

A decidable quantified fragment of set theory with ordered pairs and some undecidable extensions*

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In this paper we address the decision problem for a fragment of set theory with restricted quantification which extends the language studied in [4] with pair related quantifiers and constructs, in view of possible applications in the field of *knowledge representation*. We will also show that the decision problem for our language has a non-deterministic exponential time complexity. However, for the restricted case of formulae whose quantifier prefixes have length bounded by a constant, the decision problem becomes NP-complete. We also observe that in spite of such restriction, several useful set-theoretic constructs, mostly related to maps, are expressible. Finally, we present some undecidable extensions of our language, involving any of the operators domain, range, image, and map composition.

1 Introduction

The intuitive formalism of set theory has helped providing solid and unifying foundations to such diverse areas of mathematics as geometry, arithmetic, analysis, and so on. Hence, positive solutions to the decision problem for fragments of set theory can have considerable applications to the automation of mathematical reasoning and therefore in any area which can take advantage of automated deduction capabilities.

The decision problem in set theory has been intensively studied in the context of *Computable Set Theory* (see [5, 9, 18]), and decision procedures or undecidability results have been provided for several sublanguages of set theory. *Multi-Level Syllogistic* (in short MLS, cf. [12]) was the first unquantified sublanguage of set theory that has been shown to have a solvable satisfiability problem. We recall that MLS is the Boolean combinations of atomic formulae involving the set predicates \in , \subseteq , $=$, and the Boolean set operators \cup , \cap , \setminus . Numerous extensions of MLS with various combinations of operators (such as singleton, powerset, unionset, etc.) and predicates (on finiteness, transitivity, etc.) have been proved to be decidable. Sublanguages of set theory admitting explicit quantification (see for example [4, 16, 17, 6]) are of particular interest, since, as reported in [4], they allow one to express several set-theoretical constructs using only the basic predicates of membership and equality among sets.

Applications of Computable Set Theory to *knowledge representation* have been recently investigated in [8, 6], where some interrelationships between (decidable) fragments of set theory and description logics have been exploited.¹ As knowledge representation mainly focuses on representing relationships among items of a particular domain, any set-theoretical language of interest to knowledge representation should include a suitable collection of operators on *multi-valued maps*.²

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¹We recall that description logics are a well-established framework for knowledge representation; see [1] for an introduction.

²According to [19], we use the term ‘maps’ to denote sets of ordered pairs.

Non-deterministic exponential time decision procedures for two unquantified fragments of set theory involving map related constructs have been provided in [13, 10]. As in both cases the map domain operator is allowed together with all the constructs of MLS, it turns out that both fragments have an EXPTIME-hard decision problem (cf. [7]). On the other hand, the somewhat less expressive fragment $\text{MLSS}_{2,m}^\times$ has been shown to have an NP-complete decision problem in [7], where $\text{MLSS}_{2,m}^\times$ is a two-sorted language with set and map variables, which involves various map constructs like Cartesian product, map restrictions, map inverse, and Boolean operators among maps, and predicates for single-valuedness, injectivity, and bijectivity of maps.

In [4], an extension of the quantified fragment \forall_0 (studied in the same paper—here the subscript ‘0’ denotes that quantification is restricted) with *single-valued* maps, the map domain operator, and terms of the form $f(t)$, with t a function-free term, was considered. We recall that \forall_0 -formulae are propositional combinations of restricted quantified prenex formulae $(\forall y_1 \in z_1) \cdots (\forall y_n \in z_n)p$, where p is a Boolean combination of atoms of the types $x \in y$, $x = y$, and *quantified variables nesting* is not allowed, in the sense that any quantified variable y_i can not occur at the right-hand side of a membership symbol \in in the same quantifier prefix (roughly speaking, no z_j can be a y_i). More recently, a decision procedure for a new fragment of set theory, called \forall_0^π , has been presented in [6]. The superscript “ π ” denotes the presence of operators related to ordered pairs. Formulae of the fragment \forall_0^π , to be reviewed in Section 4, involve the operator $\bar{\pi}(\cdot)$, which intuitively represents the collection of the non-pair members of its argument, and terms of the form $[x, y]$, for ordered pairs. The predicates $=$ and \in allowed in it can occur only within atoms of the forms $x = y$, $x \in \bar{\pi}(y)$, and $[x, y] \in z$; quantifiers in \forall_0^π -formulae are restricted to the forms $(\forall x \in \bar{\pi}(y))$ and $(\forall [x, y] \in z)$, and, much as in the case of the fragment \forall_0 , quantified variables nesting is not allowed.

In this paper we solve the decision problem for the extension $\forall_{0,2}^\pi$ of the fragment \forall_0 with ordered pairs and prove that, under particular conditions, our decision procedure runs in non-deterministic polynomial time. $\forall_{0,2}^\pi$ is a two-sorted (as indicated by the second subscript “2”) quantified fragment of set theory which allows restricted quantifiers of the forms $(\forall x \in y)$, $(\exists x \in y)$, $(\forall [x, y] \in f)$, $(\exists [x, y] \in f)$, and literals of the forms $x \in y$, $[x, y] \in f$, $x = y$, $f = g$, where x, y are set variables and f, g are map variables. Considerably many set-theoretic constructs are expressible in it, as shown in Table 1. In fact, the language $\forall_{0,2}^\pi$ is also an extension of $\text{MLSS}_{2,m}^\times$. However, as will be shown in Section 5, it is not strong enough to express inclusions like $x \subseteq \text{dom}(f)$, $x \subseteq \text{range}(f)$, $x \subseteq f[y]$, and $h \subseteq f \circ g$, but only those in which the operators domain, range, (multi-)image, and map composition are allowed to appear on the left-hand side of the inclusion operator \subseteq .

The paper is organized as follows. Section 2 provides some preliminary notions and definitions. In Section 3 we give the precise syntax and semantics of the language $\forall_{0,2}^\pi$. Decidability and complexity of reasoning in the language $\forall_{0,2}^\pi$ are addressed in Section 4. Some undecidable extensions of $\forall_{0,2}^\pi$ are then presented in Section 5. Finally, in Section 6 we draw our conclusions and provide some hints for future works.

2 Preliminaries

We briefly review basic notions from set theory and introduce also some definitions which will be used throughout the paper.

Let $SVars \stackrel{\text{Def}}{=} \{x, y, z, \dots\}$ and $MVars \stackrel{\text{Def}}{=} \{f, g, h, \dots\}$ be two infinite disjoint collections of *set* and *map variables*, respectively. As we will see, map variables will be interpreted as maps (i.e., sets of ordered pairs). We put $Vars \stackrel{\text{Def}}{=} SVars \cup MVars$. For a formula φ , we write $Vars(\varphi)$ for the collection of

