Interval Temporal Logics over Strongly Discrete Linear Orders: the Complete Picture

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Interval temporal logics provide a general framework for temporal reasoning about interval structures over linearly ordered domains, where intervals are taken as the primitive ontological entities. In this paper, we identify all fragments of Halpern and Shoham's interval temporal logic HS with a decidable satisfiability problem over the class of strongly discrete linear orders. We classify them in terms of both their relative expressive power and their complexity. We show that there are exactly 44 expressively different decidable fragments, whose complexity ranges from NP to EXPSPACE. In addition, we identify some new undecidable fragments (all the remaining HS fragments were already known to be undecidable over strongly discrete linear orders). We conclude the paper by an analysis of the specific case of natural numbers, whose behavior slightly differs from that of the whole class of strongly discrete linear orders. The number of decidable fragments over $\mathbb{N}$ raises up to 47: three undecidable fragments become decidable with a non-primitive recursive complexity.

1 Introduction

Interval temporal logics provide a general framework for temporal reasoning about interval structures over linearly (or partially) ordered domains. They take time intervals as the primitive ontological entities and define truth of formulas relative to time intervals, rather than time points. Interval logic modalities correspond to various relations between pairs of intervals, with the exception of Venema's CDT and its fragments, that consider ternary relations [22]. In particular, Halpern and Shoham's modal logic of time intervals HS [15] features a set of modalities that makes it possible to express all Allen's interval relations [1] (see Table 1).

Interval-based formalisms have been extensively used in many areas of computer science, such as, for instance, planning, natural language processing, constraint satisfaction, and verification of hardware and software systems. However, most of them impose severe syntactic and semantic restrictions that considerably weaken their expressive power. Interval temporal logics relax these restrictions, allowing one to cope with much more complex application domains and scenarios. Unfortunately, many of them, including HS and the majority of its fragments, turn out to be undecidable [4].

In this paper, we focus our attention on the class of strongly discrete linear orders, that is, of those linear structures characterized by the presence of finitely many points in between any two points. This class includes, for instance, $\mathbb{N}$, $\mathbb{Z}$, and all finite linear orders. We give a complete classification of all HS fragments (defined by restricting the set of modalities), reviewing known results and solving open problems; the results differ, as we will see, from those in the class of all finite linearly ordered sets [7]. The aim of such a classification is twofold: on the one hand, we identify the subset of all expressively-different decidable fragments, thus marking the decidability border; on the other hand, we determine

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Relation	Operator	Formal definition	Pictorial example
meets	$\langle A\rangle$	$[x,y]R_A[x',y'] \Leftrightarrow y = x'$	
before	$\langle L\rangle$	$[x,y]R_L[x',y'] \Leftrightarrow y < x'$	
started-by	$\langle B\rangle$	$[x,y]R_B[x',y'] \Leftrightarrow x = x', y' < y$	
finished-by	$\langle E\rangle$	$[x,y]R_E[x',y'] \Leftrightarrow y = y', x < x'$	
contains	$\langle D\rangle$	$[x,y]R_D[x',y'] \Leftrightarrow x < x', y' < y$	
overlaps	$\langle O\rangle$	$[x,y]R_O[x',y'] \Leftrightarrow x < x' < y < y'$	

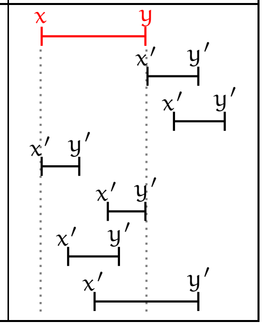

Table 1: Allen's interval relations and the corresponding HS modalities.

the exact complexity of each of them. As shown in Figure 1, $A\overline{A}B\overline{B}$ (that features modal operators for Allen's relations *meets* and *started-by*, and their inverses) and its mirror image $A\overline{A}E\overline{E}$ (that replaces relations *starts* and *started-by* by relations *finishes* and *finished-by*) are the minimal fragments including all decidable subsets of operators from the HS repository, for a total of 62 languages. Of those, 44 turn out to be decidable. As a matter of fact, the status of various fragments was already known: *(i)* D, $\overline{D}$, O, and $\overline{O}$ have been shown to be undecidable in [6, 16]; *(ii)* BE, $B\overline{E}$, $\overline{B}E$, and $\overline{BE}$ are undecidable, as they can define, respectively, $\langle D\rangle$ (by the equation $\langle D\rangle p \equiv \langle B\rangle\langle E\rangle p$), $\langle\overline{O}\rangle$ ($\langle\overline{O}\rangle p \equiv \langle B\rangle\langle\overline{E}\rangle p$), $\langle O\rangle$ ($\langle O\rangle p \equiv \langle E\rangle\langle\overline{B}\rangle p$), and $\langle\overline{D}\rangle$ ($\langle\overline{D}\rangle p \equiv \langle\overline{B}\rangle\langle\overline{E}\rangle p$); *(iii)* undecidability of $A\overline{AB}$ (resp., $A\overline{AE}$) can be shown using the same technique used in [18] to prove the undecidability of $A\overline{A}B$ (resp., $A\overline{A}E$); *(iv)* $AB\overline{BL}$ (resp., $\overline{A}E\overline{E}L$) is in EXPSPACE [10], and the proof of EXPSPACE-hardness for AB and $A\overline{B}$ (resp., $\overline{A}E$ and $\overline{AE}$) over finite linear orders [7] can be easily adapted to the case of strongly discrete linear orders; *(v)* $A\overline{A}$ (a.k.a. Propositional Neighborhood Logic) is in NEXPTIME [8, 13], and NEXPTIME-hardness already holds for A and $\overline{A}$ [9]; *(vi)* $B\overline{B}$ is NP-complete [14], and, obviously, NP-hardness already holds for B and $\overline{B}$ (both include propositional logic); *(vii)* the relative expressive power of the HS fragments we are interested in is as shown in Figure 1, whose soundness and completeness follow from the results given in [11] and in [7], respectively, as definability (resp., undefinability) results transfer from more (resp., less) general to less (resp., more) general classes.

In this paper, we complete the picture by proving the following new results: *(i)* the undecidability of $A\overline{A}B$ (resp., $A\overline{A}E$) and $A\overline{AB}$ (resp., $A\overline{AE}$) can be sharpened to $\overline{A}B$ (resp., AE) and $\overline{AB}$ (resp., $A\overline{E}$), respectively (Section 3); *(ii)* the NP-completeness (in particular, NP-membership) of $B\overline{B}$ can be extended to $B\overline{B}L\overline{L}$ (Section 4). In addition, we analyze the behavior of the various fragments over interesting subclasses of the class of all strongly discrete linearly ordered sets, taking as an example that of models based on $\mathbb{N}$ (Section 6). As $\mathbb{N}$-models are not left/right symmetric, reversing the time order and coherently replacing modalities (e.g., $\langle A\rangle$ by $\langle\overline{A}\rangle$) does not preserve, in general, the computational properties of a fragment. We show that: *(i)* $\overline{A}B$ becomes decidable (which is a direct consequence of [18]), precisely, non-primitive recursive [7]; *(ii)* the same holds for $\overline{AB}$ and $\overline{A}B\overline{B}$, but, in these cases, the decidability proof for $A\overline{A}B\overline{B}$ given in [18] must be suitably adapted; *(iii)* $\overline{A}BL$, $\overline{AB}L$, and $\overline{A}B\overline{B}L$ remain undecidable, but the original reductions must be suitably adapted. Thus, the number of decidable fragments over $\mathbb{N}$ raises up to 47, the three new decidable fragments being all non-primitive recursive. In fact, we can slightly generalize such a result, as the addition of finite linear orders (finite prefixes of $\mathbb{N}$) to $\mathbb{N}$ does not alter the picture; however, to keep presentation and proofs as simple as possible, we restrict our attention to $\mathbb{N}$-models only. Symmetric results can be obtained in the case of negative integers.

2 HS and its Fragments

Let $\mathbb{D} = \langle D, < \rangle$ be a *strongly discrete linearly ordered set*, that is, a linearly ordered set where for every pair x, y, with $x < y$, there exist at most finitely many $z_1, z_2, \ldots, z_n$ such that $x < z_1 < z_2 < \ldots < z_n < y$. According to the strict approach, we exclude intervals with coincident endpoints (point-intervals) from the semantics: an *interval* over $\mathbb{D}$ is an ordered pair $[x, y]$, with $x, y \in D$ and $x < y$.

12 different ordering relations (plus equality) between any pair of intervals are possible, often called *Allen's relations* [1]: the six relations depicted in Table 1 and their inverses. We interpret interval structures as Kripke structures and Allen's relations as accessibility relations, thus associating a modality $\langle X \rangle$ with each Allen's relation R_X. For each modality $\langle X \rangle$, its *inverse* (or *transpose*), denoted by $\langle \overline{X} \rangle$, corresponds to the inverse relation $R_{\overline{X}}$ of R_X (that is, $R_{\overline{X}} = (R_X)^{-1}$). Halpern and Shoham's logic HS is a multi-modal logic whose formulas are built on a set $\mathcal{AP}$ of proposition letters, the boolean connectives $\vee$ and $\neg$, and one modality for each Allen's relation. We associate a fragment $X_1 X_2 \ldots X_k$ of HS with every subset $\{R_{X_1}, \ldots, R_{X_k}\}$ of Allen's relations, whose formulas are defined by the following grammar:

$$\varphi ::= p \mid \neg \varphi \mid \varphi \vee \varphi \mid \langle X_1 \rangle \varphi \mid \ldots \mid \langle X_k \rangle \varphi.$$

The other boolean connectives can be viewed as abbreviations, and the dual operators $[X]$ are defined as usual ($[X]\varphi \equiv \neg\langle X \rangle \neg \varphi$). Given a formula φ, its *length* $|\varphi|$ is the number of its symbols.

