+1}, \ldots, x_{m'+p}$ such that, for $1 \leqslant i \leqslant p$, $x_{m'+i} \notin \text{Blocked}_{(A, \Psi)}^{\overline{y}}$ and $\mathcal{M}(x_{m'+i}, \overline{y})(= \mathcal{M}(w_{(A, \Psi)}, \overline{y}')) = (A, \Psi, k_\varphi)$. We show that, in

order to fulfill ψ, the labeling of at least one among $(x_{m'+1}, y), \ldots, (x_{m'+p}, y)$ can be suitably updated. To this end, it suffices to observe that $|\mathcal{R}eq_l(\mathcal{L}'(y, y))| \leqslant p$ and thus there exists $1 \leqslant j \leqslant p$ such that, for every $\theta \in \mathcal{R}eq_l(\mathcal{L}'(y, y))$, if $\theta \in \mathcal{L}'(x_{m'+j}, y)$, then $\theta \in \mathcal{L}'(x_{m'+l}, y)$, for some $0 \leqslant l \leqslant p$, with $l \neq j$, as well. Moreover, since $y + (\overline{y}' - \overline{y}) \notin \mathcal{ES}^{\overline{y}'}_{(A, \Psi)}$, for every $\phi \in \Psi$, there exists $y'(> \overline{y}) \neq y$ such that $\phi \in \mathcal{L}'(x_{m'+j}, y')$ and thus $(x_{m'+j}, y)$ is not needed to fulfill $\Diamond_r$-requests in $\mathcal{R}eq_r(\mathcal{L}'(x_{m'+j}, x_{m'+j}))$. Hence, we can safely revise $\mathcal{L}'(x_{m'+j}, y)$ putting $\mathcal{L}'(x_{m'+j}, y) = \mathcal{L}(x'_\psi, y + (\overline{y}' - \overline{y}))$;

b) there exists a witness $w_{(\overline{A}, \overline{\Psi})}$ such that $y + (\overline{y}' - \overline{y}) \in \mathcal{ES}^{\overline{y}'}_{(\overline{A}, \overline{\Psi})}$. Let (A, Ψ, k_φ) be the marked atom associated with $(x'_\psi, \overline{y}')$ in $\mathcal{G}$, and let $(x_{m'+1}, y), \ldots, (x_{m'+p}, y)$ be the p elements of case a). As above, we can show that, to fulfill ψ, the labeling of at least one among them, say $(x_{m'+j}, y)$, can be suitably updated. The irrelevance of $(x_{m'+j}, y)$ with respect to requests in $\mathcal{R}eq_l(\mathcal{L}'(y, y))$ can be proved in exactly the same way. To complete the proof, it suffices to show that $y + (\overline{y}' - \overline{y}) \notin \mathcal{ES}^{\overline{y}'}_{(A, \Psi)}$. By contradiction, assume that $y + (\overline{y}' - \overline{y}) \in \mathcal{ES}^{\overline{y}'}_{(A, \Psi)}$. This implies that there exists $x_i \in \mathcal{B}locked^{\overline{y}}_{(A, \Psi)}$ such that $\psi \in \mathcal{L}'(x_i, y)$, and thus ψ is fulfilled in $\mathcal{G}'$ (contradiction). Then, we can proceed as in case a) and rewrite $\mathcal{L}'(x_{m'+j}, y)$ as $\mathcal{L}(x'_\psi, y + (\overline{y}' - \overline{y}))$.

$\mathcal{G}'$ is a fulfilling compass φ-structure for φ. $\qquad \square$

By exploiting Lemma 1, we can prove that a formula φ is satisfiable by a finite compass structure iff it is satisfiable by a finite compass structure whose horizontal configurations are pairwise non-equivalent.

Theorem 1. *Let φ be an* MPNL*-formula. If there exists a finite fulfilling compass φ-structure $\mathcal{G} = (\mathbb{P}_\mathbb{D}, \mathcal{L})$ which features φ, then there exists a finite fulfilling compass φ-structure $\mathcal{G}' = (\mathbb{P}_{\mathbb{D}'}, \mathcal{L}')$ featuring φ such that $|D'| \leqslant \left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1 \right)^{2^{3|\varphi|}}$.*

Proof. Let $\mathcal{G} = (\mathbb{P}_\mathbb{D}, \mathcal{L})$ be a finite fulfilling compass φ-structure featuring φ and suppose that $|D| > \left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1 \right)^{2^{3|\varphi|}}$. Since the index of $\equiv$ is smaller than $|D|$, there exist $\overline{y}, \overline{y}' \in D$, with $\overline{y} < \overline{y}'$, such that $\mathcal{C}_{\overline{y}} \equiv \mathcal{C}'_y$. Then, we exploit Lemma 1 to build a smaller compass φ-structure $\mathcal{G}_1 = (\mathbb{P}_{\mathbb{D}_1}, \mathcal{L}_1)$ with $|D_1| = |D| - (\overline{y}' - \overline{y})$. By iterating such a contraction step, we eventually obtain a compass φ-structure $\mathcal{G}_n = (\mathbb{P}_{\mathbb{D}_n}, \mathcal{L}_n)$ whose horizontal configurations are pairwise non-equivalent. Since the number of equivalence classes in $\equiv$ is less than or equal to $\left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1 \right)^{2^{3|\varphi|}}$, the thesis immediately follows. $\square$

5 Decidability of MPNL over the naturals

We now extend the result of the previous section to cope with the satisfiability problem for MPNL over $\mathbb{N}$. First, we identify a subset of finite compass φ-structures, called *compass generators*, which turn out to be crucial for decidability.

Definition 6. *Let φ be an MPNL formula. An $\mathbb{N}$-compass generator for φ is a finite compass φ-structure $\mathcal{G} = (\mathbb{P}_\mathbb{D}, \mathcal{L})$, which features φ, that satisfies the following conditions:*

1. *all $\Diamond_l$-requests of every point $(x, y) \in \mathbb{P}_\mathbb{D}$ are fulfilled;*
2. *there exists y_{inf}, with $y_{max} - y_{inf} \geqslant k_\varphi$, such that:*
   (a) for every $(F, \Psi, h) \in \mathcal{A}^\mathcal{M}_\varphi$, if $\mathcal{C}_{y_{max}}(F, \Psi, h) > 0$, then $\mathcal{C}_{y_{inf}}(F, \Psi, h) > 0$, and
   (b) $\mathcal{M}(x, y_{max}) = (F, \emptyset, h)$, for every $0 \leqslant x \leqslant y_{inf}$.