$x = \emptyset$	$(\forall x' \in x)(x' \neq x')$
$x \subseteq y$	$(\forall x' \in x)(x' \in y)$
$x = y \cup z$	$y \subseteq x \wedge z \subseteq x \wedge (\forall x' \in x)(x' \in y \vee x' \in z)$
$x = y \cap z$	$x \subseteq y \wedge x \subseteq z \wedge (\forall y' \in y)(y' \in z \rightarrow y' \in x)$
$x = y \setminus z$	$x \subseteq y \wedge (\forall y' \in y)(y' \in x \leftrightarrow y' \notin z)$
$x = \{y\}$	$y \in x \wedge (\forall x' \in x)(x' = y)$
$f = \emptyset$	$(\forall [x, y] \in f)(x \neq x)$
$f \subseteq g$	$(\forall [x, y] \in f)([x, y] \in g)$
$f = g \cup h$	$g \subseteq f \wedge h \subseteq f \wedge (\forall [x, y] \in f)([x, y] \in g \vee [x, y] \in h)$
$f = g \cap h$	$f \subseteq g \wedge f \subseteq h \wedge (\forall [x, y] \in g)([x, y] \in h \rightarrow [x, y] \in f)$
$f = g \setminus h$	$f \subseteq g \wedge (\forall [x, y] \in g)([x, y] \in f \leftrightarrow [x, y] \notin h)$
$f = \{[x, y]\}$	$[x, y] \in f \wedge (\forall [x', y'] \in f)(x' = x \wedge y' = y)$
$f = g^{-1}$	$(\forall [x, y] \in f)([y, x] \in g) \wedge (\forall [x, y] \in g)([y, x] \in f)$
$f = x \times y$	$(\forall x' \in x)(\forall y' \in y)([x', y'] \in f) \wedge (\forall [x', y'] \in f)(x' \in x \wedge y' \in y)$
$f = g_{x }$	$f \subseteq g \wedge (\forall [x', y'] \in g)([x', y'] \in f \leftrightarrow x' \in x)$
$f = g_{ y}$	$f \subseteq g \wedge (\forall [x', y'] \in g)([x', y'] \in f \leftrightarrow y' \in y)$
$f = g_{x y}$	$f \subseteq g \wedge (\forall [x', y'] \in g)([x', y'] \in f \leftrightarrow x' \in x \wedge y' \in y)$
$f = \text{id}(x)$	$(\forall x' \in x)([x', x'] \in f) \wedge (\forall [x', y'] \in f)(x' = y' \wedge x' \in x)$
$f = \text{sym}(g)$	$(\forall [x, y] \in f)([x, y] \in g \vee [y, x] \in g) \wedge (\forall [x, y] \in g)([x, y] \in f \wedge [y, x] \in f)$
single_valued(f)	$(\forall [x, y] \in f)(\forall [x', y'] \in f)(x = x' \rightarrow y = y')$
injective(f)	$(\forall [x, y] \in f)(\forall [x', y'] \in f)(y = y' \rightarrow x = x')$
bijjective(f)	$(\forall [x, y] \in f)(\forall [x', y'] \in f)(x = x' \leftrightarrow y = y')$
is_transitive(f)	$(\forall [x, y] \in f)(\forall [x', y'] \in f)(y = x' \rightarrow [x, y'] \in f)$
is_irreflexive(f)	$(\forall [x, y] \in f)(x \neq y)$
is_asym(f)	$(\forall [x, y] \in f)(x = y \vee [y, x] \notin f)$
$f \circ g \subseteq h$	$(\forall [x, y] \in f)(\forall [x', y'] \in g)(y = x' \rightarrow [x, y'] \in h)$
$\text{dom}(f) \subseteq x$	$(\forall [x', y'] \in f)(x' \in x)$
$\text{range}(f) \subseteq y$	$(\forall [x', y'] \in f)(y' \in y)$
$f[x] \subseteq y$	$(\forall [x', y'] \in f)(x' \in x \rightarrow y' \in y)$

Table 1: Set-theoretic constructs expressible in $\forall^{\pi}_{0,2}$.

variables occurring free (i.e., not bound by any quantifier) in φ , and put $SVars(\varphi) \stackrel{\text{Def}}{=} Vars(\varphi) \cap SVars$ and $MVars(\varphi) \stackrel{\text{Def}}{=} Vars(\varphi) \cap MVars$.

Semantics of most of the languages studied in the context of Computable Set Theory are based on the *von Neumann standard cumulative hierarchy of sets* \mathcal{V} , which is the class containing all the *pure* sets (i.e., all sets whose members are recursively based on the empty set \emptyset). The von Neumann hierarchy \mathcal{V} is defined as follows:

$$\begin{aligned}
\mathcal{V}_0 &\stackrel{\text{Def}}{=} \emptyset \\
\mathcal{V}_{\gamma+1} &\stackrel{\text{Def}}{=} \mathcal{P}(\mathcal{V}_\gamma), \quad \text{for each ordinal } \gamma \\
\mathcal{V}_\lambda &\stackrel{\text{Def}}{=} \bigcup_{\mu < \lambda} \mathcal{V}_\mu, \quad \text{for each limit ordinal } \lambda \\
\mathcal{V} &\stackrel{\text{Def}}{=} \bigcup_{\gamma \in On} \mathcal{V}_\gamma,
\end{aligned}$$

where $\mathcal{P}(\cdot)$ is the powerset operator and On denotes the class of all ordinals. The *rank* $\text{rank}(u)$ of a set $u \in \mathcal{V}$ is defined as the least ordinal γ such that $u \in \mathcal{V}_\gamma$. We will refer to mappings from $Vars$ to \mathcal{V} as *assignments*.

Next we introduce some notions related to pairing functions and ordered pairs. Let $\pi(\cdot, \cdot)$ be a binary operation over the universe \mathcal{V} . The *Cartesian product* $u \times_\pi v$ of two sets $u, v \in \mathcal{V}$, relative to π , is defined

as $u \times_{\pi} v \stackrel{\text{Def}}{=} \{\pi(u', v') : u' \in u \wedge v' \in v\}$. When it is clear from the context, for the sake of conciseness we will omit to specify the binary operation π and simply write ‘ \times ’ in place of ‘ \times_{π} ’. A binary operation π over sets in \mathcal{V} is said to be a *pairing function* if

- (i) $\pi(u, v) = \pi(u', v') \iff u = u' \wedge v = v'$, for all $u, u', v, v' \in \mathcal{V}$, and
- (ii) the Cartesian product $u \times v$ (relative to π) is a set of \mathcal{V} , for all $u, v \in \mathcal{V}$.

In view of the replacement axiom, condition (ii) is obviously met when $\pi(u, v)$ is expressible by a set-theoretic term. This, for instance, is the case for Kuratowski’s ordered pairs, defined by $\pi_{\text{Kur}}(u, v) \stackrel{\text{Def}}{=} \{\{u\}, \{u, v\}\}$, for all $u, v \in \mathcal{V}$. Given a pairing function π and a set s , we denote with $\text{Pairs}_{\pi}(s)$ the collection of the *pairs* in s (with respect to π), namely $\text{Pairs}_{\pi}(s) \stackrel{\text{Def}}{=} \{u \in s : (\exists v_1, v_2 \in \mathcal{V})(u = \pi(v_1, v_2))\}$.

A *pair-aware interpretation* $I = (M^I, \pi^I)$ consists of a pairing function π^I and an assignment M^I such that $\text{Pairs}_{\pi^I}(M^I f) = M^I f$ holds for every map variable $f \in MVars$ (i.e., map variables can only be assigned sets of ordered pairs, or the empty set). For conciseness, in the rest of the paper we will refer to *pair-aware* interpretations just as interpretations. An interpretation $I = (M^I, \pi^I)$ associates sets to variables and pair terms, respectively, as follows:

$$\begin{aligned} Ix &\stackrel{\text{Def}}{=} M^I x, \\ I[x, y] &\stackrel{\text{Def}}{=} \pi^I(Ix, Iy), \end{aligned} \quad (1)$$

for all $x, y \in Vars$. Let $W \subseteq Vars$ be a finite collection of variables, and let M, M' be two assignments. We say that M' is a W -variant of M if $Mx = M'x$ for all $x \in Vars \setminus W$. For two interpretations $I = (M^I, \pi^I)$ and $J = (M^J, \pi^J)$, we say that J is a W -variant of I if M^J is a W -variant of M^I and $\pi^J = \pi^I$.