The semantics of HS is given in terms of *interval models* $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$, where $\mathbb{I}(\mathbb{D})$ is the set of all intervals over $\mathbb{D}$. The *valuation function* $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$ assigns to every $p \in \mathcal{AP}$ the set of intervals $V(p)$ over which p holds. The *truth* of a formula over a given interval $[x, y]$ of an interval model M is defined by structural induction on formulas:

- $M, [x, y] \Vdash p$ iff $[x, y] \in V(p)$, for all $p \in \mathcal{AP}$;
- $M, [x, y] \Vdash \neg\psi$ iff it is not the case that $M, [x, y] \Vdash \psi$;
- $M, [x, y] \Vdash \varphi \vee \psi$ iff $M, [x, y] \Vdash \varphi$ or $M, [x, y] \Vdash \psi$;
- $M, [x, y] \Vdash \langle X \rangle \psi$ iff there exists an interval $[x', y']$ such that $[x, y] R_X [x', y']$ and $M, [x', y'] \Vdash \psi$, where R_X is the relation corresponding to $\langle X \rangle$.

An HS-formula ϕ is *valid*, denoted by $\Vdash \phi$, if it is true over every interval of every interval model.

In this paper, we study expressiveness and computational complexity of HS fragments over the class of strongly discrete linear orders. Given a fragment $\mathcal{F} = X_1 X_2 \ldots X_k$ and a modality $\langle X \rangle$, we write $\langle X \rangle \in \mathcal{F}$ if $X \in \{X_1, \ldots, X_k\}$. Given two fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if $\langle X \rangle \in \mathcal{F}_1$ implies $\langle X \rangle \in \mathcal{F}_2$, for every modality $\langle X \rangle$.

Definition 1. *We say that an* HS *modality* $\langle X \rangle$ *is* definable *in an* HS *fragment* $\mathcal{F}$ *if there exists a formula* $\psi(p) \in \mathcal{F}$ *such that* $\langle X \rangle p \leftrightarrow \psi(p)$ *is valid, for any fixed proposition letter* p*. In such a case, the equivalence* $\langle X \rangle p \equiv \psi(p)$ *is called an* inter-definability equation for $\langle X \rangle$ in $\mathcal{F}$.

Definition 2. *Let* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *be two* HS *fragments. We say that (i)* $\mathcal{F}_2$ *is* at least as expressive as $\mathcal{F}_1$ *(*$\mathcal{F}_1 \preceq \mathcal{F}_2$*) if every modality* $\langle X \rangle \in \mathcal{F}_1$ *is definable in* $\mathcal{F}_2$*; (ii)* $\mathcal{F}_1$ *is* strictly less expressive *than* $\mathcal{F}_2$*, (*$\mathcal{F}_1 \prec \mathcal{F}_2$*) if* $\mathcal{F}_1 \preceq \mathcal{F}_2$*, but not* $\mathcal{F}_2 \preceq \mathcal{F}_1$*; (iii)* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are* equally expressive*, or* expressively equivalent *(*$\mathcal{F}_1 \equiv \mathcal{F}_2$*), if* $\mathcal{F}_1 \preceq \mathcal{F}_2$ *and* $\mathcal{F}_2 \preceq \mathcal{F}_1$*; (iv)* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are* expressively incomparable *(*$\mathcal{F}_1 \not\equiv \mathcal{F}_2$*) if neither* $\mathcal{F}_1 \preceq \mathcal{F}_2$ *nor* $\mathcal{F}_2 \preceq \mathcal{F}_1$*.*

We denote each HS fragment $\mathcal{F}$ by the list of its modalities in alphabetical order, omitting those modalities which are definable in terms of the others. As a matter of fact, in our setting, only $\langle L \rangle$ and $\langle \overline{L} \rangle$ turn out to be definable in some fragments. Any fragment $\mathcal{F}$ can be transformed into its mirror image by reversing the time order and simultaneously replacing (each occurrence of) $\langle A \rangle$ by $\langle \overline{A} \rangle$, $\langle L \rangle$ by $\langle \overline{L} \rangle$, $\langle B \rangle$

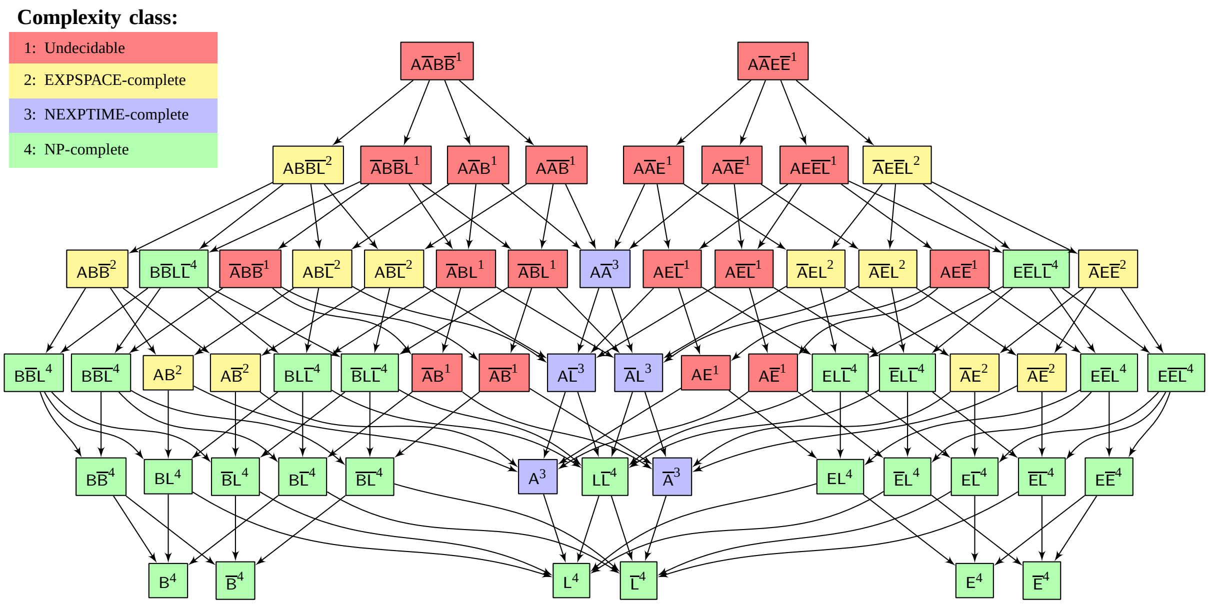

Figure 1: Hasse diagram of fragments of $A\overline{A}B\overline{B}$ and $A\overline{A}E\overline{E}$ over strongly discrete linear orders.

by $\langle E\rangle$, and $\langle \overline{B}\rangle$ by $\langle \overline{E}\rangle$. In the considered class of linear orders, the mirroring operation can be applied to any fragment preserving all its computational properties. Thus, all results given in this paper, except for the ones in Section 6, hold both for the considered fragments and their mirror images. When the considered class of models is not left/right symmetric, as it happens with $\mathbb{N}$ (Section 6), this is no longer true. The rest of the paper, with the exception of Section 6, is devoted to prove the following theorem.

Theorem 1. *The Hasse diagram in Figure 1 correctly shows all the decidable fragments of* HS *over the class of strongly discrete linear orders, their relative expressive power, and the precise complexity class of their satisfiability problem.*

3 Relative Expressive Power and Undecidability

The most basic definability results in HS, e.g., HS $\equiv A\overline{A}B\overline{B}E\overline{E}$, are known since [15]. In order to show that a given modality is not definable in a specific HS fragment, we make use of the standard notion of bisimulation and the invariance of modal formulas with respect to bisimulations (see, e.g., [2]). In particular, we exploit the fact that, given a modal logic $\mathcal{F}$, any $\mathcal{F}$-bisimulation preserves the truth of all formulas in $\mathcal{F}$. Thus, in order to prove that a modality $\langle X\rangle$ is not definable in $\mathcal{F}$, it suffices to construct a pair of interval models M and M' and an $\mathcal{F}$-bisimulation between them that relates a pair of intervals $[x,y] \in M$ and $[x',y'] \in M'$ such that $M,[x,y] \Vdash \langle X\rangle p$ and $M',[x',y'] \nVdash \langle X\rangle p$.