Theorem 2. *An MPNL formula φ is satisfiable over $\mathbb{N}$ iff there exists an $\mathbb{N}$-compass generator for it.*

Proof. To prove the left-to-right direction, suppose φ to be satisfiable over $\mathbb{N}$, and let $\mathcal{G} = (\mathbb{P}_{\mathbb{N}}, \mathcal{L})$ be a fulfilling compass φ-structure which features φ. Since the index of $\equiv$ is finite, there must exist an infinite sequence $\mathcal{S} = y_1 < y_2 < \ldots$ in $\mathbb{N}$ such that $\mathcal{C}_{y_i} \equiv \mathcal{C}_{y_j}$ for every $i, j \in \mathbb{N}$. Consider now the first element y_1 in $\mathcal{S}$, and let $(x, y_1) \in \mathbb{P}_{\mathbb{N}}$ be a point on the row y_1. Suppose $\mathcal{M}(x, y_1) = (F, \Psi, k_\varphi)$. Since $\mathcal{G}$ is fulfilling, for every $\psi \in \Psi$, there exists $y_\psi > y_1$ such that $\psi \in \mathcal{L}(x, y_\psi)$. Let $\overline{y}$ be the maximum of such y_ψ with respect to every $x \leqslant y_1$ and every $\psi \in \Psi$, and let y_j be the smallest element in $\mathcal{S}$ such that $\overline{y} < y_j$ and $y_j - y_1 \geqslant k_\varphi$. By the definition of the marking function $\mathcal{M}$, we have that $\mathcal{M}(x, y_j) = (F, \emptyset, h)$, for every $0 \leqslant x \leqslant y_1$. Consider now the restriction $\mathcal{G}'$ of $\mathcal{G}$ to $D = \{0, 1, \ldots, y_j\}$. It is straightforward to check that, given $y_{max} = y_j$, y_1 satisfies the conditions for y_{inf} of Definition 6, and thus $\mathcal{G}'$ is an $\mathbb{N}$-compass generator featuring φ ($(0, 0)$ belongs to $\mathcal{G}'$).

To prove the right-to-left direction, suppose that $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ is an $\mathbb{N}$-compass generator for φ. We build a fulfilling compass φ-structure $\mathcal{G}_\omega = (\mathbb{P}_{\mathbb{N}}, \mathcal{L}_\omega)$ as the (infinite) union of an appropriate sequence of $\mathbb{N}$-compass generators $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \ldots$. First, we take $\mathcal{G}_0 = \mathcal{G}$. Then, for every $i \geqslant 0$, we build $\mathcal{G}_{i+1} = (\mathbb{P}_{\mathbb{D}_{i+1}}, \mathcal{L}_{i+1})$ starting from $\mathcal{G}_i = (\mathbb{P}_{\mathbb{D}_i}, \mathcal{L}_i)$ as follows. Let $y_{inf} \in D_i$ satisfy the conditions of Definition 6. We put $D_{i+1} = \{0, 1, \ldots, y_{max}, \ldots, y_{max} + (y_{max} - y_{inf})\}$ and we define $\mathcal{L}_{i+1}$ as follows:

1. for every $(x, y) \in \mathbb{P}_{\mathbb{D}_i}$, we put $\mathcal{L}_{i+1}(x, y) = \mathcal{L}_i(x, y)$;

2. for every $(x, y) \in \mathbb{P}_{\mathbb{D}_{i+1}}$ such that $x > y_{max} - k_\varphi$ and $y > y_{max}$, we put $\mathcal{L}_{i+1}(x, y) = \mathcal{L}_i(x - (y_{max} - y_{inf}), y - (y_{max} - y_{inf}))$;

3. for every $(x, y) \in \mathbb{P}_{\mathbb{D}_{i+1}}$ such that $y_{inf} - k_\varphi \geqslant x \geqslant 0$ and $y > y_{max}$, we put $\mathcal{L}_{i+1}(x, y) = \mathcal{L}_i(x, y - (y_{max} - y_{inf}))$;

4. for every $(x, y) \in \mathbb{P}_{\mathbb{D}_{i+1}}$ such that $y_{max} - k_\varphi \geqslant x > y_{inf} - k_\varphi$ and $y > y_{max}$, we put $\mathcal{L}_{i+1}(x, y) = \mathcal{L}_i(x', y - (y_{max} - y_{inf}))$, for some x' such that $\mathcal{M}(x', y_{inf}) = \mathcal{M}(x, y_{max})$ (the existence of such an x' is guaranteed by property (a) of Definition 6).

By construction, for every $(F, \Psi, h) \in \mathcal{A}_\varphi^{\mathcal{M}}$, if $\mathcal{C}_{y_{max} + (y_{max} - y_{inf})}(F, \Psi, h) > 0$, then $\mathcal{C}_{y_{max}}(F, \Psi, h) > 0$, Moreover, $\mathcal{M}(x, y_{max} + (y_{max} - y_{inf})) = (A, \emptyset, h)$, for every $0 \leqslant x \leqslant y_{max}$, and thus $\mathcal{G}_{i+1}$ is a $\mathbb{N}$-compass generator for φ.

The fulfilling compass φ-structure satisfying φ on $\mathbb{N}$ we were looking for is $\mathcal{G}_\omega = \bigcup_{i \geqslant 0} \mathcal{G}_i$. $\square$

Theorem 3. *Let φ be an MPNL formula. If there exists an $\mathbb{N}$-compass generator $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ that features φ, then there exists an $\mathbb{N}$-compass generator $\mathcal{G}' = (\mathbb{P}_{\mathbb{D}'}, \mathcal{L}')$, that features φ, with $|D'| \leqslant$*
$$\left(2^{3|\varphi|} + 2\right) \cdot \left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1\right)^{2^{3|\varphi|}} + 1.$$

Proof. Let $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ be an $\mathbb{N}$-compass generator which features φ, and let $y_{inf} \in D$ satisfy the conditions of Definition 6. We define a minimal set $S = \{\overline{y}_0, \ldots, \overline{y}_m\}$ of elements in D such that (i) $\overline{y}_0 = 0$, (ii) $\overline{y}_j < \overline{y}_{j+1}$, for each $0 \leqslant j < m$, (iii) $\overline{y}_{m-1} = y_{inf}$, (iv) $\overline{y}_m = y_{max}$, and (v) for every $(F, \Psi, h) \in \mathcal{A}_\varphi^{\mathcal{M}}$, if $\mathcal{C}_{y_{inf}}(F, \Psi, h) > 0$, then there exists $\overline{y}_j$ such that $\mathcal{M}(\overline{y}_j, y_{inf}) = (F, \Psi, h)$. From the minimality requirement, it follows that $m \leqslant 2^{3|\varphi|} + 3$.