In the next section we introduce the precise syntax and semantics of the language $\forall_{0,2}^{\pi}$.

3 The language $\forall_{0,2}^{\pi}$

The language $\forall_{0,2}^{\pi}$ consists of the denumerable infinity of variables $Vars = SVars \cup MVars$, the binary *pairing* operator $[\cdot, \cdot]$, the predicate symbols $\in, =$, the Boolean connectives of propositional logic $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, parentheses, and *restricted* quantifiers of the forms $(\forall x \in y)$, $(\forall [x, y] \in f)$, $(\exists x \in y)$, and $(\exists [x, y] \in f)$. *Atomic $\forall_{0,2}^{\pi}$ -formulae* are expressions of the following four types

$$x \in y, \quad x = y, \quad [x, y] \in f, \quad f = g, \quad (2)$$

with $x, y \in SVars$ and $f, g \in MVars$. *Quantifier-free $\forall_{0,2}^{\pi}$ -formulae* are propositional combinations of atomic $\forall_{0,2}^{\pi}$ -formulae. *Prenex $\forall_{0,2}^{\pi}$ -formulae* are expressions of the following two forms

$$(\forall x_1 \in z_1) \dots (\forall x_h \in z_h) (\forall [x_{h+1}, y_{h+1}] \in f_{h+1}) \dots (\forall [x_n, y_n] \in f_n) \delta, \quad (3)$$

$$(\exists x_1 \in z_1) \dots (\exists x_h \in z_h) (\exists [x_{h+1}, y_{h+1}] \in f_{h+1}) \dots (\exists [x_n, y_n] \in f_n) \delta, \quad (4)$$

where $x_i, y_i, z_i \in SVars$, $f_j \in MVars$, and δ is a quantifier-free $\forall_{0,2}^{\pi}$ -formula. We will refer to the variables z_1, \dots, z_h as the *domain variables* of the formulae (3) and (4). Notice that quantifier-free $\forall_{0,2}^{\pi}$ -formulae can also be regarded as prenex $\forall_{0,2}^{\pi}$ -formulae with an empty quantifier prefix. A prenex $\forall_{0,2}^{\pi}$ -formula is said to be *simple* if nesting among quantified variables is not allowed, i.e., if no quantified variable can occur also as a domain variable. Finally, $\forall_{0,2}^{\pi}$ -formulae are Boolean combinations of simple-prenex $\forall_{0,2}^{\pi}$ -formulae.

Semantics of $\forall_{0,2}^{\pi}$ -formulae is given in terms of interpretations. An interpretation $I = (M^I, \pi^I)$ *evaluates* a $\forall_{0,2}^{\pi}$ -formula φ into a truth value $I\varphi \in \{\mathbf{true}, \mathbf{false}\}$ in the following recursive manner.

First of all, interpretation of quantifier-free $\forall_{0,2}^\pi$ -formulae is carried out following the rules of propositional logic, where atomic formulae (2) are interpreted according to the standard meaning of the predicates \in and $=$ in set theory and the pair operator $[\cdot, \cdot]$ is interpreted as in (1). Thus, for instance, $I([x, y] \in f \rightarrow x \in y) = \mathbf{true}$, provided that either $\pi^I(Ix, Iy) \notin If$ or $Ix \in Iy$. Then, evaluation of simple-prenex $\forall_{0,2}^\pi$ -formulae is defined recursively as follows:

- $I(\forall x \in z)\varphi = \mathbf{true}$, provided that $J\varphi = \mathbf{true}$, for every $\{x\}$ -variant J of I such that $Jx \in Jz$;
- $I(\forall [x, y] \in f)\varphi = \mathbf{true}$, provided that $J\varphi = \mathbf{true}$, for every $\{x, y\}$ -variant J of I such that $J[x, y] \in Jf$;
- $I(\exists x \in z)\varphi = \mathbf{true}$, provided that $I(\forall x \in z)\neg\varphi = \mathbf{false}$; and
- $I(\exists [x, y] \in f)\varphi = \mathbf{true}$, provided that $I(\forall [x, y] \in f)\neg\varphi = \mathbf{false}$.

Finally, evaluation of $\forall_{0,2}^\pi$ -formulae is carried out following the rules of propositional logic.

If an interpretation I evaluates a $\forall_{0,2}^\pi$ -formula to **true** we say that I is a *model* for φ (and write $I \models \varphi$). A $\forall_{0,2}^\pi$ -formula φ is said to be *satisfiable* if and only if it admits a model. Two $\forall_{0,2}^\pi$ -formulae are said to be *equivalent* if they have exactly the same models. Two $\forall_{0,2}^\pi$ -formulae φ and φ' are said to be *equisatisfiable* provided that φ is satisfiable if and only if so is φ' . The *satisfiability problem* (s.p., for short) for the theory $\forall_{0,2}^\pi$ is the problem of establishing algorithmically whether any given $\forall_{0,2}^\pi$ -formula is satisfiable or not.

By way of a simple normalization procedure based on disjunctive normal form, the s.p. for $\forall_{0,2}^\pi$ -formulae can be reduced to that for *conjunctions* of simple-prenex $\forall_{0,2}^\pi$ -formulae of the types (3) and (4). Moreover, since any such conjunction of the form

$$\psi \wedge (\exists x_1 \in z_1) \dots (\exists x_h \in z_h) (\exists [x_{h+1}, y_{h+1}] \in f_{h+1}) \dots (\exists [x_n, y_n] \in f_n) \delta$$

is equisatisfiable with $\psi \wedge \delta'_+$, where δ'_+ is obtained from the quantifier-free formula

$$\delta_+ =_{\text{def}} \bigwedge_{i=1}^h x_i \in z_i \wedge \bigwedge_{j=h+1}^n [x_j, y_j] \in f_j \wedge \delta$$

by a suitable renaming of the (quantified) variables $x_1, \dots, x_n, y_{h+1}, \dots, y_n$, it turns out that the s.p. for $\forall_{0,2}^\pi$ -formulae can be reduced to the s.p. for *conjunctions* of simple-prenex $\forall_{0,2}^\pi$ -formulae of the type (3) only, which we call *normalized $\forall_{0,2}^\pi$ -conjunctions*.

Satisfiability of normalized $\forall_{0,2}^\pi$ -conjunctions does not depend strictly on the pairing function of the interpretation, provided that suitable conditions hold, as proved in the following technical lemma.

Lemma 1. *Let φ be a normalized $\forall_{0,2}^\pi$ -conjunction, and let I and J be two interpretations such that*

- (a) $Ix = Jx$, for all $x \in SVars$,
- (b) $\pi^I(u, v) \in If \iff \pi^J(u, v) \in Jf$, for all $u, v \in \mathcal{V}$ and $f \in MVars$.

Then $I \models \varphi \iff J \models \varphi$.

Proof. It is enough to prove that

$$I \models \psi \iff J \models \psi \tag{5}$$

holds, for every (universal) simple-prenex conjunct ψ occurring in φ . We shall proceed by induction on the length of the quantifier prefix of ψ . We begin with observing that, by (a), I and J evaluate to the same truth values all atomic formulae of the types $x \in y$ and $x = y$, for all $x, y \in SVars$. Likewise,

$$I \models f = g \iff J \models f = g \quad \text{and} \quad I \models [x, y] \in f \iff J \models [x, y] \in f$$

follow directly from (a) and (b). Thus (5) follows easily when ψ is quantifier-free, i.e., when the length of its quantifier prefix is 0.