In the following, in order to prove that Figure 1 is sound and complete for the class of all strongly discrete linear orders, we focus our attention on fragments of $A\overline{A}B\overline{B}$ and of its mirror image $A\overline{A}E\overline{E}$, and we show that the set of nodes of the graph in Figure 1 is the set of all expressively different fragments of $A\overline{A}B\overline{B}$ and $A\overline{A}E\overline{E}$ (including $A\overline{A}B\overline{B}$ and $A\overline{A}E\overline{E}$ themselves). Nodes are partitioned with respect to the complexity of their satisfiability problem: nodes corresponding to undecidable fragments are identified by a red rectangle and by the superscript 1, while nodes corresponding to EXPSPACE-complete (resp., NEXPTIME-complete, NP-complete) fragments are identified by a yellow rectangle and the superscript

2 (resp., blue rectangle/superscript 3, green rectangle/superscript 4). All HS fragments that do not appear in the picture are undecidable. Graph edges represent the relative expressive power of two fragments: if two nodes, labeled by the fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, are connected by a path going from $\mathcal{F}_1$ to $\mathcal{F}_2$, then $\mathcal{F}_2 \prec \mathcal{F}_1$; if two fragments $\mathcal{F}_1$ and $\mathcal{F}_2$ are not connected by a path, then $\mathcal{F}_1 \not\equiv \mathcal{F}_2$. Thus, to show that Figure 1 is sound and complete, we need to prove that *(i)* each fragment $\mathcal{F}_1$ connected to a fragment $\mathcal{F}_2$ by an arrow is strictly more expressive than $\mathcal{F}_2$; *(ii)* pairs of fragments in Figure 1, which are not connected by a path, are expressively incomparable; and *(iii)* the complexity of the satisfiability problem for the considered fragments is correctly depicted in Figure 1. Conditions *(i)* and *(ii)* are direct consequences of the following lemma, whose proof, given in [7], makes use of bisimulations based on finite linearly ordered sets. As the class of all strongly discrete linearly ordered sets includes that of finite linearly ordered sets, all results immediately apply.

Lemma 1 ([7])**.** *The only definability equations for the* HS *fragment* $\mathsf{A}\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}$*, over the class of all strongly discrete linear orders, are* $\langle \mathsf{L}\rangle p \equiv \langle \mathsf{A}\rangle\langle \mathsf{A}\rangle p$ *and* $\langle \overline{\mathsf{L}}\rangle p \equiv \langle \overline{\mathsf{A}}\rangle\langle \overline{\mathsf{A}}\rangle p$.

Hence, we can restrict our attention to condition *(iii)*. The rest of the section is devoted to prove the undecidability of all fragments marked as undecidable in Figure 1. All fragments which are not referred to in the figure have already been proved undecidable over the class of strongly discrete linearly ordered sets [6, 16]. All decidable fragments of HS over the class of strongly discrete linear orders are thus depicted in Figure 1. Section 4 and 5 will be devoted to the identification of the exact complexity of these decidable fragments.

The undecidability result we give here resembles those in [7, 18]. Nevertheless, the required modifications are far from being trivial. From [18, 20], we know that there exists a reduction from the structural termination problem for lossy counter automata, which is known to be undecidable [17], to the satisfiability problem for $\mathsf{A}\overline{\mathsf{A}}\mathsf{B}$ and $\mathsf{A}\overline{\mathsf{AB}}$. Here, we consider the nonemptiness problem for incrementing counter automata over infinite words, which is known to be undecidable [12], and we show that it can be reduced to the satisfiability problem for the fragments $\overline{\mathsf{A}}\mathsf{B}$, $\overline{\mathsf{AB}}$, AE, and $\mathsf{A}\overline{\mathsf{E}}$. For the sake of brevity, we will work out all the details of the reduction for AE only. Since AE and $\overline{\mathsf{A}}\mathsf{B}$ are completely symmetric with respect to the class of strongly discrete linearly ordered sets, the reduction for AE basically works for $\overline{\mathsf{A}}\mathsf{B}$ as well. Moreover, adapting it to $\mathsf{A}\overline{\mathsf{E}}$ (and therefore, by symmetry, to $\overline{\mathsf{AB}}$) is straightforward. Incrementing counter automata can be viewed as a variant of lossy counter automata where faulty transitions increase the values of counters instead of decrementing them. Hence, some of the basic concepts of the reduction given in [18, 20] can be exploited. A comprehensive survey on faulty machines and on the relevant complexity, decidability, and undecidability results can be found in [3]. Formally, an *incrementing counter automaton* is a tuple $\mathcal{A} = (\Sigma, Q, q_0, C, \Delta, F)$, where Σ is a finite alphabet, Q is a finite set of control *states*, $q_0 \in Q$ is the initial state, $C = \{c_1, \ldots, c_k\}$ is the set of *counters*, whose values range over $\mathbb{N}$, Δ is a *transition relation*, and $F \subseteq Q$ is the set of final states. Let us denote by ϵ the *empty word* (we assume $\epsilon \notin \Sigma$). The relation Δ is a subset of $Q \times (\Sigma \cup \{\epsilon\}) \times L \times Q$, where L is the *instruction set* $L = \{inc, dec, ifz\} \times \{1, \ldots, k\}$. A *configuration* of $\mathcal{A}$ is a pair $(q, \overline{v})$, where $q \in Q$ and $\overline{v}$ is the vector of counter values. A *run* of an incrementing counter automaton is an infinite sequence of configurations such that, for every pair of consecutive configurations $(q, \overline{v}), (q', \overline{v}')$ an *incrementing transition* $(q, \overline{v}) \xrightarrow{l,a}_{\dagger} (q', \overline{v}')$ has been taken. We say that $(q, \overline{v}) \xrightarrow{l,a}_{\dagger} (q', \overline{v}')$ has been taken if there exist $\overline{v}_{\dagger}, \overline{v}'_{\dagger}$ such that $\overline{v} \leqslant \overline{v}_{\dagger}$, $(q, \overline{v}_{\dagger}) \xrightarrow{l,a} (q', \overline{v}'_{\dagger})$, and $\overline{v}'_{\dagger} \leqslant \overline{v}'$, where $(q, \overline{v}) \xrightarrow{l,a} (q', \overline{v}')$ iff $(q, a, l, q') \in \Delta$ and if $l = (inc, i)$ (resp., (dec, i), (ifz, i)), then $v'_i = v_i + 1$ (resp., $v'_i = v_i - 1$, $v'_i = v_i = 0$) (the ordering $\overline{v} \leqslant \overline{v}'$ is defined component-wise in the obvious way). Notice that once an incrementing transition $(q, \overline{v}) \xrightarrow{l,a}_{\dagger} (q', \overline{v}')$ has been taken, counter values may have been increased nondeterministically before or after the execution of the basic transition $(q, \overline{v}) \xrightarrow{l,a} (q', \overline{v}')$ by an arbitrary natural number. We say

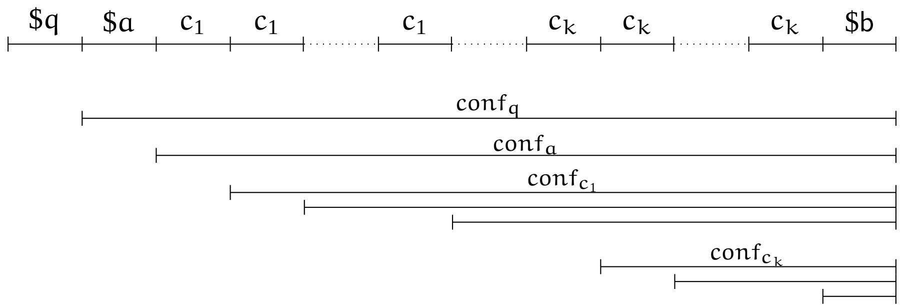

Figure 2: Encoding of a configuration of an incrementing counter automaton in AE.

that an infinite run of $\mathcal{A}$ over an ω-word $w \in \Sigma^\omega$ is *accepting* iff it traverses a state in F infinitely often. The nonemptiness problem for increasing counter automata is the problem of deciding whether there exists at least one ω-word accepted by it. In Section 6, we will show that when we restrict our attention to $\mathbb{N}$-models, the situation becomes slightly different, as symmetry does not hold anymore.

Lemma 2. *There exists a reduction from the nonemptiness problem for incrementing counter automata over ω-words to the satisfiability problem for* AE *over the class of strongly discrete linear orders.*

Proof. Let $\mathcal{A} = (\Sigma, Q, q_0, C, \Delta, F)$ be an incrementing counter automaton. We write an AE formula $\varphi_\mathcal{A}$ which is satisfiable over the class of strongly discrete linear orders iff there is at least one ω-word over Σ accepted by $\mathcal{A}$. Let us assume that $|Q| = \mu + 1$, $|\Sigma| = \nu$, $|F| = \eta$, and $|C| = k$, and there are *(i)* $\mu + 1$ proposition letters $q_0, q_1, \ldots, q_\mu$, one for each state in Q (q_0 being the initial state); *(ii)* ν proposition letters $a_1, \ldots, a_\nu$, one for each symbol in Σ; and *(iii)* k proposition letters $c_1, \ldots, c_k$, one for each counter in C. Moreover, to simplify the formula, we introduce a proposition letter $\$q$ (resp., $\$a$, $\$c$) which holds at some interval iff at least one q_i (resp., a_i, c_i) holds at that interval. Finally, a proposition letter $conf$ is used to denote a configuration. Additional auxiliary proposition letters will be introduced later on.