We build a finite sequence of $\mathbb{N}$-compass generators $\mathcal{G}_0 \supset \mathcal{G}_1 \supset \ldots \supset \mathcal{G}_n$, whose last element is a small enough $\mathbb{N}$-compass generator $\mathcal{G}_n$, as follows. We start with $\mathcal{G}_0 = \mathcal{G}$. Now, let $\mathcal{G}_i = (\mathbb{P}_{\mathbb{D}_i}, \mathcal{L}_i)$ be the i-th compass generator in the sequence, and let $S_i = \{\overline{y}_0, \ldots, \overline{y}_m\}$ be the above-defined minimal set of elements in D_i. If there exist no y, y', with $\overline{y}_j \leqslant y < y' < \overline{y}_{j+1}$ for some $0 \leqslant j < m$, such that $\mathcal{C}_y \equiv \mathcal{C}_{y'}$, we terminate the construction and put $n = i$, that is, $\mathcal{G}_i$ is the last $\mathbb{N}$-compass generator in the
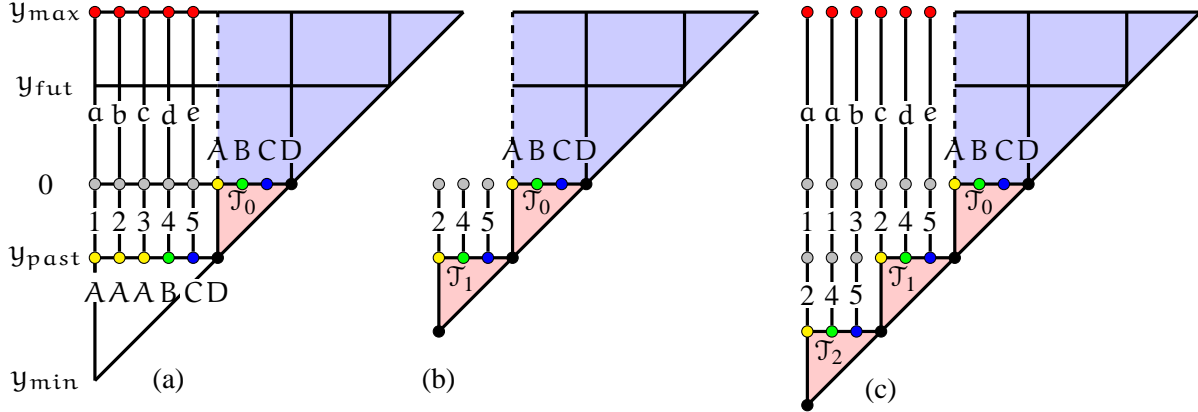
Figure 2: From a $\mathbb{Z}$-compass generator to a compass structure over $\mathbb{Z}$.

sequence. Otherwise, we must distinguish two cases. If $y_{inf} \leqslant y, y' < y_{max}$, then the application of (the construction of) Lemma 1 to the pair of positions y and y' produces an $\mathbb{N}$-compass generator $\mathcal{G}_{i+1} = (\mathbb{P}_{\mathbb{D}_{i+1}}, \mathcal{L}_{i+1})$, with $|D_{i+1}| = |D_i| - (y' - y)$. It can be easily checked that the resulting structure satisfies the conditions of Definition 6 (notice that some triples may disappear from y_{max}, that is, $\mathcal{C}_{y_{max}}(F, \Psi, h)$ may become equal to 0 for some triple (F, Ψ, h)). If $\overline{y}_j \leqslant y, y' < \overline{y}_{j+1}$ for some $j \leqslant \overline{y}_{m-2}$, we can still apply (the construction of) Lemma 1 to the pair of positions y and y', but we must guarantee that all triples belonging to the row y_{inf} in D_i are preserved. This can be done by an appropriate choice of the witnesses at step 4 of (the construction of) Lemma 1. It is worth noticing that in both cases, while positions between $\overline{y}_{j+1}$ and $\overline{y}_{m-2}$ (if any) remain unchanged (they are only shifted), those between $\overline{y}_1$ and $\overline{y}_j$ may change from S_i to S_{i+1}.

At the end of the procedure, all the horizontal configurations in between two consecutive elements $\overline{y}_j, \overline{y}_{j+1} \in S$ are pairwise non-equivalent. From this, it immediately follows that the final $\mathbb{N}$-compass generator $\mathcal{G}_n = (\mathbb{P}_{\mathbb{D}_n}, \mathcal{L}_n)$ is such that $|D_n| \leqslant \left(2^{3|\varphi|} + 2\right) \cdot \left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1\right)^{2^{3|\varphi|}} + 1$. $\square$

6   Decidability of MPNL over the integers

In this section, we generalize the notion of *compass generator* in order to prove the decidability of the satisfiability problem for MPNL over $\mathbb{Z}$.

Definition 7. *Let φ be an MPNL formula. A $\mathbb{Z}$-compass generator for φ is a finite compass φ-structure $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ such that there exist $y_{fut}, y_{past} \in D$, with $y_{past} < 0 < y_{fut}$, $y_{past} - y_{min} \geqslant k_\varphi$, and $y_{max} - y_{fut} \geqslant k_\varphi$, which satisfy the following conditions:*

1. *all $\Diamond_l$-requests of every point $(y, y) \in \mathbb{P}_{\mathbb{D}}$, with $y_{past} \leqslant y \leqslant y_{max}$, are fulfilled;*

2. *for every $(F, \Psi, h) \in \mathcal{A}_\varphi^{\mathcal{M}}$, if $\mathcal{C}_{y_{max}}(F, \Psi, h) > 0$, then $\mathcal{C}_{y_{fut}}(F, \Psi, h) > 0$, and $\mathcal{M}(x, y_{max}) = (F, \emptyset, h)$, for every $y_{min} \leqslant x \leqslant y_{fut}$;*

3. *for every $(F, \Psi, h) \in \mathcal{A}_\varphi^{\mathcal{M}}$, if $\mathcal{C}_{y_{past}}(F, \Psi, h) > 0$, then there exists $y_{past} \leqslant x \leqslant 0$ such that $\mathcal{M}(x, 0) = (F, \Psi, h)$.*