Next, let $\psi = (\forall x \in y)\psi_0$, for some $x, y \in SVars$, where ψ_0 is a universally quantified simple-prenex $\forall_{0,2}^\pi$ -formula with one less quantifier than ψ and containing no quantified occurrence of y . We prove that I_u is a model for ψ_0 if and only if so is J_u , for every $u \in Iy = Jy$, where I_u and J_u denote, respectively, the $\{x\}$ -variants of I and J such that $I_u x = J_u x = u$. But, for each $u \in Iy = Jy$, I_u and J_u satisfy conditions (a) and (b) of the lemma, so that, by inductive hypothesis, we have $I_u \models \psi_0 \iff J_u \models \psi_0$. Hence $I \models (\forall x \in y)\psi_0 \iff J \models (\forall x \in y)\psi_0$.

The case in which $\psi = (\forall [x, y] \in f)\psi_0$, with $x, y \in SVars$, $f \in MVars$, and ψ_0 a universally quantified simple-prenex $\forall_{0,2}^\pi$ -formula containing no quantified occurrence of x and y , can be dealt with much in the same manner, thus concluding the proof of the lemma. \square

In the following section we show that the s.p. for normalized $\forall_{0,2}^\pi$ -conjunctions is solvable.

4 A decision procedure for $\forall_{0,2}^\pi$

We solve the s.p. for $\forall_{0,2}^\pi$ -formulae by reducing the s.p. for normalized $\forall_{0,2}^\pi$ -conjunctions to the s.p. for the fragment of set theory \forall_0^π , studied in [6]. Following [6], \forall_0^π -formulae are finite conjunctions of *simple-prenex \forall_0^π -formulae*, namely expressions of the form

$$(\forall x_1 \in \bar{\pi}(z_1)) \dots (\forall x_h \in \bar{\pi}(z_h)) (\forall [x_{h+1}, y_{h+1}] \in z_{h+1}) \dots (\forall [x_n, y_n] \in z_n) \delta,$$

where $x_i, y_i, z_i \in SVars$, for $i = 1, \dots, n$, no domain variable z_i can occur quantified, and δ is a quantifier-free Boolean combination of atomic formulae of the types $x \in \bar{\pi}(z)$, $[x, y] \in z$, $x = y$, with $x, y, z \in SVars$.³ Intuitively, a term of the form $\bar{\pi}(z)$ represents the set of the *non-pair* members of z . Notice that \forall_0^π -formulae involve only set variables.

Semantics for \forall_0^π -formulae is given by extending interpretations also to terms of the form $\bar{\pi}(x)$ as indicated below:

$$I\bar{\pi}(x) =_{\text{Def}} Ix \setminus \text{Pairs}_{\pi'}(Ix),$$

where $x \in SVars$. Evaluation of \forall_0^π -formulae is carried out much in the same way as for $\forall_{0,2}^\pi$ -formulae. In particular, we also put $I(\forall x \in \bar{\pi}(y))\varphi = \mathbf{true}$, provided that $J\varphi = \mathbf{true}$, for every $\{x\}$ -variant J of I such that $Jx \in I\bar{\pi}(y)$.

We recall that satisfiability of \forall_0^π -formulae can be tested in non-deterministic exponential time. Additionally, the s.p. for \forall_0^π -formulae with quantifier prefixes of length at most h , for any fixed constant $h \geq 0$, is NP-complete (cf. [6]).

The s.p. for normalized $\forall_{0,2}^\pi$ -conjunctions can be reduced to the s.p. for \forall_0^π -formulae. To begin with, we define a syntactic transformation $\tau(\cdot)$ on normalized $\forall_{0,2}^\pi$ -conjunctions. More specifically, $\tau(\varphi)$ is obtained from a given normalized $\forall_{0,2}^\pi$ -conjunction φ by replacing

- each restricted universal quantifier $(\forall x \in y)$ in φ by the quantifier $(\forall x \in \bar{\pi}(y))$,
- each atomic formula $x \in y$ in φ by the literal $x \in \bar{\pi}(y)$, and
- each map variable f occurring in φ by a fresh set variable x_f , thus identifying an application $f \mapsto x_f$ from $MVars(\varphi)$ into $SVars$, which will be referred to as *map-variable renaming for $\tau(\varphi)$* .

³Thus, normalization is already built-in into \forall_0^π -formulae, and we could have called them *normalized \forall_0^π -conjunctions*.

Thus, for instance, if

$$\varphi = (\forall x' \in x)([x, x] \in f) \wedge (\forall [x', y'] \in f)(x' = y' \wedge x' \in x)$$

then

$$\tau(\varphi) = (\forall x' \in \bar{\pi}(x))([x, x] \in x_f) \wedge (\forall [x', y'] \in x_f)(x' = y' \wedge x' \in \bar{\pi}(x)),$$

where x_f is a set variable distinct from x , x' , and y' .

The following lemma provides a useful semantic relation between universal simple-prenex $\forall_{0,2}^\pi$ -formulae and their corresponding \forall_0^π -formula via τ .

Lemma 2. *Let ψ be a universal simple-prenex $\forall_{0,2}^\pi$ -formula and let $I = (M^I, \pi^I)$ be an interpretation such that*

- (i) $\text{Pairs}_{\pi^I}(\{Ix : x \in S\text{Vars}(\psi)\}) = \emptyset$ (i.e., Ix is not a pair, for any free variable x of ψ), and
- (ii) $\text{Pairs}_{\pi^I}(Ix) = \emptyset$, for every domain variable x of ψ .

Then $I \models \psi$ if and only if $I \models \tau(\psi)$.

Proof. We proceed by induction on the quantifier prefix length $\ell \geq 0$ of the formula ψ . To begin with, we observe that in force of (i) we have $Ix \in Iy$ if and only if $Ix \in I\bar{\pi}(y)$, for any two free variables x and y of ψ , so that, given any atomic formula α involving only variables in $S\text{Vars}(\psi)$, $I \models \alpha$ if and only if $I \models \tau(\alpha)$. Hence the lemma follows directly from propositional logic if ψ is quantifier-free, i.e., $\ell = 0$.

Next, let $\psi = (\forall x \in y)\psi_0$, where ψ_0 is a universal simple-prenex $\forall_{0,2}^\pi$ -formula with $\ell - 1$ quantifiers, x, y are set variables occurring neither as domain nor as bound variables in ψ_0 . Observe that, by (ii), $Iy = I\bar{\pi}(y)$, since y is a domain variable of ψ . Thus it will be enough to prove that

$$I_u \models \psi_0 \iff I_u \models \tau(\psi_0) \tag{6}$$

holds for every $\{x\}$ -variant I_u of I such that $I_u x = u$, with $u \in Iy$. But $I_u x$ can not be a pair (with respect to the pairing function π^I), as it is a member of Iy and y is a domain variable of φ . Thus (6) follows by applying the inductive hypothesis to ψ_0 and to every interpretation I_u such that $u \in Iy$.