To encode the components of a configuration, we use intervals of the form $[x, x+1]$ (unit intervals), which are univocally identified by the AE formula $[E]\bot$. A configuration is modeled by a (non-unit) interval $[x, x+s]$, labeled with $conf$, consisting of a sequence of unit intervals labeled as follows: $[x, x+1]$ is labeled with (a proposition letter for) a state in Q, $[x+1, x+2]$ by a letter in Σ, and all the remaining unit intervals, but the last one (for technical reasons, $[x+s-1, x+s]$ is labeled with a special proposition letter $\$b$), are labeled with counters in C. Figure 2 depicts (part of) the encoding of a configuration. We constrain any configuration interval $[x, x+s]$ to contain one unit interval labeled with a state, one labeled with an alphabet letter, and, for $1 \leqslant i \leqslant k$, as many unit intervals labeled with c_i as the value of counter c_i is in that configuration. Without loss of generality, we can assume all counter values to be initialized to 0 ($\overline{v} = \overline{0}$), and thus the initial configuration contains no counter proposition letters.

Let $[U]\varphi$ be a shorthand for the formula $[U]\varphi = \varphi \wedge [A]\varphi \wedge [A][A]\varphi$ (*universal* modality). We first constrain proposition letters that denote states (in Q), input symbols (in Σ), and counter values to be correctly placed.

$$[U](\$q \leftrightarrow \bigvee_{i=0}^{\mu} q_i \wedge \$a \leftrightarrow \bigvee_{i=1}^{\nu} a_i \wedge \$c \leftrightarrow \bigvee_{i=1}^{k} c_i) \qquad \textit{placeholders are correctly set} \qquad (1)$$

$$[U]([E]\bot \leftrightarrow \$q \vee \$a \vee \$c \vee \$b) \qquad \textit{placeholders are unit intervals} \qquad (2)$$

$$[U] \bigwedge_{p \in \{q,a,c,b\}} (\$p \rightarrow \neg \bigvee_{p' \in \{q,a,c,b\}, p' \neq p} \$p') \qquad \textit{exactly one placeholder per unit interval} \qquad (3)$$

$$[U]\big(\bigwedge_{i\neq j}(q_i \to \neg q_j) \wedge \bigwedge_{i\neq j}(a_i \to \neg a_j) \wedge \bigwedge_{i\neq j}(c_i \to \neg c_j)\big) \qquad \textit{exactly one state, letter, counter} \tag{4}$$

Next, we encode the sequence of configurations as a (unique) infinite chain that starts at the ending point of the interval where $\varphi_{\mathcal{A}}$ is evaluated, and we constrain the counter values of the initial configuration to be equal to 0. To force such a chain to be unique and to prevent configurations from containing or overlapping other configurations, we introduce an additional proposition letter $conf'$, which holds over all and only those intervals which are suffixes of a $conf$-interval.

$$\langle A\rangle(conf \wedge \langle E\rangle\langle E\rangle\top \wedge [E][E][E]\bot) \qquad \textit{the initial configuration has two internal points only} \tag{5}$$

$$[U](conf \to \langle A\rangle conf \wedge \langle E\rangle\langle E\rangle\top) \qquad \textit{a chain of } conf\textit{s; each } conf \textit{ has room for state and letter} \tag{6}$$

$$[U]((conf \to [E]conf') \wedge (conf' \to \neg conf)) \qquad conf\textit{s are ended by } conf'\textit{s which are not } conf \tag{7}$$

$$[U]\big((\langle A\rangle conf' \to \neg conf) \wedge (conf' \to \langle A\rangle conf \wedge \neg\langle E\rangle conf)\big) \qquad conf \textit{ neither overlap nor contain other } conf\textit{s; } conf'\textit{s end } conf\textit{s} \tag{8}$$

Now, we force configurations to be properly structured: they must start with a unit interval labeled with a state (the initial configuration with q_0), followed by a unit interval labeled with an input letter, possibly followed by a number of unit intervals labeled with counters, followed by a last unit interval labeled with $\$b$. As modalities $\langle A\rangle$ and $\langle E\rangle$ do not allow one, in general, to refer to the subintervals of a given interval, a little technical detour is necessary. We introduce the auxiliary proposition letters $conf_q$, $conf_a$, and $conf_{c_i}$ (one for each type of counter), and we label the suffix of a configuration interval met by a unit interval labeled with $\$q$ (resp., $\$a$, c_i) with $conf_q$ (resp., $conf_a$, $conf_{c_i}$). In such a way, modality $\langle E\rangle$ can be exploited to get an indirect access to the components of a configuration. As an example, we use it to force every configuration to include at most one state and one input letter. Notice that proposition letter $\$b$ plays an essential role here: it allows us to associate the last c_i of each configuration with the corresponding $conf_{c_i}$.

$$\langle A\rangle q_0 \wedge [U](\langle A\rangle conf \leftrightarrow \langle A\rangle \$q) \qquad conf \textit{ starts with state (the initial one with } q_0\textit{)} \tag{9}$$

$$[U]((\$q \to \langle A\rangle\$a) \wedge (\$a \vee \$c \to \langle A\rangle(\$c \vee \$b)) \wedge (\$b \to \langle A\rangle\$q)) \qquad conf \textit{ is properly structured} \tag{10}$$

$$[U]((\$q \to [A](conf' \to conf_q)) \wedge (\$a \to [A](conf' \to conf_a))) \qquad \$q \textit{ meets } conf_q\textit{, } \$a \textit{ meets } conf_a \tag{11}$$

$$[U](\neg(conf_q \wedge \langle E\rangle conf_q) \wedge \neg(conf_a \wedge \langle E\rangle conf_a)) \qquad \textit{at most one state and one letter per } conf \tag{12}$$

$$[U]\big(\bigwedge_{i=1}^{k}(c_i \to [A](conf' \to conf_{c_i}))\big) \qquad c_i \textit{ meets } conf_{c_i} \tag{13}$$

To model decrements and increments, auxiliary proposition letters $c_{dec}, c_{new}, conf_{dec}$, and $conf_{new}$ are introduced. c_{dec}, which labels at most one unit interval c_i of a given configuration, constrains the value of the i-th counter to be decremented by 1 by the next transition, provided that Δ contains such a transition. Similarly, we constrain c_{new} to label a (unique) unit interval c_i added by the last transition to represent an increment by 1 of the value of the i-th counter, provided that Δ contains such a transition.

$$[U]\big(\bigwedge_{l\in\{new,dec\}}(c_l \to (\$c \wedge [A](conf' \to conf_l)))\big) \qquad \textit{if } c_l\textit{, then } conf_l \tag{14}$$

$$[U]\Big(\bigwedge_{l\in\{new,dec\}}(([E]\bot\wedge\langle A\rangle conf_l)\to c_l)\Big) \qquad \textit{if } conf_l\textit{, then } c_l \tag{15}$$

$$[U](\neg(conf_{dec}\wedge\langle E\rangle conf_{dec})\wedge\neg(conf_{new}\wedge\langle E\rangle conf_{new})) \qquad \textit{at most one } conf_l \textit{ per } conf \tag{16}$$

To constrain the values that counters may assume in consecutive configurations, we introduce three auxiliary proposition letters $corr$, $corr'$, and $corr_{conf}$. To model the faulty behavior of $\mathcal{A}$, that can increment, but not decrement, the values of counters non-deterministically, we allow two $corr$-intervals to start, but not to end, at the same point.

$$[A](\langle A\rangle c_{new}\to\neg\langle E\rangle corr) \qquad new \textit{ counters have not a counterpart in previous } conf \tag{17}$$

$$[U]((\$q\vee\$a\vee c_{dec})\to[A]\neg corr) \qquad q\textit{s, } a\textit{s, and } dec \textit{ counters have not a counterpart in next } conf \tag{18}$$

$$[U]((\$c\wedge\neg c_{dec})\to\langle A\rangle corr) \qquad \textit{non } dec \textit{ counters have a counterpart in next } conf \tag{19}$$

$$[U](([E]\bot\wedge\langle A\rangle corr)\to\$c) \qquad corr \textit{ are met by a counter} \tag{20}$$

$$[U]((corr\to[E]corr'\wedge\langle A\rangle\$c)\wedge \wedge(\langle A\rangle conf\to[A](corr'\to corr_{conf}))) \qquad corr\textit{s are ended by } corr'\textit{s and meet a counter, some } corr'\textit{s are } corr_{conf}\textit{s} \tag{21}$$

$$[U](\neg(corr_{conf}\wedge\langle E\rangle corr_{conf})\wedge \wedge(corr\to\langle E\rangle corr_{conf})) \qquad corr \textit{ connects counters of consecutive } conf \tag{22}$$

$$[U](\langle A\rangle corr_{conf}\to\langle A\rangle conf) \qquad corr_{conf} \textit{ begins } conf \tag{23}$$

$$[U]\Big(\bigwedge_{i=1}^{k}(c_i\to[A](corr\to\langle A\rangle c_i))\Big) \qquad \textit{each } corr \textit{ corresponds to some counter} \tag{24}$$

$$[U]\neg(corr\wedge\langle E\rangle corr) \qquad \textit{no } corr \textit{ ends } corr \tag{25}$$

Finally, we constrain consecutive configurations to be related by some transition (q,a,l,q') in Δ.