Theorem 4. *An MPNL formula φ is satisfiable over $\mathbb{Z}$ iff there exists a $\mathbb{Z}$-compass generator for it.*

Proof. We start with the left-to-right direction. From the satisfiability of φ over $\mathbb{Z}$, it follows that there exists a fulfilling compass φ-structure $\mathcal{G} = \langle \mathbb{P}_\mathbb{Z}, \mathcal{L} \rangle$ which features φ. Hence, to prove the claim it suffices to show that there exist $y_{min}, y_{past}, y_{fut}$, and y_{max}, with $y_{min} < y_{past} < 0 < y_{fut} < y_{max}$, that satisfy the conditions of Definition 7. Since the index of $\equiv$ is finite, there exists an infinite-to-the-past sequence of elements $\mathcal{S} = y_{-1} > y_{-2} > \dots$ such that, for every $i, j \in \mathbb{N}$, $\mathcal{C}_{y_i} \equiv \mathcal{C}_{y_j}$. Without loss of generality, we can assume that $y_{-1} = 0$. Since $\mathcal{S}$ is infinite to the past, there exists $j < -1$ such that, for every $(F, \Psi, h) \in \mathcal{A}_\varphi^\mathcal{M}$ with $\mathcal{C}_{y_j}(F, \Psi, h) > 0$, there exists $y_j \leqslant x \leqslant y_{-1}$, with $\mathcal{M}(x, y_{-1}) = (F, \Psi, h)$. We put $y_{past} = y_j$. The elements y_{max} and y_{fut} can be selected using the very same argument of the proof of Theorem 2 guaranteeing that $0 < y_{fut} < y_{max}$. Next, we take an element $\overline{y} < y_{past}$ such that, for every $y_{past} \leqslant y \leqslant y_{max}$ and every $\psi \in \mathcal{R}eq_l(\mathcal{L}(y, y))$, there exists an element $\overline{y} \leqslant x \leqslant y$ such that $\psi \in \mathcal{L}(x, y)$. We put $y_{min} = \overline{y}$. Let $\mathcal{G}' = \langle \mathbb{P}_\mathbb{D}, \mathcal{L}' \rangle$ be a compass φ-structure such that $D = \{y_{min}, \dots, y_{max}\}$ and, for every $(x, y) \in \mathbb{P}_\mathbb{D}$, $\mathcal{L}'(x, y) = \mathcal{L}(x, y)$. It can be easily checked that $\mathcal{G}'$ is a $\mathbb{Z}$-compass generator for φ.

The right-to-left direction is much more involved with respect to the case of $\mathbb{N}$. We give a sketch of the proof only, making use of the pictorial representation given in Figure 6. Figure 6.a depicts a $\mathbb{Z}$-compass generator $\mathcal{G} = \langle \mathbb{P}_\mathbb{D}, \mathcal{L} \rangle$ for some MPNL formula φ. The vertical segments that will be used to fill in the gaps that will appear during the construction of the infinite prefix are suitably numbered; lowercase letters will be used to identify the vertical segments that will be exploited to fill in the gaps in between 0 and y_{max}; upper case letters identify the marked atoms.

We first define the labeling of points (x, y), with $x \leqslant y \leqslant y_{max}$ (the infinite prefix). To this end, we leave the labeling of points (x, y), with $y_{past} \leqslant x \leqslant y \leqslant y_{max}$, unchanged, and we define the labeling of the other points as follows (in particular, we suitably redefine the labeling of points (x, y), with $y_{min} \leqslant x < y_{past}$ and $y_{min} \leqslant y \leqslant y_{max}$).

Let us first consider the $\Diamond_l$-requests of points (x, x), with $y_{past} \leqslant x \leqslant 0$. By condition 1 of Definition 7, all of them are satisfied in $\mathcal{G}$. We rearrange the structure of $\mathcal{G}$ in order to generate a fulfilling infinite-to-the-past compass φ-structure $\mathcal{G}'$. To give an intuitive account of the construction, suppose that the set of points that satisfy the $\Diamond_l$-requests is included in the set of points belonging to the vertical segments $1, \dots, 5$ of Figure 6.a. By exploiting condition 3 of Definition 7, we generate a sufficient number of copies $\mathcal{T}_1, \dots, \mathcal{T}_n$ of the triangle $\mathcal{T}_0$ (2 copies in Figure 6.b), and we append them one below the other starting from $\mathcal{T}_0$ ($\mathcal{T}_1$ immediateley below $\mathcal{T}_0$, $\mathcal{T}_2$ immediately below $\mathcal{T}_1$, and so on). $\Diamond_l$-requests involving length constraints with $k < k_\varphi$ are satisfied by points belonging to the vertical segments rooted at the right end of the horizontal edge of $\mathcal{T}_1$ only (segments 4, and 5); the other $\Diamond_l$-requests are satisfied by points belonging to the vertical segments rooted at the left end of the horizontal edge of $\mathcal{T}_1$, at $\mathcal{T}_2, \dots$, and at $\mathcal{T}_n$ (segments $1, 2$, and 3). Notice that vertical segments in $\mathcal{G}$ which are sufficiently far way from the diagonal (points (x, y) such that $y - x \geqslant k_\varphi$) are insensitive to $\overset{R}{\longrightarrow}$-preserving changes of the labels of their endpoints (segments $1, 2$, and 3 in Figure 6.c).

Let us consider now points (x, x), with $0 \leqslant x \leqslant y_{max}$, and suppose that the set of points that satisfy their $\Diamond_l$-requests in $\mathcal{G}$ is included in the set of points belonging to the vertical segments a, b, c, d, and e of Figure 6.a. In $\mathcal{G}'$, these $\Diamond_l$-requests are satisfied by (re)introducing the vertical segments a, b, c, d, and e above the appropriate vertical segments $1, 2, 3, 4$, and 5, possibly duplicating some of them (this is the case with a in Figure 6.c). As before, vertical segments in $\mathcal{G}$ which are sufficiently far way from the diagonal (points (x, y) such that $y - x \geqslant k_\varphi$) are insensitive to $\overset{R}{\longrightarrow}$-preserving changes of the labels of their endpoints (segments b and c in Figure 6.c).

The procedure that we applied to fulfill the $\Diamond_l$-requests of points (x, x), with $y_{past} \leqslant x \leqslant 0$, can then be applied to satisfy the $\Diamond_l$-requests of points (x, x), with $2 \cdot y_{past} \leqslant x \leqslant y_{past}$, of points (x, x),

with $3 \cdot y_{past} \leqslant x \leqslant 2 \cdot y_{past}$, and so on, to obtain a correct labeling for all points (x, x) of the infinite prefix.