Finally, the case in which $\psi = (\forall [x, y] \in f)\psi_0$, where ψ_0 is a universal simple-prenex $\forall_{0,2}^\pi$ -formula, x, y are set variables not occurring as domain variables in ψ_0 , and f is a map variable, can be dealt with much in the same way as the previous case, and is left to the reader. \square

In the following theorem we use the transformation $\tau(\cdot)$ to reduce the s.p. for normalized $\forall_{0,2}^\pi$ -conjunctions to the s.p. for \forall_0^π -formulae.

Theorem 1. *The s.p. for normalized $\forall_{0,2}^\pi$ -conjunctions can be reduced in linear time to the s.p. for \forall_0^π -formulae, and therefore it is in NEXPTIME.*

Proof. We prove the theorem by showing that, given any normalized $\forall_{0,2}^\pi$ -conjunction ψ , we can construct in linear time a corresponding \forall_0^π -formula ψ' which is equisatisfiable with ψ .

So, let ψ be a normalized $\forall_{0,2}^\pi$ -conjunction and let $f \mapsto x_f$ be the map-variable renaming for $\tau(\psi)$. We define the corresponding \forall_0^π -formula ψ' as follows:

$$\psi' \stackrel{\text{Def}}{=} \tau(\psi) \wedge \bigwedge_{z \in S\text{Vars}(\psi)} (\forall [x, y] \in z)(x \neq x) \wedge \bigwedge_{f \in M\text{Vars}(\psi)} (\forall x \in \bar{\pi}(x_f))(x \neq x) \wedge \bigwedge_{z \in S\text{Vars}(\psi)} (z \in \bar{\pi}(U)),$$

where U is a fresh set variable. Plainly, the size of ψ' is linear in the size of ψ .

Let us first assume that ψ' admits a model $J = (M^J, \pi^J)$. For each $z \in SVars(\psi)$ we have $\text{Pairs}_{\pi^J}(Jz) = \emptyset$, as $J(\forall[x, y] \in z)(x \neq x) = \mathbf{true}$, for $z \in SVars(\psi)$. Likewise, for each $f \in MVars(\psi)$ we have $Jx_f = \text{Pairs}_{\pi^J}(Jx_f)$, as $J(\forall x \in \bar{\pi}(x_f))(x \neq x) = \mathbf{true}$, for $f \in MVars(\psi)$. Finally, for each $x \in SVars(\psi)$, we have $Jx \in JU \setminus \text{Pairs}_{\pi^J}(JU)$, so that $\text{Pairs}_{\pi^J}(\{Jx : x \in SVars(\psi)\}) = \emptyset$. We define I as the $MVars(\psi)$ -variant of J such that $If = Jx_f$, for $f \in MVars(\psi)$. Plainly, $I \models \tau(\psi)$ so that, by Lemma 2, $I \models \psi$ as well.

For the converse direction, let $I = (M^I, \pi^I)$ be a model for ψ . We shall exhibit an interpretation J' which satisfies ψ' . To begin with, we define a new pairing function π^J by putting

$$\pi^J(u, v) =_{\text{Def}} \{\pi_{\text{Kur}}(u, v), \{D_\varphi\}\},$$

for every $u, v \in \mathcal{V}$, where π_{Kur} is the Kuratowski's pairing function and $D_\varphi =_{\text{Def}} \{Ix : x \in SVars(\psi)\}$. Then we define M^J as the $MVars(\psi)$ -variant of the assignment M^I such that $M^J f = \{\pi^J(u, v) : u, v \in \mathcal{V} \text{ and } \pi^I(u, v) \in M^I f\}$, for each $f \in MVars(\psi)$. From Lemma 1, it follows that the interpretation $J = (M^J, \pi^J)$ satisfies ψ . Moreover, we have

$$\text{Pairs}_{\pi^J}(Jz) = \emptyset, \quad (7)$$

for each $z \in SVars(\psi)$. Indeed, if for some $u, v \in \mathcal{V}$ and $z \in SVars(\psi)$ we had $\pi^J(u, v) \in Jz$, then

$$Iz \in D_\varphi \in \{D_\varphi\} \in \{\pi_{\text{Kur}}(u, v), \{D_\varphi\}\} = \pi^J(u, v) \in Jz = Iz,$$

contradicting the regularity axiom of set theory. Next, let $W =_{\text{Def}} \{x_f : f \in MVars(\psi)\} \cup \{U\}$ and let J' be the W -variant of J , where $J'x_f = Jf$, for $f \in MVars(\psi)$, and $J'U = \{Jz : z \in SVars(\psi)\}$. In view of (7), it is an easy matter to verify that

$$J' \models \tau(\psi). \quad (8)$$

From (7), we have immediately that $\text{Pairs}_{\pi^J}(J'z) = \emptyset$, so that

$$J' \models \bigwedge_{z \in SVars(\psi)} (\forall[x, y] \in z)(x \neq x). \quad (9)$$

Likewise, by reasoning much in the same manner as for the proof of (7), one can prove that

$$J' \models \bigwedge_{f \in MVars(\psi)} (\forall x \in \bar{\pi}(x_f))(x \neq x) \wedge \bigwedge_{z \in SVars(\psi)} (z \in \bar{\pi}(U)). \quad (10)$$

From (8), (9), and (10), it follows at once that $J' \models \psi'$, completing the proof that ψ and ψ' are equisatisfiable.

Since the s.p. for \forall_0^π -formulae is in NEXPTIME, as was shown in [6, Section 3.1], it readily follows that the s.p. for normalized $\forall_{0,2}^\pi$ -conjunctions is in NEXPTIME as well. \square

Corollary 1. *The s.p. for $\forall_{0,2}^\pi$ -formulae is in NEXPTIME.*

Proof. Let φ be a satisfiable $\forall_{0,2}^\pi$ -formula. We may assume without loss of generality that all existential simple-prenex $\forall_{0,2}^\pi$ -formulae of the form (4) have already been rewritten in terms of equivalent universal simple-prenex $\forall_{0,2}^\pi$ -formulae of the form (3), so that φ is a propositional combination of universal simple-prenex $\forall_{0,2}^\pi$ -formulae. In addition, by suitably renaming variables, we may assume that all quantified variables in φ are pairwise distinct and that they are also distinct from free variables.

Let $\Sigma_\varphi = \{\psi_1, \dots, \psi_n\}$ be the collection of the universal simple-prenex $\forall_{0,2}^\pi$ -formulae occurring in φ . By traversing the syntax tree of φ , one can find in linear time the propositional skeleton P_φ of φ and a substitution σ from the propositional variables p_1, \dots, p_n of P_φ into Σ_φ , such that $P_\varphi \sigma = \varphi$, where $P_\varphi \sigma$ is the result of substituting each propositional variable p_i in P_φ by the universal simple-prenex $\forall_{0,2}^\pi$ -formula $\sigma(p_i)$. Then to check the satisfiability of φ one can perform the following non-deterministic procedure:

- guess a Boolean valuation v of the propositional variables p_1, \dots, p_n of P_φ such that $v(P_\varphi) = \mathbf{true}$;
- form the $\forall_{0,2}^\pi$ -conjunction

$$\bigwedge_{v(p_i)=\mathbf{true}} \sigma(p_i) \wedge \bigwedge_{v(p_i)=\mathbf{false}} \neg\sigma(p_i); \quad (11)$$

- transform each conjunct

$$\neg(\forall x_1 \in z_1) \dots (\forall x_h \in z_h)(\forall [x_{h+1}, y_{h+1}] \in f_{h+1}) \dots (\forall [x_n, y_n] \in f_n) \delta$$

of the form $\neg\sigma(p_i)$ in (11), where $v(p_i) = \mathbf{false}$, into the equisatisfiable formula

$$\bigwedge_{i=1}^h x_i \in z_i \wedge \bigwedge_{j=h+1}^n [x_j, y_j] \in f_j \wedge \neg\delta.$$

Let φ' be the normalized $\forall_{0,2}^\pi$ -conjunction so obtained. Plainly, $\varphi' \rightarrow \varphi$ is satisfied by any interpretation.