$$\bigvee_{(q,a,(inc,i),q')\in\Delta}\Big(\langle A\rangle(q\wedge\langle A\rangle a)\wedge\langle A\rangle(conf\wedge\langle A\rangle q'\wedge \langle A\rangle(conf\wedge\langle E\rangle(conf_{c_i}\wedge conf_{new}))))\Big) \qquad \textit{instruction } (inc,i) \tag{26}$$

$$\bigvee_{(q,a,(dec,i),q')\in\Delta}\Big(\langle A\rangle(q\wedge\langle A\rangle a)\wedge\langle A\rangle(conf\wedge\langle A\rangle q'\wedge \langle E\rangle(conf_{c_i}\wedge conf_{dec})))\Big) \qquad \textit{instruction } (dec,i) \tag{27}$$

$$\bigvee_{(q,a,(ifz,i),q')\in\Delta}\Big(\langle A\rangle(q\wedge\langle A\rangle a)\wedge\langle A\rangle(conf\wedge\langle A\rangle q'\wedge[E]\neg conf_{c_i})\Big) \qquad \textit{instruction } (ifz,i) \tag{28}$$

$$[U]\Big(\langle A\rangle conf\to\big((26)\vee(27)\vee(28)\big)\Big) \qquad \textit{an instruction} \tag{29}$$

We define $\varphi_{\mathcal{A}}$ as the conjunction of all above formulas paired with the condition that the infinite computation passes through a final state infinitely often.

$$\varphi_{\mathcal{A}}=(1)\wedge\ldots\wedge(25)\wedge(29)\wedge[A]\langle A\rangle\langle A\rangle\bigvee_{q_f\in F}q_f$$

It is straightforward to prove that $\varphi_{\mathcal{A}}$ is satisfiable iff $\mathcal{A}$ accepts at least one ω-word. $\square$

4 NP-Completeness

In this section, we prove that NP-completeness of $\mathsf{B}\overline{\mathsf{B}}$ [14] can be extended to $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$. Since the satisfiability problem for propositional logic is NP-complete, every proper fragment of $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ including it is at least NP-hard. Unlike the rest of the sections, the core of this one is a membership proof, namely, a proof of NP-membership: by a model-theoretic argument, it shows that satisfiability of a $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$-formula φ can be reduced to its satisfiability in a periodic model where the lengths of prefixes and periods have a bound which is polynomial in $|\varphi|$.

For the sake of simplicity, we consider the case of $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ interpreted over $\mathbb{N}$. The proof can be generalized to the whole class of strongly discrete linear orders. Moreover, it can be shown that satisfiability of a $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$-formula φ over $\mathbb{N}$ can be reduced to satisfiability of the formula $\tau(\varphi) = \langle L\rangle\langle\overline{L}\rangle\varphi$ over the interval $[0,1]$, that is, $M,[x,y] \Vdash \varphi$ for some $[x,y]$ if and only $M,[0,1] \Vdash \tau(\varphi)$. Thus, we can safely restrict our attention to the problem of satisfiability over $[0,1]$ (*initial* satisfiability). As a preliminary step, we introduce some useful notation and notions, including that of periodic model.

Definition 3. *An interval model $M = \langle \mathbb{I}(\mathbb{N}), V\rangle$ is* ultimately periodic, *with prefix Pre and period Per, if for every interval $[x,y] \in \mathbb{I}(\mathbb{N})$ and proposition letter $p \in \mathcal{AP}$, (i) if $x \geqslant Pre$, then $[x,y] \in V(p)$ iff $[x+Per, y+Per] \in V(p)$ and (ii) if $y \geqslant Pre$, then $[x,y] \in V(p)$ iff $[x, y+Per] \in V(p)$.*

Let us consider a $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$-formula φ. We define $Cl(\varphi)$ as the set of all subformulas of φ and of their negations. Let M be a model such that $M,[0,1] \Vdash \varphi$. For every point x of the model, let $\mathcal{R}_L(x)$ (resp., $\mathcal{R}_{\overline{L}}(x)$) be the maximal subset of $Cl(\varphi)$ consisting of all and only those $\langle L\rangle$-formulas (resp., $\langle\overline{L}\rangle$-formulas) and their negations that are satisfied over intervals ending (resp., beginning) at x[1]. Let $\mathcal{R}(x) = \mathcal{R}_L(x) \cup \mathcal{R}_{\overline{L}}(x)$. $\mathcal{R}(x)$ must be consistent, that is, it cannot contain a formula and its negation. Let $\mathcal{R}$ be the subset of $Cl(\varphi)$ that contains all possible $\langle L\rangle$- and $\langle\overline{L}\rangle$-formulas. It is immediate to see that $|\mathcal{R}| \leqslant 2|\varphi|$. In the following, we will also compare intervals with respect to satisfiability of $\langle B\rangle$- and $\langle\overline{B}\rangle$-formulas. Given a model M, we say that two intervals $[x,y]$ and $[x',y']$ are B-equivalent (denoted $[x,y] \equiv_B [x',y']$) if for every $\langle B\rangle\psi \in Cl(\varphi)$, $M,[x,y] \Vdash \langle B\rangle\psi$ iff $M,[x',y'] \Vdash \langle B\rangle\psi$ and for every $\langle\overline{B}\rangle\psi \in Cl(\varphi)$, $M,[x,y] \Vdash \langle\overline{B}\rangle\psi$ iff $M,[x',y'] \Vdash \langle\overline{B}\rangle\psi$. We denote by m_B the number of $\langle B\rangle$- and $\langle\overline{B}\rangle$-formulas in $Cl(\varphi)$. To prove that the satisfiability problem for $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$ is in NP, we first prove that every satisfiable formula φ has an ultimately periodic model, and then we show how to possibly contract such a model to obtain a model whose prefix and period are polynomial in $|\varphi|$.

Lemma 3. *Let φ be a $\mathsf{B}\overline{\mathsf{B}}\mathsf{L}\overline{\mathsf{L}}$-formula and $M = \langle \mathbb{I}(\mathbb{N}), V\rangle$ be such that $M,[0,1] \Vdash \varphi$. Then, there exists an ultimately periodic model $M^* = \langle \mathbb{I}(\mathbb{N}), V^*\rangle$ that satisfies φ.*

Proof. Let $M = \langle \mathbb{I}(\mathbb{N}), V\rangle$ be such that $M,[0,1] \Vdash \varphi$. If M is not ultimately periodic, we turn it into an ultimately periodic model as follows. First, by transitivity of $\langle L\rangle$ and $\langle\overline{L}\rangle$, there must exist a point $\overline{x} > 1$ such that $\mathcal{R}(y) = \mathcal{R}(\overline{x})$ for every $y \geqslant \overline{x}$. We take $\overline{x}$ as the prefix Pre. Then, we take as the period of the model a value $Per > m_B$ that satisfies the following conditions: *(i)* for every point $x \leqslant Pre$ and formula $\langle L\rangle\psi \in \mathcal{R}(x)$, there exists an interval $[x_\psi, y_\psi]$ such that $M,[x_\psi,y_\psi] \Vdash \psi$ and $x < x_\psi < y_\psi < Pre + Per$; *(ii)* for every interval $[x,y]$ such that $x < Pre$ and $y \geqslant Pre + Per$ and every formula $\langle\overline{B}\rangle\psi$ such that $M,[x,y] \Vdash \langle\overline{B}\rangle\psi$, there exists an interval $[x,y_\psi]$ such that $[x,y] \equiv_B [x,y_\psi]$, $M,[x,y_\psi] \Vdash \psi$, and $Pre \leqslant y_\psi < Pre + Per$. The existence of such a Per is guaranteed by transitivity of $\langle B\rangle$ and $\langle\overline{B}\rangle$. To force the model to be periodic, the following additional condition is necessary: *(iii)* for every interval $[x,y]$ such that $Pre \leqslant x < Pre + Per$ and $y \geqslant Pre + 2Per$ and every formula $\langle\overline{B}\rangle\psi$ such that $M,[x,y] \Vdash \langle\overline{B}\rangle\psi$,

[1] It is easy to see that all intervals ending (resp., beginning) at the same point satisfy the same $\langle L\rangle$-formulas (resp., $\langle\overline{L}\rangle$-formulas).

there exists an interval $[x, y_\psi]$ such that $[x,y] \equiv_B [x, y_\psi]$, $M, [x, y_\psi] \Vdash \psi$, and $y_\psi < Pre + 2Per$. If this is not the case, we can change the valuation V to force condition *(iii)* to be satisfied as follows. Let $[x,y]$ be an interval that does not satisfy condition *(iii)*. We choose a finite set of "witness points" $\{y_1 < \ldots < y_k\}$ such that (a) for every interval $[x,y']$ and every formula $\langle B\rangle\psi$, if $M,[x,y'] \Vdash \langle B\rangle\psi$, then there exists a witness point $x < y_i < y'$ such that $M,[x,y_i] \Vdash \psi$, and (b) for every interval $[x,y'']$ and every formula $\langle\overline{B}\rangle\theta$, if $M,[x,y''] \Vdash \langle\overline{B}\rangle\theta$, then there exists a witness point y_j such that $M,[x,y_j] \Vdash \psi$ and either $y_j > y''$ or $[x,y_j] \equiv_B [x,y'']$. By transitivity of $\langle B\rangle$ and $\langle\overline{B}\rangle$, it follows that the number of witness points is less than or equal to m_B (the number of $\langle B\rangle$- and $\langle\overline{B}\rangle$-formulas in $Cl(\varphi)$).