To complete the labeling of $\mathcal{G}'$, we need to specify the labeling of points (x, y), with $y_{max} < y$ (the infinite suffix). To this end, we apply the procedure of Theorem 2 to y_{fut} and y_{max}. The resulting compass φ-structure $\mathcal{G}' = \langle \mathbb{P}_{\mathbb{Z}}, \mathcal{L}' \rangle$ is a fulfilling compass φ-structure featuring φ. $\qquad\square$

Theorem 5. *Let φ be an MPNL formula. If there exists a $\mathbb{Z}$-compass generator $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ that features φ, then there exists a $\mathbb{Z}$-compass generator $\mathcal{G}' = (\mathbb{P}_{\mathbb{D}'}, \mathcal{L}')$, that features φ, with $|D'| \leqslant \left(2^{3|\varphi|+1} + 4\right) \cdot \left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1\right)^{2^{3|\varphi|}} + 1.$*

Proof. Let $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ be a $\mathbb{Z}$-compass generator, that features φ, and let y_{fut} and $y_{past} \in D$ satisfy the conditions of Definition 7. We define a minimal set $S = \{\overline{y}_0, \ldots, \overline{y}_m\}$ of elements in D such that (i) $\overline{y}_0 = y_{min}$, (ii) $\overline{y}_m = y_{max}$, (iii) $\overline{y}_{m-1} = y_{fut}$, (iv) $\overline{y}_j < \overline{y}_{j+1}$, for each $0 \leqslant j < m$, (v) $\overline{y}_j = 0$, for some $1 < j < m$, (vi) $\overline{y}_{j'} = y_{past}$, for some $1 < j' < j$, (vii) $\overline{y}_{j''} = y_{fut}$, for some $j < j'' < m$, (viii) for every $(F, \Psi, h) \in \mathcal{A}_\varphi^{\mathcal{M}}$, if $\mathcal{C}_{y_{past}}((F, \Psi, h)) > 0$, then there exists $l \leqslant j$ such that $\mathcal{M}(\overline{y}_l, 0) = (F, \Psi, h)$, and (ix) for every $(F, \Psi, h) \in \mathcal{A}_\varphi^{\mathcal{M}}$, if $\mathcal{C}_{y_{fut}}(F, \Psi, h) > 0$, then there exists $\overline{y}_l \leqslant \overline{y}_{fut}$ such that $\mathcal{M}(\overline{y}_l, y_{fut}) = (F, \Psi, h)$. From the minimality requirement, it follows that $m \leqslant 2^{3|\varphi|+1} + 5$.

We build a finite sequence of $\mathbb{Z}$-compass generators $\mathcal{G}_0 \supset \mathcal{G}_1 \supset \ldots \supset \mathcal{G}_n$, whose last element is a small enough $\mathbb{Z}$-compass generator $\mathcal{G}_n$. We start with $\mathcal{G}_0 = \mathcal{G}$. Now, let $\mathcal{G}_i = (\mathbb{P}_{\mathbb{D}_i}, \mathcal{L}_i)$ be the i-th compass generator in the sequence and let $S_i = \{\overline{y}_0, \ldots, \overline{y}_m\}$ be the above-defined minimal set of elements in D_i. If there exist no y, y', with $\overline{y}_j \leqslant y < y' < \overline{y}_{j+1}$ for some $0 \leqslant j < m$, such that $\mathcal{C}_y \equiv \mathcal{C}_{y'}$, we put $n = i$, and we end the construction. Otherwise, as in Theorem 3, we apply (the construction of) Lemma 1 to y and y' to obtain a compass generator $\mathcal{G}_{i+1} = (\mathbb{P}_{\mathbb{D}_{i+1}}, \mathcal{L}_{i+1})$, with $|D_{i+1}| = |D_i| - (y' - y)$.

At the end of the procedure, all the horizontal configurations in between two consecutive elements $\overline{y}_j, \overline{y}_{j+1} \in S$ are pairwise non-equivalent. From this, it immediately follows that the final $\mathbb{Z}$-compass generator $\mathcal{G}_n = (\mathbb{P}_{\mathbb{D}_n}, \mathcal{L}_n)$ is such that $|D_n| \leqslant \left(2^{3|\varphi|+1} + 4\right) \cdot \left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1\right)^{2^{3|\varphi|}} + 1$. $\qquad\square$

7   An EXPSPACE decision procedure

In this section, we describe a decision procedure that solves the satisfiability problem for MPNL over the integer numbers. Both the procedure for the finite case and that for the natural numbers can be easily tailored from it. Let φ be the MPNL formula to check for satisfiability. In order to establish whether or not there exists a finite model satisfying φ, we can proceed as follows. First, we represent a finite model in $\mathbb{Z}$ by means of the following formula:

$$\begin{aligned}
\psi_{fin} = \#_{all} \wedge \# \wedge \Box_r \Box_l (\neg \pi \wedge \Box_l \neg \#_{all}) \wedge \Box_l \Box_r (\neg \pi \wedge \Box_r \neg \#_{all}) \wedge \\
\Box_r (\neg \pi \wedge \Box_r \Box_l \neg \#_{all}) \wedge \Box_l (\neg \pi \wedge \Box_l \Box_r \neg \#_{all}) \wedge \\
[G](\# \leftrightarrow (\#_{all} \vee \Diamond_r \Diamond_l \#_{all} \vee \Diamond_l \Diamond_r \#_{all} \vee (\Diamond_r \Diamond_r \Diamond_l \#_{all} \wedge \Diamond_l \Diamond_l \Diamond_r \#_{all}))),
\end{aligned}$$

where $[G]$ is the commonly-used universal modality [13], $\#_{all}$ holds over one and one interval that collects all points of the finite model and $\#$ holds over all and only the subintervals of such a $\#_{all}$-interval.

