

# Complexity of validity for propositional dependence logics

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We study the validity problem for propositional dependence logic, modal dependence logic and extended modal dependence logic. We show that the validity problem for propositional dependence logic is NEXPTIME-complete. In addition, we establish that the corresponding problem for modal dependence logic and extended modal dependence logic is NEXPTIME-hard and in  $\text{NEXPTIME}^{\text{NP}}$ .

## 1 Introduction

Dependencies occur in many scientific disciplines. For example, in physics there are dependencies in experimental data, and in social science they can occur between voting extrapolations. For example, one might want to express whether a value of a certain physical measurement is determined by the values of some other measurements. More concretely, is it the case that in some collection of experimental data, the temperature of some object is completely determined by the solar activity and the distance between the object and the sun. One might also want to know whether the voting pattern of some single constituency always determines the election results.

With the aim to express such dependencies Väänänen introduced first-order *dependence logic* [27] and its modal variant *modal dependence logic* [28]. First-order dependence logic extends first-order logic by novel atomic formulae called *dependence atoms*. Modal dependence logic, in turn, extends modal logic with *propositional dependence atoms*. A dependence atom, denoted by  $=(x_1, \dots, x_n, y)$ , intuitively states that the value of the variable  $y$  is solely determined by the values of the variables  $x_1, \dots, x_n$ . The intuitive meaning of the propositional dependence atom  $\text{dep}(p_1, \dots, p_n, q)$  is that the truth value of the proposition  $q$  is functionally determined by the truth values of the propositions  $p_1, \dots, p_n$ . One of the core ideas in these logics of dependence is the use of team semantics. Väänänen realized that dependencies do not manifest themselves in a single assignment nor in a single point. To manifest dependencies one must look at sets of assignment or collections of points. These sets of assignments or points are called teams. Thus whereas in the standard semantics for first-order logic formulae are evaluated with respect to first-order models and assignments, in team semantics of dependence logic formulae are evaluated with respect to first-order models and sets of assignments. Analogously, in team semantics for modal logic formulae are evaluated with respect to Kripke models and sets of points. For example, the formula

$$=(x_{\text{activity}}, x_{\text{dist}}, x_{\text{temp}}),$$

where the values of the variables  $x_{\text{activity}}$ ,  $x_{\text{dist}}$ , and  $x_{\text{temp}}$  range over the magnitude of solar activity, distance to the sun, and temperature, respectively, expresses that in some set of data the temperature is completely determined by the solar activity and the distance to the sun. Sets of data are captured by teams. Each assignment in a team corresponds to one record of data.

Team semantics was originally defined by Hodges [14] as a means to obtain compositional semantics for the independence-friendly logic of Hintikka and Sandu [13]. Later on Väänänen adopted team semantics as a central notion for his dependence logic.

Modal dependence logic was the first step in combining functional dependence and modal logic. The logic however lacks the ability to express temporal dependencies; there is no mechanism in modal dependence logic to express dependencies that occur between different points of the model. This is due to the restriction that only proposition symbols are allowed in the dependence atoms of modal dependence logic. To overcome this defect Ebbing et al. [6] introduced the *extended modal dependence logic* by extending the scope of dependence atoms to arbitrary modal formulae, i.e., dependence atoms in extended modal dependence logic are of the form  $\text{dep}(\varphi_1, \dots, \varphi_n, \psi)$ , where  $\varphi_1, \dots, \varphi_n, \psi$  are formulae of modal logic. For example when interpreted in a temporal model, the formula

$$\text{dep}(\diamond_P q, \diamond_P \diamond_P q, \diamond_P \diamond_P \diamond_P q, q)$$

expresses that the truth of  $q$ , at this moment, only depends of the truth of  $q$  in the previous 3 time steps.

It was shown in [6] that extended modal dependence logic is strictly more expressive than modal dependence logic. Furthermore Hella et al. [12] established that exactly the properties of teams that are downward closed and closed under the so-called team  $k$ -bisimulation, for some finite  $k$ , are definable in extended modal dependence logic. The characterization of Hella et al. truly demonstrates the naturality of extended modal dependence logic. In recent years the research around modal dependence logic has bloomed, for recent work see e.g. [6–9, 20, 21, 24].

Team semantics in propositional context is also closely related to the inquisitive logic of Groenendijk [11]. In inquisitive logic the meaning of formulae is defined on sets of assignments for proposition symbols. This connection between propositional dependence logic and inquisitive logic has already been noted in the recent Ph.D. thesis of Fan Yang [29]. For recent work related to inquisitive logic, see e.g. [3, 23].