We concentrate our attention on those witness points $\{y_j < \ldots < y_k\}$ that are greater than $Pre + Per$, and we turn V into a new valuation V' such that all intervals starting at x fulfills condition *(iii)* as follows: *(1)* for every $p \in \mathcal{AP}$ and every $x < y' \leqslant Pre + Per$, we put $[x,y'] \in V'(p)$ iff $[x,y'] \in V(p)$; *(2)* for every $p \in \mathcal{AP}$ and every $j \leqslant i \leqslant k$, we put $[x, Pre + Per + i] \in V'(p)$ iff $[x,y_i] \in V(p)$; *(3)* for every $p \in \mathcal{AP}$ and every $Pre + Per + k < y' \leqslant y_k$, we put $[x,y'] \in V'(p)$ iff $[x,y_k] \in V(p)$; *(4)* the valuation of all other intervals remains unchanged. Once such a rewriting has been completed, no other interval $[x,y']$ starting at x can falsify property *(iii)*. By repeating such a procedure a sufficient number of times, we obtain a model for φ that satisfies all the required properties (notice that properties (1) and (2) are not affected by the rewriting).

The ultimately periodic model $M^* = \langle \mathbb{I}(\mathbb{N}), V^*\rangle$ can be built as follows. First, we define the valuation function V^* for some intervals in the prefix and some intervals in the first occurrence of the period: *(1)* for every $p \in \mathcal{AP}$ and every $[x,y]$ such that $y < Pre + Per$, $[x,y] \in V^*(p)$ iff $[x,y] \in V'(p)$; *(2)* for every $p \in \mathcal{AP}$ and every $[x,y]$ such that $Pre \leqslant x < Pre + Per$ and $y \leqslant x + Per$, $[x,y] \in V^*(p)$ iff $[x,y] \in V'(p)$. Then, we extend V^* to cover the entire model: *(1)* for every $p \in \mathcal{AP}$ and every $[x,y]$ such that $x < Pre$ and $y \geqslant Pre + Per$, $[x,y] \in V^*(p)$ iff $[x, y - Per] \in V^*(p)$; *(2)* for every $p \in \mathcal{AP}$ and every $[x,y]$ such that $Pre \leqslant x < Pre + Per$ and $y > x + Per$, $[x,y] \in V^*(p)$ iff $[x, y - Per] \in V^*(p)$; *(3)* for every $p \in \mathcal{AP}$ and every $[x,y]$ such that $x \geqslant Pre + Per$, $[x,y] \in V^*(p)$ iff $[x - Per, y - Per] \in V^*(p)$. It is straightforward to prove that $M^*, [0,1] \Vdash \varphi$, and thus M^* is the ultimately periodic model we were looking for. $\square$

By applying a point-elimination technique similar to the one used in [7] to show NP-membership of $B\overline{B}L\overline{L}$ over finite linear orders, we can reduce the length of the prefix and the period of an ultimately periodic model to a size polynomial in $|\varphi|$, as proved by the following lemma.

Lemma 4. *Let φ be a $B\overline{B}L\overline{L}$-formula. Then, φ is initially satisfiable over $\mathbb{N}$ iff it is initially satisfiable over an ultimately periodic model $M = \langle \mathbb{I}(\mathbb{N}), V\rangle$, with prefix Pre and period Per, such that $Pre + Per \leqslant (m_L + 2) \cdot m_B + m_L + 4$, where $m_L = 2|\mathcal{R}|$.*

Proof. By Lemma 3, we can assume that φ is initially satisfied over an ultimately periodic model $M = \langle \mathbb{I}(\mathbb{N}), V\rangle$. If $Pre + Per > (m_L + 2) \cdot m_B + m_L + 4$, then we proceed as follows.

Let us consider all points $1 < x < Pre + 2Per$. For each $\psi \in Cl(\varphi)$ such that $\langle L\rangle\psi \in \mathcal{R}(x)$ for some x in such a set, we select $1 < x^\psi_{max} \leqslant Pre + Per$ and $y^\psi_{max} < Pre + 2Per$ such that $[x^\psi_{max}, y^\psi_{max}]$ satisfies ψ and for each $x^\psi_{max} < x \leqslant Pre + Per$ no interval starting at x satisfies ψ. We collect all such points into a set (of L-*blocked* points) $Bl_L \subset \{0, \ldots, Pre + 2Per\}$. Then, for each $\psi \in Cl(\varphi)$ such that $\langle\overline{L}\rangle\psi \in \mathcal{R}(x)$ for some $1 < x < Pre + 2Per$, we select an interval $[x^\psi_{min}, y^\psi_{min}]$ that satisfies ψ and such that for each $y < y^\psi_{min}$ no interval ending at y satisfies it. We collect all points $x^\psi_{min}, y^\psi_{min}$ into a set (of $\overline{L}$-*blocked* points) $Bl_{\overline{L}} \subset \{0, \ldots, Pre\}$. Let $Bl = Bl_L \cup Bl_{\overline{L}} \cup \{Pre, Pre + Per\}$. We have that $|Bl| \leqslant m_L + 2$. Now, let us assume $Bl = \{x_1 < x_2 < \ldots < x_n\}$. For each $0 < i < n$, let $Bl_i = \{x | x_i < x < x_{i+1}\}$; similarly, let $Bl_0 = \{x | 0 < x < x_1\}$ and $Bl_n = \{x | x_n < x < Pre + 2Per\}$. We prove that if $y, y' \in Bl_i$, for some i, then $\mathcal{R}(y) = \mathcal{R}(y')$. The proof is by contradiction. Let us assume $\mathcal{R}(y) \neq \mathcal{R}(y')$. Since $\mathcal{R}(x)$ is the same for all points $x > Pre$ (it immediately follows from periodicity), at least one between y and y' must belong to the prefix of M. If $\langle L\rangle\psi \in \mathcal{R}(y)$ and $\langle L\rangle\psi \notin \mathcal{R}(y')$, then, by definition, $[L]\neg\psi \in \mathcal{R}(y')$. This implies that $y < y'$, as $\langle L\rangle$ is transitive. It immediately follows that $y < Pre$. Let us consider now the above-defined interval $[x^{\psi}_{max}, y^{\psi}_{max}]$. Two cases may arise: either $x^{\psi}_{max} < y$ or $x^{\psi}_{max} > y'$. In the former case, since $\langle L\rangle\psi \in \mathcal{R}(y)$, there must exist an interval $[x'', y'']$ satisfying ψ and such that $x^{\psi}_{max} < x'' \leqslant y'$, thus violating the definition of x^{ψ}_{max}. In the latter case, $[L]\neg\psi \notin \mathcal{R}(y')$, against the hypothesis. The case in which $\langle\overline{L}\rangle\psi \in \mathcal{R}(y)$ and $\langle\overline{L}\rangle\psi \notin \mathcal{R}(y')$ can be proved in a similar way. Since by assumption $Pre + Per > (m_L + 2) \cdot m_B + m_L + 4$, by a simple combinatorial argument there must exist $x_{i+1} (\leqslant Pre + Per)$ in Bl such that $|Bl_i| > m_B$. Let $\overline{x}$ be the smallest point in Bl_i. We show that we can build a model $M' = \langle \mathbb{I}(\mathbb{N} \setminus \{\overline{x}\}), V'\rangle$, where $\overline{x}$ has been removed and V' is a suitable adaptation of V, such that $M', [0,1] \Vdash \varphi$.

Let $M'' = \langle \mathbb{I}(\mathbb{N} \setminus \{\overline{x}\}), V''\rangle$, where V'' is the projection of V over the intervals that neither start nor end at $\overline{x}$. By definition, replacing M by M'' does not affect satisfaction of box-formulas (from $Cl(\varphi)$). The only possible problem is the presence of some diamond-formulas which were satisfied in M and are not satisfied anymore in M''. Let $[x, y]$, with $y < \overline{x}$, be such that $M, [x, y] \Vdash \langle L\rangle\psi$. By definition of Bl, there exists an interval $[x^{\psi}_{max}, y^{\psi}_{max}]$, with $x^{\psi}_{max}, y^{\psi}_{max} \in Bl$ and $x^{\psi}_{max} \leqslant Pre + Per$, such that ψ holds over $[x^{\psi}_{max}, y^{\psi}_{max}]$ and there exists no interval $[x', y']$, with $x^{\psi}_{max} < x' \leqslant Pre + Per$, such that ψ holds over $[x', y']$. It follows that either $x^{\psi}_{max} > y$ or there exists an interval $[x', y']$ such that $M, [x', y'] \Vdash \psi$ and $x' > Pre + Per$. Therefore, $M'', [x, y] \Vdash \langle L\rangle\psi$. A symmetric argument applies to the case of $\langle\overline{L}\rangle\psi$. Hence, the removal of point $\overline{x}$ does not cause any problem with diamond-formulas of the forms $\langle L\rangle\vartheta$ or $\langle\overline{L}\rangle\vartheta$. Assume now that, for some $y < x < \overline{x}$ (resp., $y < \overline{x} < x$) and some formula $\langle\overline{B}\rangle\psi$ (resp., $\langle B\rangle\psi$) in $Cl(\varphi)$, it is the case that $M, [y, x] \Vdash \langle\overline{B}\rangle\psi$ (resp., $M, [y, x] \Vdash \langle B\rangle\psi$) and that $[y, \overline{x}]$ was the only interval starting at y (in M) satisfying ψ. Since $\overline{x}$ is the smallest point in Bl_i, $M, [y, x_i] \Vdash \langle\overline{B}\rangle\psi$ (resp., $M, [y, x_{i+1}] \Vdash \langle B\rangle\psi$) by transitivity of $\langle\overline{B}\rangle$ (resp., $\langle B\rangle$). Consider now the first m_B successors of $\overline{x}$: $\overline{x} + 1, \ldots, \overline{x} + m_B$. Since $|Bl_i| > m_B$, we have that all those points belong to Bl_i. It is possible to prove that there exists a point among them, say, $\overline{x} + k$, that satisfies the following properties: *(i)* for every $\langle B\rangle\xi \in Cl(\varphi)$, if $M, [y, \overline{x} + k + 1] \Vdash \langle B\rangle\xi$, then $M, [y, \overline{x} + k] \Vdash \langle B\rangle\xi$, and *(ii)* for every $\langle\overline{B}\rangle\zeta \in Cl(\varphi)$, if $M, [y, \overline{x} + k - 1] \Vdash \langle\overline{B}\rangle\zeta$, then $M, [y, \overline{x} + k] \Vdash \langle\overline{B}\rangle\zeta$. To prove it, it suffices to observe that, by the transitivity of $\langle B\rangle$, if $M, [y, \overline{x} + k + 1] \Vdash \langle B\rangle\xi$ then $M, [y, x'] \Vdash \langle B\rangle\xi$ for every $x' \geqslant \overline{x} + k + 1$. Hence, if $\overline{x} + k$ does not satisfy property *(i)* for ξ, all its successors are forced to satisfy it for ξ. Symmetrically, by the transitivity of $\langle\overline{B}\rangle$, if $M, [y, \overline{x} + k - 1] \Vdash \langle\overline{B}\rangle\zeta$, but $M, [y, \overline{x} + k] \nVdash \langle\overline{B}\rangle\zeta$, then $M, [y, x'] \nVdash \langle\overline{B}\rangle\zeta$ for every $x' \geqslant \overline{x} + k$. Hence, all successors of $\overline{x} + k$ trivially satisfy property *(ii)* for ζ. Since the number of $\langle B\rangle$- and $\langle\overline{B}\rangle$-formulas is limited by m_B, a point with the required properties can always be found. We fix the defect by defining the labeling V' as follows: we put $[y, \overline{x} + h] \in V'(p)$ if and only if $[y, \overline{x} + h - 1] \in V(p)$, for every proposition letter p and $1 \leqslant t \leqslant h$. The labeling of the other intervals remain unchanged. By definition of Bl, it follows that this change in the labeling does not introduce any new defect.