Under the assumption that $\#_{all}$ and $\#$ do not appear in φ, we can replace φ by a formula $tr(\varphi)$ such that φ has a finite model if and only if $\psi_{fin} \wedge \Diamond_l \Diamond_r \Diamond_r tr(\varphi)$ has a model in $\mathbb{Z}$. The formula $tr(\varphi)$ is inductively defined as follows: (i) if $\varphi = p$ or $\varphi = len_{<k}$, then $tr(\varphi) = \varphi \wedge \#$, (ii) if $\varphi = \neg \psi$, then

```
proc GUESSCONFIGURATION()
  for all (F, Ψ, h) ∈ 𝒜_φ^𝓜, 𝒞(F, Ψ, h) ← 0;
  let S_r ⊆ {ψ ∈ 𝒞l(φ | ◇_r ψ ∈ 𝒞l(φ))};
  let S_l ⊆ {ψ ∈ 𝒞l(φ | ◇_l ψ ∈ 𝒞l(φ))};
  for all 1 ≤ i < k_φ
    let F an atom s.t. ℛeq_r(F) = S_r and Len(F) = i;
    let Ψ ⊆ {ψ ∈ 𝒞l(φ) | ◇_r ψ ∈ 𝒞l(φ))};
    𝒞̄(F, Ψ, i) ← 1;
  for all (F, Ψ, k_φ) ∈ 𝒜_φ^𝓜 s.t. ℛeq_r(F) = S_r
    let 0 ≤ i ≤ k_φ,  𝒞̄(F, Ψ, h) ← i
  return 𝒞̄;

proc MERGE(𝒞, 𝒞')
  for all (F, Ψ, h) ∈ 𝒜_φ^𝓜
    {𝒞̄(F, Ψ, h) ← 𝒞(F, Ψ, h) + 𝒞'(F, Ψ, h);
  return 𝒞̄;

proc LEN(F)
  if ∃1 ≤ h < k_φ s.t. ¬len_<h ∈ F ∧ len_<h+1 ∈ F
    then return h
    else return k_φ
```

```
proc MA_SET(𝒞)
  R = {(F, Ψ, h) | 𝒞(F, Ψ, h) > 0};
  return R;

proc NC_ZEROTOFUT(𝒞^current)
  let S_r ⊆ {ψ ∈ 𝒞l(φ | ◇_r ψ ∈ 𝒞l(φ))};
  let S_l ⊆ {ψ ∈ 𝒞l(φ | ◇_l ψ ∈ 𝒞l(φ))};
  let F_π an atom with len_<1 ∈ F_π, ℛeq_r(F_π) = S_r,
        and ℛeq_l(F_π) = S_l;
  for all (F, Ψ, h) ∈ 𝒜_φ^𝓜, 𝒞(F, Ψ, h) ← 0;
  𝒞(F_π, ℛeq_r(F) \ F_π, 1) ← 1;
  for all (F, Ψ, h) ∈ 𝒜_φ^𝓜
    for (1 ≤ i ≤ 𝒞^current(G, Ψ, h))
      if h = k_φ
        then k ← k_φ
        else k ← h+1
      let G s.t. Len(G) = k, ℛeq_r(G) = S_r,
            and ℛeq_l(G) = ℛeq_l(F);
      𝒞(G, Ψ \ G, k) ← 𝒞(G, Ψ \ G, k) + 1;
  if ( ∃ψ ∈ S_l s. t. ∀(F, Ψ, h) ∈ 𝒜_φ^𝓜 with ψ ∈ A
       we have 𝒞(F, Ψ, h) = 0 )
    then return false ;
  return 𝒞;
```

Figure 3: Auxiliary procedures for checking the satisfiability of φ over the integers.

$tr(\varphi) = \neg\# \vee \neg tr(\psi)$, (iii) if $\varphi = \psi_1 \vee \psi_2$, then $tr(\varphi) = (\psi_1 \wedge \#) \vee (\psi_2 \wedge \#)$, (iv) if $\varphi = \Diamond_r \psi$, then $tr(\varphi) = \Diamond_r(\# \wedge \psi)$, (v) if $\varphi = \Diamond_l \psi$, then $tr(\varphi) = \Diamond_l(\# \wedge \psi)$.

Similarly, it is possible to prove that an MPNL formula φ has a model over the linear order of natural numbers if and only if $\psi_{nat} \wedge \Diamond_l \Diamond_r \Diamond_r tr(\varphi)$ has a model in $\mathbb{Z}$, where $tr(\varphi)$ is defined as above and ψ_{nat} is defined as follows:

$$\psi_{nat} = \# \wedge \Box_l \neg\# \wedge \Box_l \Box_l \neg\# \wedge [G]((\neg\# \wedge \Diamond_r \#) \rightarrow (\Box_r \# \wedge \Box_r \Box_r \#))$$

The detailed code of the decision procedure is reported in Figure 7. It builds a tentative $\mathbb{Z}$-compass generator for φ starting from y_{min} and exploring two consecutive horizontal configurations at every step. Every configuration is represented using an exponential number of counters, bounded by the maximum size for a $\mathbb{Z}$-compass generator given in Theorem 5 (doubly exponential in the size of $|\varphi|$). However, assuming that the values of all counters are encoded in binary, the maximum value for each counter takes an exponential storage space. The very same argument can be used to provide an exponential space bound for the $steps$ counter. Moreover, the procedure needs to keep track of a constant number of horizontal configurations only ($\mathcal{C}^{min}, \mathcal{C}^{past}, \mathcal{C}^0, \mathcal{C}^{fut}, \mathcal{C}^{max}, \overline{\mathcal{C}}, \mathcal{C}, \mathcal{C}', \mathcal{C}^{right}$, and $\mathcal{C}^{left}$). Pairing this result with the EXPSPACE-hardness given in [4], we can state the following theorem.

Theorem 6. *The satisfiability problem for MPNL, interpreted over (any subsets of) the integers is EXP-SPACE-complete.*

Acknowledgements

proc NC_MINTOPAST$(\mathcal{C}^{current})$
$\quad$**let** $S_r \subseteq \{\psi \in \mathcal{Cl}(\varphi \mid \Diamond_r\psi \in \mathcal{Cl}(\varphi))\}$;
$\quad$**let** $S_l \subseteq \{\psi \in \mathcal{Cl}(\varphi \mid \Diamond_l\psi \in \mathcal{Cl}(\varphi))\}$;
$\quad$**let** F_π **an atom with** $len_{<1} \in F_\pi$, $\mathcal{Req}_r(F_\pi) = S_r$, **and** $\mathcal{Req}_l(F_\pi) = S_l$;
$\quad$**for all** $(F,\Psi,h) \in \mathcal{A}_\varphi^{\mathcal{M}}\,\mathcal{C}(F,\Psi,h) \leftarrow 0$;
$\quad$$\mathcal{C}(F_\pi, \mathcal{Req}_r(F) \setminus F_\pi, 1) \leftarrow 1$;
$\quad$**for all** $(F,\Psi,h) \in \mathcal{A}_\varphi^{\mathcal{M}}$
$\quad\quad$**for** $(1 \leqslant i \leqslant \mathcal{C}^{current}(G,\Psi,h))$
$\quad\quad\quad$**if** $h = k_\varphi$
$\quad\quad\quad\quad$**then** $k \leftarrow k_\varphi$
$\quad\quad\quad\quad$**else** $k \leftarrow h+1$
$\quad\quad\quad$**let** G **s.t.** $Len(G) = k$, $\mathcal{Req}_r(G) = S_r$, **and** $\mathcal{Req}_l(G) = \mathcal{Req}_l(F)$;
$\quad\quad\quad$$\mathcal{C}(G,\Psi \setminus G, k) \leftarrow \mathcal{C}'(G,\Psi \setminus G, k) + 1$;
$\quad$**return** $\mathcal{C}$;