In this paper we study the computational complexity of the validity problem for propositional dependence logic, modal dependence logic and extended modal dependence logic. The study of computational complexity of the satisfiability problem and the model checking problem for logics of dependence has been very active. For research related to fragments of first-order dependence logic and related formalisms see [2, 10, 15, 16, 26]. For work on variants of propositional and modal dependence logics see [6, 7, 20, 24, 29]. However, there is not much research done on the validity problem of these logics. We wish to mend this shortcoming. Note that since the logics of dependence are not closed under negation, the traditional connection between the satisfiability problem and the validity problem fails. In this article we establish that the validity problem for propositional dependence logic is NEXPTIME-complete. In addition, we obtain that the corresponding problem for modal dependence logic and extended modal dependence logic is contained in  $\text{NEXPTIME}^{\text{NP}}$ .

The article is structured as follows. In section 2 we define the basic concepts and results relevant to this article. In section 3 we introduce a variant of QBF, called dependency quantified Boolean formulae, for which the decision problem whether a given formula is true is NEXPTIME-complete. We start Section 4 with compact definitions of satisfiability, validity and model checking in the context of team semantics. The rest of the section is devoted for the study of the complexity of the validity problem for propositional dependence logic. In Section 5 we consider the validity problem of modal dependence logic and extended modal dependence logic.

## 2 Preliminaries

In this section we define the basic concepts and results relevant to this article. We assume that the reader is familiar with propositional logic PL and modal logic ML.

### 2.1 Propositional logics

Let  $\mathbb{Z}_+$  denote the set of positive integers, and let  $\text{PROP} = \{p_i \mid i \in \mathbb{Z}_+\}$  be the set of exactly all *proposition symbols*. Let  $D$  be a finite, possibly empty, subset of  $\text{PROP}$ . A function  $s : D \rightarrow \{0, 1\}$  is called an *assignment*. A set  $X$  of assignments  $s : D \rightarrow \{0, 1\}$  is called a *propositional team*. The set  $D$  is the *domain* of  $X$ . Note that the empty team  $\emptyset$  does not have a unique domain; any subset of  $\text{PROP}$  is a domain of the empty team.

Most of the logics considered in this article are not closed under negation, thus we adopt the convention that a syntax of a logic is always defined in negation normal form, i.e., negations are allowed only in front of proposition symbols. This convention is widely used in the dependence logic community. Formula that is not in negation normal form is regarded as a shorthand for the formula obtained by pulling all the negations to the atomic level.

Let  $\Phi$  be a set of proposition symbols. The syntax for propositional logic  $\text{PL}(\Phi)$  is defined as follows.

$$\varphi ::= p \mid \neg p \mid (\varphi \wedge \psi) \mid (\varphi \vee \psi),$$

where  $p \in \Phi$ . We will now give the team semantics for propositional logic. As we will see below, the team semantics and the ordinary semantics for propositional logic defined via assignments, in a rather strong sense, coincide.

**Definition 2.1.** *Let  $\Phi$  be a set of atomic propositions and let  $X$  be a propositional team. The satisfaction relation  $X \models \varphi$  is defined as follows. Note that, we always assume that the proposition symbols that occur in  $\varphi$  are also in the domain of  $X$ .*

$$\begin{aligned} X \models p &\Leftrightarrow \forall s \in X : s(p) = 1. \\ X \models \neg p &\Leftrightarrow \forall s \in X : s(p) = 0. \\ X \models (\varphi \wedge \psi) &\Leftrightarrow X \models \varphi \text{ and } X \models \psi. \\ X \models (\varphi \vee \psi) &\Leftrightarrow Y \models \varphi \text{ and } Z \models \psi, \text{ for some } Y, Z \text{ such that } Y \cup Z = X. \end{aligned}$$

**Proposition 2.2** ([24]). *Let  $\varphi$  be a formula of propositional logic and let  $X$  be a propositional team. Then*

$$X \models \varphi \quad \text{iff} \quad \forall s \in X : s \models_{\text{PL}} \varphi.$$

Here  $\models_{\text{PL}}$  refers to the ordinary satisfaction relation of propositional logic defined via assignments.

The syntax of *propositional dependence logic*  $\text{PD}(\Phi)$  is obtained by extending the syntax of  $\text{PL}(\Phi)$  by the grammar rule

$$\varphi ::= \text{dep}(p_1, \dots, p_n, q),$$

where  $p_1, \dots, p_n, q \in \Phi$ . The intuitive meaning of the *propositional dependence atom*  $\text{dep}(p_1, \dots, p_n, q)$  is that the truth value of the proposition symbol  $q$  solely depends on the truth values of the proposition symbols  $p_1, \dots, p_n$ . The semantics for the propositional dependence atom is defined as follows:

$$\begin{aligned} X \models \text{dep}(p_1, \dots, p_n, q) &\Leftrightarrow \forall s, t \in X : s(p_1) = t(p_1), \dots, s(p_n) = t(p_n) \\ &\quad \text{implies that } s(q) = t(q). \end{aligned}$$

The next proposition is very useful. The proof is very easy and the result is stated, for example, in [29].