By iterating the above-described operation, we obtain an interval model $\overline{M} = \langle \mathbb{I}(\mathbb{N}), \overline{V}\rangle$, with $Pre + Per \leqslant (m_L + 2) \cdot m_B + m_L + 4$. However, since all changes that we did so far are limited to the portion of the model in between 0 and $Pre + 2Per$, we are not guaranteed that $\overline{M}$ is actually a model for φ. To turn it into a model for φ, we must propagate the changes to the rest of the interval model. We proceed as in the proof of Lemma 3, building an ultimately periodic model $M^* = \langle \mathbb{I}(\mathbb{N}), V^*\rangle$ as follows: *(i)* for every $p \in \mathcal{AP}$ and every $[x, y]$ such that $y \leqslant Pre + Per$, $[x, y] \in V^*(p)$ iff $[x, y] \in \overline{V}(p)$; *(ii)* for every $p \in \mathcal{AP}$ and every $[x, y]$ such that $Pre < x \leqslant Pre + Per$ and $y \leqslant x + Per$, $[x, y] \in V^*(p)$ iff $[x, y] \in \overline{V}(p)$; *(iii)* for every $p \in \mathcal{AP}$ and every $[x, y]$ such that $x \leqslant Pre$ and $y > Pre + Per$, $[x, y] \in V^*(p)$ iff $[x, y - Per] \in V^*(p)$; *(iv)* for every $p \in \mathcal{AP}$ and every $[x, y]$ such that $Pre < x \leqslant Pre + Per$ and

$y > x + Per$, $[x,y] \in V^*(p)$ iff $[x, y - Per] \in V^*(p)$; *(v)* for every $p \in \mathcal{AP}$ and every $[x,y]$ such that $x \geqslant Pre + Per$, $[x,y] \in V^*(p)$ iff $[x - Per, y - Per] \in V^*(p)$. This concludes the proof. $\square$

5 NEXPTIME- and EXPSPACE-Completeness

The cases of NEXPTIME-complete and EXPSPACE-complete fragments have been already fully worked out. In the following, we briefly summarize them. NEXPTIME-membership of $\mathsf{A}\overline{\mathsf{A}}$ has been proved in [5], while NEXPTIME-hardness of A over $\mathbb{N}$ has been shown in [9]. It is immediate to show that the latter result holds also for the class of strongly discrete linear orders; moreover, it can be easily adapted to the case of $\overline{\mathsf{A}}$, thus proving NEXPTIME-hardness of any HS fragment featuring $\langle A \rangle$ or $\langle \overline{A} \rangle$. As for EXPSPACE-complete fragments, we know from [10] that $\mathsf{AB}\overline{\mathsf{BL}}$ is EXPSPACE-complete. In [19], Montanari et al. prove EXPSPACE-hardness of the fragment AB over $\mathbb{N}$ by a reduction from the exponential-corridor tiling problem, which is known to be EXPSPACE-complete [21]. The reduction immediately applies to the case of strongly discrete linear orders. Moreover, it can be easily adapted to the fragment $\mathsf{A}\overline{\mathsf{B}}$ (a similar adaptation has been provided for finite linear orders in [7]). Given a tuple $\mathcal{T} = (T, t_\perp, t_\top, H, V, n)$, where T is a finite set of tile types, $t_\perp \in T$ is the bottom tile, $t_\top \in T$ is the top tile, H and V are two binary relations over T, that specify the horizontal and vertical constraints, and $n \in \mathbb{N}$, the exponential-corridor tiling problem consists of deciding whether there exists a tiling function f from a discrete corridor of height exponential in n to T that associates the tile $t_\perp$ (resp., $t_\top$) with the bottom (resp., top) row of the corridor and that satisfies the horizontal and vertical constraints H and V. The reduction exploits the correspondence between the points inside the corridor and the intervals of the model. It makes use of $|T|$ proposition letters to represent the tiling function f; moreover, a binary encoding of each row of the corridor is provided by means of additional proposition letters; finally, local constrains on the tiling function f are enforced by using modalities.

6 Decidability and Complexity over $\mathbb{N}$

As we already pointed, the asymmetry of $\mathbb{N}$-models, which are left-bounded and right-unbounded, is reflected in the computational behavior of (some of) the fragments of $\mathsf{A}\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}$ and its mirror image $\mathsf{A}\overline{\mathsf{A}}\mathsf{E}\overline{\mathsf{E}}$. More precisely: *(i)* $\overline{\mathsf{A}}\mathsf{B}$, but not AE, becomes decidable (non-primitive recursive) [18]; *(ii)* $\overline{\mathsf{AB}}$ and $\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}$, but not $\mathsf{A}\overline{\mathsf{E}}$ nor $\mathsf{AE}\overline{\mathsf{E}}$, become decidable (this can be shown by a suitable adaptation of the argument given in [18]); *(iii)* $\overline{\mathsf{A}}\mathsf{BL}$ and $\overline{\mathsf{AB}}\mathsf{L}$ remain undecidable, but the proof given in [18] must be suitably adapted.

Theorem 2. *The Hasse diagram in Figure 3 correctly shows all the decidable fragments of* HS *over* $\mathbb{N}$, *their relative expressive power, and the precise complexity class of their satisfiability problem.*

The main ingredients of the decidability proof for $\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}$ (and thus for $\overline{\mathsf{AB}}$ and $\overline{\mathsf{A}}\mathsf{B}$) can be summarized as follows. Let φ be a satisfiable $\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}$-formula and let $M = \langle \mathbb{I}(\mathbb{N}), V \rangle$ be a model such that $M, [x_\varphi, y_\varphi] \Vdash \varphi$ for some interval $[x_\varphi, y_\varphi]$. It can be easily checked that modalities $\langle \overline{A} \rangle$, $\langle B \rangle$, and $\langle \overline{B} \rangle$ do not allow one to access any interval $[x,y]$, with $x > x_\varphi$, starting from $[x_\varphi, y_\varphi]$, and thus valuation over such intervals can be safely ignored. By exploiting such a limitation, we can reduce the search for a model of φ to a set of ultimately periodic models only, as it is possible to prove that, for each satisfiable $\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}$-formula, there exist an ultimately periodic model $M^* = \langle \mathbb{I}(\mathbb{N}), V^* \rangle$ and an interval $[x_\varphi, y_\varphi]$ such that $M, [x_\varphi, y_\varphi] \Vdash \varphi$, $y_\varphi < Pre$, and $Per \leqslant m_B$, where m_B is the number of $\langle B \rangle$- and $\langle \overline{B} \rangle$-formulas in $Cl(\varphi)$. To guess the non-periodic part of the model, the algorithm for satisfiability checking of $\mathsf{A}\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}$ formulas over finite linear orders can be used [18]. Then, the algorithm for satisfiability checking of $\mathsf{A}\mathsf{B}\overline{\mathsf{B}}$ formulas over