- Check that φ' is satisfiable by a NEXPTIME procedure for normalized $\forall_{0,2}^\pi$ -conjunctions (cf. Theorem 1).

Since φ' can be constructed in non-deterministic linear time, the corollary follows. \square

Next we consider $(\forall_{0,2}^\pi)^{\leq h}$ -formulae, namely $\forall_{0,2}^\pi$ -formulae whose simple-prenex subformulae have quantifier-prefix lengths bounded by the constant $h \geq 0$. By reasoning much as in the proofs of Theorem 1 and Corollary 1, it is immediate to check that the s.p. for $(\forall_{0,2}^\pi)^{\leq h}$ -formulae can be reduced in non-deterministic linear time to the s.p. of $(\forall_{\mathbf{0}}^\pi)^{\leq h}$ -formulae, and thus, by [6, Corollary 4], it can be decided in non-deterministic polynomial time. On the other hand, it is an easy matter to show that the s.p. for $(\forall_{0,2}^\pi)^{\leq h}$ -formulae is NP-hard. Indeed, given a propositional formula Q , consider the $(\forall_{0,2}^\pi)^{\leq 0}$ -formula ψ_Q , obtained from Q by replacing each propositional variable p in Q with the atomic $\forall_{0,2}^\pi$ -formula $x_p \in X$, where X and the x_p 's are distinct set variables. Plainly, Q is propositionally satisfiable if and only if the $\forall_{0,2}^\pi$ -formula ψ_Q is satisfiable. The following lemma summarizes the above considerations.

Lemma 3. *For any integer constant $h \geq 0$, the s.p. for $(\forall_{0,2}^\pi)^{\leq h}$ -formulae is NP-complete.* \square

It is noticeable that, despite of the large collection of set-theoretic constructs which are expressible by $\forall_{0,2}^\pi$ -formulae (see Table 1), some very common map-related operators like domain, range, and map image can not be expressed by $\forall_{0,2}^\pi$ -formulae in full generality, but only in restricted contexts. In the next section we prove that dropping any of such restrictions triggers undecidability.

5 Some undecidable extensions of $\forall_{0,2}^\pi$

In this section we prove the undecidability of any extension of $\forall_{0,2}^\pi$ which allows one to express literals of the form $x \subseteq \text{dom}(f)$. As we will see, analogous undecidability results hold also for similar extensions of $\forall_{0,2}^\pi$ in the case of other map related constructs such as range, map image, and map composition. Our proof will be carried out via a reduction of the *Domino Problem*, a well-known undecidable problem studied in [2] (see also [3]), which asks for a tiling of the quadrant $\mathbb{N} \times \mathbb{N}$ subject to a finite set of constraints.

Definition 1 (Domino problem). A domino system is a triple $\mathbb{D} = (D, H, V)$, where $D = \{d_1, \dots, d_\ell\}$ is a finite nonempty set of domino types, and H and V , respectively the horizontal and vertical compatibility conditions, are two functions which associate to each domino type $d \in D$ a subset of D , respectively $H(d)$ and $V(d)$.

A tiling t for a domino system $\mathbb{D} = (D, H, V)$ is any mapping which associates a domino type in D to each ordered pair of natural numbers in $\mathbb{N} \times \mathbb{N}$. A tiling t is said to be compatible if and only if $t[m+1, n] \in H(t[m, n])$ and $t[m, n+1] \in V(t[m, n])$ for all $n, m \in \mathbb{N}$. The domino problem consists in determining whether a domino system admits a compatible tiling. \square

In order to reformulate the domino problem in set-theoretic terms, we make use of the following set-theoretic variant of Peano systems (see, for instance, [14]).

Definition 2 (Peano systems). Let π be a pairing-function and let $\mathcal{N}, \mathcal{Z}, \mathcal{S}$ be three sets in the von Neumann hierarchy of sets. The tuple $\mathbb{S} = (\mathcal{N}, \mathcal{Z}, \mathcal{S}, \pi)$ is said to be a Peano system if it satisfies the following conditions:

- (P1) \mathcal{N} is a set to which \mathcal{Z} belongs;
- (P2) $\mathcal{S} \subseteq \mathcal{N} \times_{\pi} \mathcal{N}$ is a total function over \mathcal{N} , i.e., a single-valued map with domain \mathcal{N} ;
- (P3) \mathcal{S} is injective;
- (P4) \mathcal{Z} is not in the range of \mathcal{S} ;
- (P5) for each $X \subseteq \mathcal{N}$ the following holds:

$$(\mathcal{Z} \in X \wedge (\forall n \in \mathcal{N})(n \in X \longrightarrow \mathcal{S}n \in X)) \longrightarrow X = \mathcal{N}.$$

The first Peano system was devised by G. Peano himself. It can be characterized as $\mathbb{S}_0 = (\mathcal{N}_0, \mathcal{S}_0, \emptyset, \pi_{\text{Kur}})$, where \mathcal{N}_0 is the minimal set containing the empty set \emptyset and satisfying $(\forall u \in \mathcal{N}_0)(\{u\} \in \mathcal{N}_0)$, and \mathcal{S}_0 is the relation over \mathcal{N}_0 such that $\pi_{\text{Kur}}(u, v) \in \mathcal{S}_0$ if and only if $u \in v$.⁴

The domino problem can be easily reformulated in pure set-theoretic terms. To this purpose, we observe that any tiling t for a domino system induces a partitioning of the integer plane $\mathbb{N} \times \mathbb{N}$, as it associates exactly one domino type to each pair $\langle n, m \rangle \in \mathbb{N} \times \mathbb{N}$. Hence, given a domino system $\mathbb{D} = (\{d_1, \dots, d_\ell\}, H, V)$, the domino problem for \mathbb{D} can be expressed in set-theoretic terms as the problem of deciding whether there exists a partitioning $P = (A_1, \dots, A_\ell)$ of $\mathcal{N} \times_{\pi} \mathcal{N}$, for some fixed Peano system $\mathbb{S} = (\mathcal{N}, \mathcal{Z}, \mathcal{S}, \pi)$, such that for all $u, v, u', v' \in \mathcal{N}$, and for all $1 \leq i, j \leq \ell$ such that $\pi(u, v) \in A_i$ and $\pi(u', v') \in A_j$,

- (D1) if $\pi(u, u') \in \mathcal{S}$ (i.e., u' is the successor of u) and $v = v'$ then $d_j \in H(d_i)$, and
- (D2) if $\pi(v, v') \in \mathcal{S}$ (i.e., v' is the successor of v) and $u = u'$ then $d_j \in V(d_i)$.

Notice that from the properties of Peano systems it follows that if a domino system \mathbb{D} admits a compatible tiling t then we can construct a partitioning of the integer plane which satisfies (D1) and (D2) however the Peano system is chosen.