proc NC_LEFTRIGHT$(\mathcal{C}^{left}, \mathcal{C}^{right})$
$\quad$**let** $S_r \subseteq \{\psi \in \mathcal{Cl}(\varphi \mid \Diamond_r\psi \in \mathcal{Cl}(\varphi))\}$;
$\quad$**let** $S_l \subseteq \{\psi \in \mathcal{Cl}(\varphi \mid \Diamond_l\psi \in \mathcal{Cl}(\varphi))\}$;
$\quad$**let** F_π **an atom with** $len_{<1} \in F_\pi$, $\mathcal{Req}_r(F_\pi) = S_r$, **and** $\mathcal{Req}_l(F_\pi) = S_l$;
$\quad$**for all** $(F,\Psi,h) \in \mathcal{A}_\varphi^{\mathcal{M}}\,\overline{\mathcal{C}}^{right}(F,\Psi,h) \leftarrow 0$;
$\quad$**for all** $(F,\Psi,h) \in \mathcal{A}_\varphi^{\mathcal{M}}\,\overline{\mathcal{C}}^{left}(F,\Psi,h) \leftarrow 0$;
$\quad$$\overline{\mathcal{C}}^{right}(F_\pi, \mathcal{Req}_r(F) \setminus F_\pi, 1) \leftarrow 1$;
$\quad$**for all** $(F,\Psi,h) \in \mathcal{A}_\varphi^{\mathcal{M}}$
$\quad\quad$**for** $(1 \leqslant i \leqslant \mathcal{C}^{right}(G,\Psi,h))$
$\quad\quad\quad$**if** $h = k_\varphi$
$\quad\quad\quad\quad$**then** $k \leftarrow k_\varphi$
$\quad\quad\quad\quad$**else** $k \leftarrow h+1$
$\quad\quad\quad$**let** G **s.t.** $Len(G) = k$, $\mathcal{Req}_r(G) = S_r$, **and** $\mathcal{Req}_l(G) = \mathcal{Req}_l(F)$;
$\quad\quad\quad$$\overline{\mathcal{C}}^{right}(G,\Psi \setminus G, k) \leftarrow \overline{\mathcal{C}}^{right}(G,\Psi \setminus G, k) + 1$;
$\quad\quad$**for** $(1 \leqslant i \leqslant \mathcal{C}^{left}(G,\Psi,h))$
$\quad\quad\quad$**if** $h = k_\varphi$
$\quad\quad\quad\quad$**then** $k \leftarrow k_\varphi$
$\quad\quad\quad\quad$**else** $k \leftarrow h+1$
$\quad\quad\quad$**let** G **s.t.** $Len(G) = k$, $\mathcal{Req}_r(G) = S_r$, **and** $\mathcal{Req}_l(G) = \mathcal{Req}_l(F)$;
$\quad\quad\quad$$\overline{\mathcal{C}}^{left}(G,\Psi \setminus G, k) \leftarrow \overline{\mathcal{C}}^{left}(G,\Psi \setminus G, k) + 1$;
$\quad$**if** $\left(\exists \psi \in S_l \text{ s. t. } \forall (F,\Psi,h) \in \mathcal{A}_\varphi^{\mathcal{M}} \text{ with } \psi \in A \text{ we have } \overline{\mathcal{C}}^{left}(F,\Psi,h) = \overline{\mathcal{C}}^{right}(F,\Psi,h) = 0 \right)$
$\quad\quad$**then return false** ;
$\quad$**return** $(\overline{\mathcal{C}}^{left}, \overline{\mathcal{C}}^{right})$;