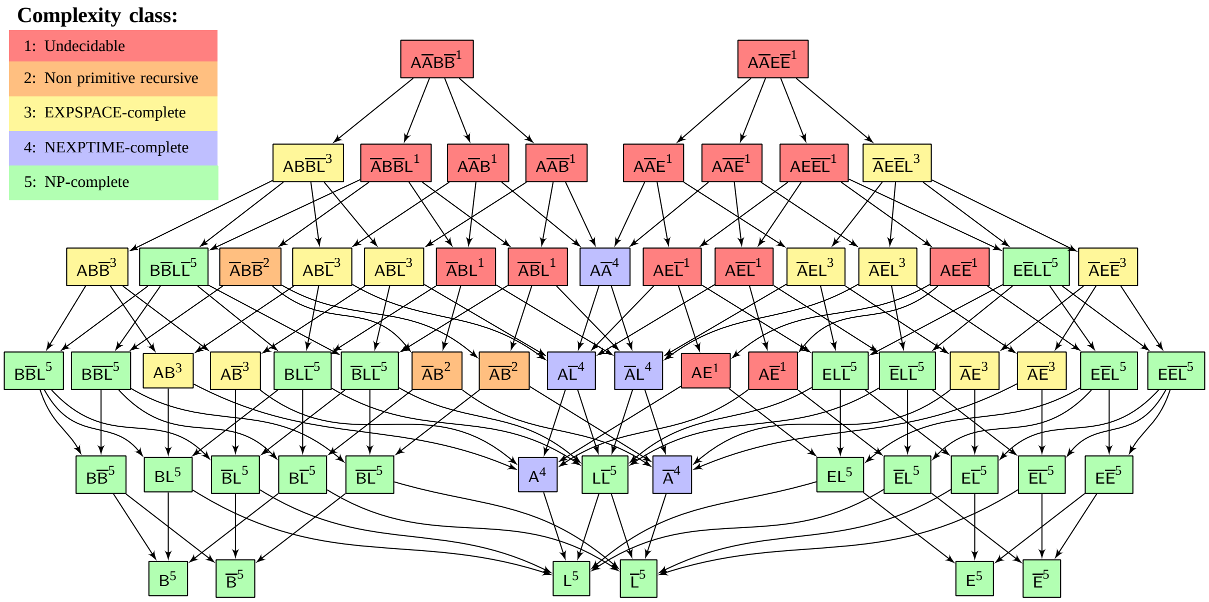

Figure 3: Hasse diagram of all fragments of $A\overline{A}B\overline{B}$ and $A\overline{A}E\overline{E}$ over the natural numbers.

$\mathbb{N}$ [19] can be applied to check whether the guessed prefix can be extended to a complete model over $\mathbb{I}(\mathbb{N})$ by guessing the valuation of intervals $[x,y]$ with $x < \mathrm{Pre}$ and $\mathrm{Pre} \leqslant y \leqslant \mathrm{Pre}+\mathrm{Per}$. To prove termination of the algorithm, it suffices to observe that if the guessed prefix is not *minimal* (in the sense of [18]), we can shrink it into a smaller one that satisfies the minimality condition (see Proposition 2 and Figure 3 in [18]). Since the number of minimal prefix models is bounded, and so is the length of the period, we can conclude that the satisfiability problem for $\overline{A}B\overline{B}$ over $\mathbb{N}$ is decidable. Non-primitive recursiveness has been already shown in [7].

In a very similar way, it is not difficult to adapt the reduction given in [18] to prove the undecidability of $\overline{A}BL$ and $\overline{A}\overline{B}L$ over $\mathbb{N}$. In this case, we reduce the structural termination problem for lossy counter automata [17] to the satisfiability problem for $\overline{A}BL$ and $\overline{A}\overline{B}L$. Since the universal modality $[U]$ can be expressed in $\overline{A}BL$ and $\overline{A}\overline{B}L$ as $[U]\varphi = \varphi \wedge [L]([\overline{A}]\varphi \wedge [\overline{A}][\overline{A}]\varphi)$, one can repeat the entire construction from [18] to encode an infinite computation of the lossy counter automata, using $\langle L \rangle$ to impose the required properties on final states.

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References

[1] J. F. Allen (1983): *Maintaining Knowledge about Temporal Intervals*. *Communications of the ACM* 26(11), pp. 832–843, doi:`10.1145/182.358434`.

[2] P. Blackburn, M. de Rijke & Y. Venema (2002): *Modal Logic*. Cambridge University Press.

[3] P. Bouyer, N. Markey, J. Ouaknine, P. Schnoebelen & J. Worrell (2008): *On Termination for Faulty Channel Machines*. In: *Proc. of the 29th STACS*, pp. 121–132, doi:`10.4230/LIPIcs.STACS.2008.1339`.

[4] D. Bresolin, D. Della Monica, V. Goranko, A. Montanari & G. Sciavicco (2008): *Decidable and Undecidable Fragments of Halpern and Shoham's Interval Temporal Logic: Towards a Complete Classification*. In: *Proc. of the 15th LPAR*, *LNCS* 5330, Springer, pp. 590–604, doi:10.1007/978-3-540-89439-1_41.

[5] D. Bresolin, V. Goranko, A. Montanari & G. Sciavicco (2009): *Propositional Interval Neighborhood Logics: Expressiveness, Decidability, and Undecidable Extensions*. *Annals of Pure and Applied Logic* 161(3), pp. 289–304, doi:10.1016/j.apal.2009.07.003.

[6] D. Bresolin, D. Della Monica, V. Goranko, A. Montanari & G. Sciavicco (2011): *The Dark Side of Interval Temporal Logic: Sharpening the Undecidability Border*. In: *Proc. of the 18th TIME*, pp. 131–138, doi:10.1109/TIME.2011.21.

[7] D. Bresolin, D. Della Monica, A. Montanari, P. Sala & G. Sciavicco (2012): *Interval Temporal Logics over Finite Linear Orders: The Complete Picture*. In: *Proc. of the 20th ECAI*.

[8] D. Bresolin, A. Montanari, P. Sala & G. Sciavicco (2011): *Optimal Tableau Systems for Propositional Neighborhood Logic over All, Dense, and Discrete Linear Orders*. In: *Proc. of the 20th TABLEAUX*, *LNCS* 6793, Springer, pp. 73–87, doi:10.1007/978-3-642-22119-4_8.

[9] D. Bresolin, A. Montanari & G. Sciavicco (2007): *An optimal decision procedure for Right Propositional Neighborhood Logic*. *Journal of Automated Reasoning* 38(1-3), pp. 173–199, doi:10.1007/s10817-006-9051-0.

[10] D. Bresolin, P. Sala & G. Sciavicco (2012): *On Begins, Meets, and Before*. *International Journal on Foundations of Computer Science* 23(3), pp. 559–583, doi:10.1142/S012905411240028X.

[11] D. Della Monica, V. Goranko, A. Montanari & G. Sciavicco (2011): *Expressiveness of the Interval Logics of Allens Relations on the Class of All Linear Orders: Complete Classification*. In: *Proc. of the 20th IJCAI*, pp. 845–850.

[12] S. Demri & R. Lazic (2006): *LTL with the Freeze Quantifier and Register Automata*. In: *Proc. of the 21st LICS*, IEEE Computer Society, pp. 17–26, doi:10.1109/LICS.2006.31.

[13] V. Goranko, A. Montanari & G. Sciavicco (2003): *Propositional interval neighborhood temporal logics*. *Journal of Universal Computer Science* 9(9), pp. 1137–1167, doi:10.3217/jucs-009-09-1137.

[14] V. Goranko, A. Montanari & G. Sciavicco (2004): *A road map of interval temporal logics and duration calculi*. *Journal of Applied Non-Classical Logics* 14(1–2), pp. 9–54, doi:10.3166/jancl.14.9-54.

[15] J. Halpern & Y. Shoham (1991): *A propositional modal logic of time intervals*. *Journal of the ACM* 38(4), pp. 935–962, doi:10.1145/115234.115351.

[16] J. Marcinkowski & J. Michaliszyn (2011): *The Ultimate Undecidability Result for the Halpern-Shoham Logic*. In: *Proc. of the 26th LICS*, IEEE Computer Society, pp. 377–386, doi:10.1109/LICS.2011.21.

[17] R. Mayr (2003): *Undecidable problems in unreliable computations*. *Theoretical Computer Science* 297(1–3), pp. 337–354, doi:10.1016/S0304-3975(02)00646-1.

[18] A. Montanari, G. Puppis & P. Sala (2010): *Maximal Decidable Fragments of Halpern and Shoham's Modal Logic of Intervals*. In: *Proc. of the 37th ICALP*, *LNCS* 6199, Springer, pp. 345–356, doi:10.1007/978-3-642-14162-1_29.

[19] A. Montanari, G. Puppis, P. Sala & G. Sciavicco (2010): *Decidability of the Interval Temporal Logic* $\mathsf{AB\overline{B}}$ *over the Natural Numbers*. In: *Proc. of the 31st STACS*, pp. 597–608, doi:10.4230/LIPIcs.STACS.2010.2488.

[20] P. Sala (2010): *Decidability of Interval Temporal Logics*. Ph.D. thesis, University of Udine.

[21] P. Van Emde Boas (1997): *The Convenience of Tilings*. In: *Complexity, Logic and Recursion Theory*, *Lecture Notes in Pure and Applied Mathematics* 187, Marcel Dekker Inc., pp. 331–363.

[22] Y. Venema (1991): *A modal logic for chopping intervals*. *Journal of Logic and Computation* 1(4), pp. 453–476, doi:10.1093/logcom/1.4.453.