All instances of the domino problem can be formalized with normalized $\forall_{0,2}^{\pi}$ -conjunctions extended with two positive literals of the form $x \subseteq \text{dom}(f)$, with $x \in S\text{Vars}$ and $f \in M\text{Vars}$, where the obvious semantics for the operator $\text{dom}(\cdot)$ is $I(\text{dom}(f)) =_{\text{def}} \{u \in \mathcal{V} : [u, v] \in I f, \text{ for some } v \in \mathcal{V}\}$, for any interpretation I . In view of the undecidability of the domino problem, this yields the undecidability of the s.p. for the class $\forall_{0,2}^{\pi+2\text{dom}}$ of normalized $\forall_{0,2}^{\pi}$ -conjunctions extended with two positive literals of the form $x \subseteq \text{dom}(f)$, proved in the following theorem.

⁴In the original definition the pairing function was not specified.

Theorem 2. *The s.p. for $\forall_{0,2}^{\pi+2\text{dom}}$, namely the class of normalized $\forall_{0,2}^{\pi}$ -conjunctions extended with two positive literals of the form $x \subseteq \text{dom}(f)$, is undecidable.*

Proof. Let $\mathbb{D} = (D, H, V)$, with $D = \{d_1, \dots, d_\ell\}$, be a domino system. We will show how to construct in polynomial time a formula $\varphi_{\mathbb{D}}$ of $\forall_{0,2}^{\pi+2\text{dom}}$ which is satisfiable if and only if there exists a partitioning of the integer plane which satisfies conditions **(D1)** and **(D2)**, so that the undecidability of the s.p. for $\forall_{0,2}^{\pi+2\text{dom}}$ will follow directly from the undecidability of the domino problem.

Let N, Z be two distinct set variables, and let S be a map variable. In addition, let Q_1, \dots, Q_ℓ be pairwise distinct map variables, which are also distinct from S . These are intended to represent the blocks of the partition of the integer plane induced by a tiling. To enhance the readability of the formula $\varphi_{\mathbb{D}}$ we are about to construct, we introduce some abbreviations which will also make use of some map constructs defined in Table 1. To begin with, we put

$$\text{partition}(Q_1, \dots, Q_\ell; N \times N) \stackrel{\text{Def}}{=} N \times N \subseteq Q_1 \cup \dots \cup Q_\ell \wedge \bigwedge_{i \neq j} (Q_i \cap Q_j = \emptyset).$$

Plainly, for every interpretation I , we have $I \models \text{partition}(Q_1, \dots, Q_\ell; N \times N)$ if and only if (IQ_1, \dots, IQ_ℓ) partitions $I(N \times N)$. Next we define the formulae hor_i and ver_i , for $i = 1, \dots, \ell$, which will encode respectively the horizontal and the vertical compatibility constraints:

$$\text{hor}_i \stackrel{\text{Def}}{=} S^{-1} \circ Q_i \subseteq \bigcup_{d_j \in H(d_i)} Q_j, \quad \text{ver}_i \stackrel{\text{Def}}{=} Q_i \circ S \subseteq \bigcup_{d_j \in V(d_i)} Q_j.$$

Finally, we denote with $\text{is_Peano}(N, Z, S)$ the following formula:

$$\text{is_Peano}(N, Z, S) \stackrel{\text{Def}}{=} Z \in N \wedge \text{bijective}(S) \wedge \text{dom}(S) = N \wedge \text{range}(S) = (N \setminus \{Z\}) \wedge (\forall [x, y] \in S)(x \in y).$$

Notice that $\text{range}(S) = (N \setminus \{Z\})$ is equivalent to $\text{dom}(S^{-1}) = (N \setminus \{Z\})$. In addition, a literal of the form $x = \text{dom}(f)$ can obviously be expressed by the conjunction $(\forall [x', y'] \in f)(x' \in x) \wedge x \subseteq \text{dom}(f)$.

Next we show that the formula $\text{is_Peano}(N, Z, S)$ is satisfiable and correctly characterizes Peano systems, in the sense that if $I \models \text{is_Peano}(N, Z, S)$ for an interpretation I , then (IN, IZ, IS, π^I) is a Peano system. Given any interpretation I such that $IN = \mathcal{N}_0$, $IS = \mathcal{S}_0$, $IZ = \emptyset$, and $\pi^I = \pi_{\text{Kur}}$, $I \models \text{is_Peano}(N, Z, S)$ follows from the very definition of \mathcal{S}_0 , so that $\text{is_Peano}(N, Z, S)$ is satisfiable. In addition, if $I \models \text{is_Peano}(N, Z, S)$ for an interpretation I , it can easily be proved that (IN, IZ, IS, π^I) is a Peano system. Indeed **(P1)**, **(P2)**, **(P3)**, and **(P4)** follow readily from the first four conjuncts of $\text{is_Peano}(N, Z, S)$. Concerning **(P5)**, we proceed by contradiction. Thus, let us assume that there exists a proper subset X of IN such that the following holds

$$IZ \in X \wedge (\forall n, n' \in IN)((n \in X \wedge \pi^I(n, n') \in IS) \longrightarrow n' \in X) \quad (12)$$

and let u be a set in $IN \setminus X$ with minimal rank. We must have $u \neq IZ$, in force of the first conjunct of (12), and thus $u \in \text{range}(IS)$ must hold, as we assumed that I correctly models the conjunct $\text{range}(S) = (N \setminus \{Z\})$ of the formula $\text{is_Peano}(N, Z, S)$. Hence, there must exist a set v such that $\pi(v, u) \in IS$. Since $I \models (\forall [x, y] \in S)(x \in y)$, v must have rank strictly less than u , so that $v \in X$ must hold, as by assumption u has minimal rank in $IN \setminus X$. But (12) would yield $u \in X$, which contradicts our initial assumption $u \in IN \setminus X$.

We are now ready to define the formula $\varphi_{\mathbb{D}}$ of $\forall_{0,2}^{\pi+2\text{dom}}$ intended to express that the domino system $\mathbb{D} = (D, H, V)$ admits a compatible tiling. This is:

$$\varphi_{\mathbb{D}} \stackrel{\text{Def}}{=} \text{is_Peano}(N, Z, S) \wedge \text{partition}(Q_1, \dots, Q_\ell; N \times N) \wedge \bigwedge_{i=1}^{\ell} \text{hor}_i \wedge \bigwedge_{i=1}^{\ell} \text{ver}_i.$$

Observe that $\varphi_{\mathbb{D}}$ can be expanded so as to involve only two literals of the form $x \subseteq \text{dom}(f)$.

We show next that $\varphi_{\mathbb{D}}$ is satisfiable if and only if the domino system \mathbb{D} admits a compatible tiling. Let us first assume that $\varphi_{\mathbb{D}}$ is satisfiable, and let I be a model for $\varphi_{\mathbb{D}}$. Plainly, (IN, IZ, IS, π^I) is a Peano system, as $\text{is_Peano}(\mathbb{N}, \mathbb{Z}, \mathbb{S})$ is a conjunct of $\varphi_{\mathbb{D}}$. In addition, (IQ_1, \dots, IQ_ℓ) partitions $IN \times IN$, since $I \models \text{partition}(Q_1, \dots, Q_\ell; \mathbb{N} \times \mathbb{N})$. It remains to prove that the partition (IQ_1, \dots, IQ_ℓ) is induced by a compatible tiling of the domino system \mathbb{D} , i.e., that properties **(D1)** and **(D2)** hold. Thus let $u, u', v \in IN$ such that $\pi^I(u, v) \in IQ_i$, $\pi^I(u', v) \in IQ_j$, and $\pi^I(u, u') \in IS$, for some $1 \leq i, j \leq \ell$. Plainly $\pi^I(u', v) \in I(S^{-1} \circ Q_i)$, so that from $I \models \text{hor}_i$ it follows $d_j \in H(d_i)$, proving **(D1)**. Likewise, let $u, v, v' \in IN$ be such that $\pi^I(u, v) \in IQ_i$, $\pi^I(u, v') \in IQ_j$, and $\pi^I(v, v') \in IS$, for some $1 \leq i, j \leq \ell$. Thus $\pi^I(u, v') \in I(Q_i \circ S)$, so that from $I \models \text{ver}_i$ we obtain $d_j \in V(d_i)$, proving **(D2)**.