proc MPNL-INTEGER-SAT(φ)
$\quad$$BOUND \leftarrow \left(2^{3|\varphi|+1} + 4\right) \cdot \left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1\right)^{2^{3|\varphi|}}$;
$\quad$**let** $S_r \subseteq \{\psi \in \mathcal{Cl}(\varphi \mid \Diamond_r\psi \in \mathcal{Cl}(\varphi))\}$;
$\quad$**let** $S_l \subseteq \{\psi \in \mathcal{Cl}(\varphi \mid \Diamond_l\psi \in \mathcal{Cl}(\varphi))\}$;
$\quad$**let** F_π **an atom with** $len_{<1} \in F_\pi$, $\mathcal{Req}_r(F_\pi) = S_r$, **and** $\mathcal{Req}_l(F_\pi) = S_l$;
$\quad$**for all** $(F,\Psi,h) \in \mathcal{A}_\varphi^{\mathcal{M}}$, $\mathcal{C}^{min}(F,\Psi,h) \leftarrow 0$;
$\quad$$\mathcal{C}^{min}(F_\pi, \mathcal{Req}_r(F) \setminus F_\pi, 1) \leftarrow 1$;
$\quad$$\mathcal{C}^{past} \leftarrow GuessConfiguration()$;
$\quad$$\mathcal{C} \leftarrow \mathcal{C}^{min}$;
$\quad$$steps \leftarrow 0$;
$\quad$**while** $(\mathcal{C} \not\equiv \mathcal{C}^{past} \vee steps < k_\varphi)$
$\quad\quad$**if** $steps > BOUND$
$\quad\quad\quad$**then return false**
$\quad\quad$$\mathcal{C} \leftarrow NC_MinToPast(\mathcal{C})$;
$\quad\quad$$steps \leftarrow steps + 1$;
$\quad$$\mathcal{C}^{left} \leftarrow \mathcal{C}$;
$\quad$**for all** $(F,\Psi,h) \in \mathcal{A}_\varphi^{\mathcal{M}}$, $\mathcal{C}^{right}(F,\Psi,h) \leftarrow 0$;
$\quad$$steps \leftarrow 0$;
$\quad$**while** $(MA\_set(\mathcal{C}^{past}) \not\subseteq MA\_set(\mathcal{C}^{right}))$
$\quad\quad$**if** $steps > BOUND$
$\quad\quad\quad$**then return false**
$\quad\quad$$(\mathcal{C}^{left}, \mathcal{C}^{right}) \leftarrow NC_LeftRight(\mathcal{C}^{left}, \mathcal{C}^{right})$;
$\quad\quad$$steps \leftarrow steps + 1$;
$\quad$$\mathcal{C}^{fut} \leftarrow GuessConfiguration()$;
$\quad$$\mathcal{C} \leftarrow Merge(\mathcal{C}^{left}, \mathcal{C}^{right})$;
$\quad$$steps \leftarrow 0$;
$\quad$**while** $(\mathcal{C} \not\equiv \mathcal{C}^{fut})$
$\quad\quad$**if** $steps > BOUND$
$\quad\quad\quad$**then return false**
$\quad\quad$$\mathcal{C} \leftarrow NC\_ZeroToFut(\mathcal{C})$;
$\quad\quad$$steps \leftarrow steps + 1$;
$\quad$$\mathcal{C}^{max} \leftarrow \mathcal{C}$;
$\quad$$\mathcal{C}^{left} \leftarrow \mathcal{C}$;
$\quad$**for all** $(F,\Psi,h) \in \mathcal{A}_\varphi^{\mathcal{M}}$, $\mathcal{C}^{right}(F,\Psi,h) \leftarrow 0$;
$\quad$$steps \leftarrow 0$;
$\quad$**while** $\left(\begin{array}{l} (MA\_set(Merge(\mathcal{C}^{left}, \mathcal{C}^{right})) \\ \supseteq MA\_set(\mathcal{C}^{max}) \to \exists (F,\Psi,h) \in \mathcal{A}_\varphi^{\mathcal{M}} \\ \text{ with } \mathcal{C}^{left}(F,\Psi,h) > 0 \wedge \Psi \neq \emptyset) \\ \vee steps > k_\varphi \end{array} \right)$
$\quad\quad$**if** $steps > BOUND$
$\quad\quad\quad$**then return false**
$\quad\quad$$(\mathcal{C}^{left}, \mathcal{C}^{right}) \leftarrow NC_LeftRight(\mathcal{C}^{left}, \mathcal{C}^{right})$;
$\quad\quad$$steps \leftarrow steps + 1$;
$\quad$**return true** ;

Figure 4: The procedure for checking the satisfiability of φ over the integers.

References

[1] J.F. Allen (1983): *Maintaining Knowledge About Temporal Intervals*. Communications of the Association for Computing Machinery 26(11), pp. 832–843, doi:`10.1145/182.358434`.

[2] D. Bresolin, D. Della Monica, V. Goranko, A. Montanari & G. Sciavicco (2011): *Metric Propositional Neighborhood Logics on Natural Numbers*. Journal of Software and System Modeling (in press) doi:`10.1007/s10270-011-0195-y`.

[3] D. Bresolin, V. Goranko, A. Montanari & G. Sciavicco (2009): *Propositional Interval Neighborhood Logics: Expressiveness, Decidability, and Undecidable Extensions*. Annals of Pure and Applied Logic 161(3), pp. 289–304, doi:`10.1016/j.apal.2009.07.003`.

[4] D. Bresolin, V. Goranko, A. Montanari & G. Sciavicco (2009): *Right Propositional Neighborhood Logic over Natural Numbers with Integer Constraints for Interval Lengths*. In: *Proc. of the 7th IEEE International Conference on Software Engineering and Formal Methods (SEFM)*, IEEE Comp. Society Press, pp. 240–249, doi:`10.1109/SEFM.2009.36`.

[5] D. Bresolin, A. Montanari & P. Sala (2007): *An optimal tableau-based decision algorithm for Propositional Neighborhood Logic*. In: *Proc. of the 24th Annual Symposium on Theoretical Aspects of Computer Science (STACS)*, LNCS 4393, Springer, pp. 549–560, doi:`10.1007978354070918347`.

[6] D. Bresolin, A. Montanari, P. Sala & G. Sciavicco (2008): *Optimal Tableaux for Right Propositional Neighborhood Logic over Linear Orders*. In: *Proc. of the 11th European Conference on Logics in AI (JELIA)*, LNAI 5293, Springer, pp. 62–75, doi:`10.100797835408780327`.

[7] D. Bresolin, A. Montanari, P. Sala & G. Sciavicco (2010): *Tableau-based decision procedures for Propositional Neighborhood Logic*. Technical Report 01, Dipartimento di Matematica e Informatica, Università di Udine, Italy.

[8] D. Bresolin, A. Montanari & G. Sciavicco (2007): *An optimal decision procedure for Right Propositional Neighborhood Logic*. Journal of Automated Reasoning 38(1-3), pp. 173–199, doi:`10.1007s1081700690510`.

[9] D. Della Monica (2011): *Expressiveness, Decidability, and Undecidability of Interval Temporal Logic*. Ph.D. thesis, Department of Mathematics and Computer Science, University of Udine, Udine, Italy.

[10] V. Goranko, A. Montanari & G. Sciavicco (2003): *Propositional interval neighborhood temporal logics*. Journal of Universal Computer Science 9(9), pp. 1137–1167, doi:`10.3217/jucs-009-09-1137`.

[11] V. Goranko, A. Montanari & G. Sciavicco (2004): *A road map of interval temporal logics and duration calculi*. Journal of Applied Non-Classical Logics 14(1–2), pp. 9–54, doi:`10.3166/jancl.14.9-54`.

[12] M. Otto (2001): *Two Variable First-Order Logic over Ordered Domains*. Journal of Symbolic Logic 66(2), pp. 685–702, doi:`10.2307/2695037`.

[13] P. Sala (2010): *Decidability of Interval Temporal Logics*. Ph.D. thesis, Department of Mathematics and Computer Science, University of Udine, Udine, Italy.

[14] Y. Venema (1990): *Expressiveness and Completeness of an Interval Tense Logic*. Notre Dame Journal of Formal Logic 31(4), pp. 529–547, doi:`10.1305/ndjfl/1093635589`.

[15] Y. Venema (1991): *A Modal Logic for Chopping Intervals*. Journal of Logic and Computation 1(4), pp. 453–476, doi:`10.1093/logcom/1.4.453`.