Conversely, let us suppose that \mathbb{D} admits a compatible tiling and let (A_1, \dots, A_ℓ) be the induced partitioning of $\mathcal{N}_0 \times \mathcal{N}_0$ which satisfies **(D1)** and **(D2)**, relative to the Peano system $\mathbb{S}_0 = (\mathcal{N}_0, \emptyset, \mathcal{S}_0, \pi_{\text{Kur}})$. We prove that $\varphi_{\mathbb{D}}$ is satisfied by any interpretation I such that

$$\pi^I = \pi_{\text{Kur}}, \quad IN = \mathcal{N}_0, \quad IZ = \emptyset, \quad IS = \mathcal{S}_0, \quad IQ_i = A_i \text{ (for } i = 1, \dots, \ell).$$

Plainly, I models correctly $\text{is_Peano}(\mathbb{N}, \mathbb{Z}, \mathbb{S})$. In addition, $I \models \text{partition}(Q_1, \dots, Q_\ell; \mathbb{N} \times \mathbb{N})$, as we assumed that $(IQ_1, \dots, IQ_\ell) = (A_1, \dots, A_\ell)$ is a partitioning of $IN \times IN = \mathcal{N}_0 \times \mathcal{N}_0$. Next we prove that I models correctly the conjuncts hor_i of $\varphi_{\mathbb{D}}$, for $i = 1, \dots, \ell$. To this purpose, let u, v be any two sets such that $\pi^I(u, v) \in I(S^{-1} \circ Q_i)$, for some $1 \leq i \leq \ell$. Then, there must exist a set u' such that $\pi^I(u', v) \in IQ_i$, and $\pi^I(u', u) \in IS = \mathcal{S}_0$. Hence $\pi^I(u, v)$ must belong to some $A_j = IQ_j$, for $1 \leq j \leq \ell$, such that $d_j \in H(d_i)$, proving $I \models \text{hor}_i$. Analogously, one can show that $I \models \text{ver}_i$, for $i = 1, \dots, \ell$, thus proving that $I \models \varphi_{\mathbb{D}}$ and in turn concluding the proof of the theorem. \square

Because of the large number of set-theoretic constructs expressible in $\forall_{0,2}^\pi$, the undecidability of various other extensions of normalized $\forall_{0,2}^\pi$ -conjunctions easily follows from Theorem 2.

Corollary 2. *The class of normalized $\forall_{0,2}^\pi$ -conjunctions extended with two literals of any of the following types is undecidable:*

$$x \subseteq \text{range}(f), \quad h \subseteq f \circ g, \quad y \subseteq f[x], \quad (13)$$

where $x, y \in S\text{Vars}$ and $f, g, h \in M\text{Vars}$.

Proof. In view of Theorem 2, it is enough to show that any literal of the form $x \subseteq \text{dom}(f)$ can be expressed with normalized $\forall_{0,2}^\pi$ -conjunctions extended with *one* literal of any of the types (13). Concerning the case of literals of the types $x \subseteq \text{range}(f)$, $h \subseteq f \circ g$ it suffices to observe that $x \subseteq \text{dom}(f)$ is equivalent to each of the two formulae $x \subseteq \text{range}(f^{-1})$ and $\text{id}(x) \subseteq f \circ f^{-1}$, and that map identity $\text{id}(x)$ and map inverse f^{-1} are expressible by $\forall_{0,2}^\pi$ -formulae, as shown in Table 1.

Finally, concerning literals of the form $y \subseteq f[x]$, it is enough to observe that for every set variable R_f distinct from x we have

- $I \models x \subseteq f^{-1}[R_f] \rightarrow x \subseteq \text{dom}(f)$, for every interpretation I ;
- if $I \models x \subseteq \text{dom}(f)$, for some interpretation I , then $J \models x \subseteq f^{-1}[R_f]$, where J is the $\{R_f\}$ -variant of I such that $JR_f = \text{range}(If)$.

Therefore, a $\forall_{0,2}^{\pi+2\text{dom}}$ -formula $\psi \stackrel{\text{Def}}{=} \varphi \wedge x \subseteq \text{dom}(f) \wedge y \subseteq \text{dom}(g)$, where φ is a normalized $\forall_{0,2}^\pi$ -conjunction, is equisatisfiable with $\varphi \wedge x \subseteq f^{-1}[R_f] \wedge y \subseteq g^{-1}[R_g]$, where R_f and R_g are two fresh distinct set variables not occurring in ψ . \square

In the proof of Theorem 1 we provided a reduction of the s.p. for normalized $\forall_{0,2}^\pi$ -conjunctions to the s.p. for \forall_0^π -formulae. Therefore, the undecidability results of Theorem 2 and Corollary 2 hold also for the corresponding extensions of \forall_0^π -formulae.

6 Conclusions and plans for future works

In this paper we presented a quantified sublanguage of set theory, called $\forall_{0,2}^\pi$, which extends the language \forall_0 studied in [4] with quantifiers involving ordered pairs. We reduced its satisfiability problem to the same problem for formulae of the fragment studied in [6]. The resulting decision procedure runs in non-deterministic exponential time. However, if one restricts to formulae with quantifier prefixes of length bounded by a constant, the decision procedure runs in non-deterministic polynomial time. It turns out that such restricted formulae still allow one to express a large number of useful set-theoretic constructs, as reported in Table 1. Finally, we also proved that by slightly extending $\forall_{0,2}^\pi$ -formulae with few literals (at least two) of any of the types $x \subseteq \text{dom}(f)$, $x \subseteq \text{range}(f)$, $x \subseteq f[y]$, and $h \subseteq f \circ g$, one runs into undecidability.

Other extensions of $\forall_{0,2}^\pi$ are to be investigated, in particular those involving the transitive closure of maps. Also, the effects of allowing nesting of quantifiers should be further studied, extending the recent results [16, 17] to our context.

In contrast with description logics, the semantics of our language is *multi-level*, as most of the languages studied in the context of Computable Set Theory. This characteristic may play a central role when applying set-theoretic languages to knowledge representation, with particular reference to the *metamodeling* issue (see [20, 15]), which affects the description logics framework. However, the multi-level feature is limited in $\forall_{0,2}^\pi$ -formulae, since clauses like $f \in x$, $[f, g] \in h$, with x a set variable and f , g , and h map variables, are not expressible in it. In light of this, we intend to investigate extensions of the theory $\forall_{0,2}^\pi$ which also admit constructs of these forms, and study applications of these in the field of knowledge representation.

Finally, we intend to study correlations between our language $\forall_{0,2}^\pi$ and *Disjunctive Datalog* (cf. [11]) in order to use some of the machinery already available for the latter to simplify the implementation of an optimized satisfiability test for the whole fragment $\forall_{0,2}^\pi$, or just for a *Horn-like* restriction of it.

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