**Proposition 2.3** (Downwards closure). *Let  $\varphi$  be a formula of propositional dependence logic and let  $Y \subseteq X$  be propositional teams. Then  $X \models \varphi$  implies  $Y \models \varphi$ .*

## 2.2 Modal logics

In this article, in order to keep the notation light, we restrict our attention to mono-modal logic, i.e., to modal logic with just two modal operators ( $\diamond$  and  $\square$ ). However this is not really a restriction, since the definitions, results, and proofs of this article generalize, in a straightforward manner, to handle also the poly-modal case.

Let  $\Phi$  be a set of atomic propositions. The set of formulae for *standard mono-modal logic*  $\text{ML}(\Phi)$  is generated by the following grammar

$$\varphi ::= p \mid \neg p \mid (\varphi \wedge \psi) \mid (\varphi \vee \psi) \mid \diamond \varphi \mid \square \varphi,$$

where  $p \in \Phi$ . Note that, since negations are allowed only in front of proposition symbols,  $\square$  and  $\diamond$  are *not* interdefinable. The syntax of *modal logic with intuitionistic disjunction*  $\text{ML}(\otimes)(\Phi)$  is obtained by extending the syntax of  $\text{ML}(\Phi)$  by the grammar rule

$$\varphi ::= (\varphi \otimes \psi).$$

The *team semantics for modal logic* is defined via *Kripke models* and *teams*. In the context of modal logic, teams are subsets of the domain of the model.

**Definition 2.4.** *Let  $\Phi$  be a set of atomic proposition symbols. A Kripke model  $\mathbf{K}$  over  $\Phi$  is a tuple  $\mathbf{K} = (W, R, V)$ , where  $W$  is a nonempty set of worlds,  $R \subseteq W \times W$  is a binary relation, and  $V : \Phi \rightarrow \mathcal{P}(W)$  is a valuation. A subset  $T$  of  $W$  is called a team of  $\mathbf{K}$ . Furthermore, define that*

$$\begin{aligned} R[T] &:= \{w \in W \mid \exists v \in T \text{ such that } vRw \text{ holds}\}, \\ R^{-1}[T] &:= \{w \in W \mid \exists v \in T \text{ such that } wRv \text{ holds}\}. \end{aligned}$$

For teams  $T, S \subseteq W$ , we write  $T[R]S$  if  $S \subseteq R[T]$  and  $T \subseteq R^{-1}[S]$ . Thus,  $T[R]S$  holds if and only if for every  $w \in T$  there exists some  $v \in S$  such that  $wRv$ , and for every  $v \in S$  there exists some  $w \in T$  such that  $wRv$ .

We are now ready to define the team semantics for modal logic and modal logic with intuitionistic disjunction. Similar to the case of propositional logic, the team semantics of modal logic, in a rather strong sense, coincides with the traditional semantics of modal logic defined via pointed Kripke models.

**Definition 2.5.** *Let  $\mathbf{K}$  be a Kripke model. The satisfaction relation  $\mathbf{K}, T \models \varphi$  for  $\text{ML}$  is defined as follows.*

$$\begin{aligned} \mathbf{K}, T \models p &\Leftrightarrow w \in V(p) \text{ for every } w \in T. \\ \mathbf{K}, T \models \neg p &\Leftrightarrow w \notin V(p) \text{ for every } w \in T. \\ \mathbf{K}, T \models (\varphi \wedge \psi) &\Leftrightarrow \mathbf{K}, T \models \varphi \text{ and } \mathbf{K}, T \models \psi. \\ \mathbf{K}, T \models (\varphi \vee \psi) &\Leftrightarrow \mathbf{K}, T_1 \models \varphi \text{ and } \mathbf{K}, T_2 \models \psi \text{ for some } T_1, T_2 \text{ such that } T_1 \cup T_2 = T. \\ \mathbf{K}, T \models \diamond \varphi &\Leftrightarrow \mathbf{K}, T' \models \varphi \text{ for some } T' \text{ such that } T[R]T'. \\ \mathbf{K}, T \models \square \varphi &\Leftrightarrow \mathbf{K}, T' \models \varphi, \text{ where } T' = R[T]. \end{aligned}$$

For  $\text{ML}(\otimes)$  we have the following additional clause:

$$\mathbf{K}, T \models (\varphi \otimes \psi) \Leftrightarrow \mathbf{K}, T \models \varphi \text{ or } \mathbf{K}, T \models \psi.$$

**Proposition 2.6** ([24]). *Let  $\varphi \in \text{ML}$ ,  $\mathbf{K}$  be a Kripke model and  $T$  a team of  $\mathbf{K}$ . Then*

$$\mathbf{K}, T \models \varphi \quad \text{iff} \quad \forall w \in T : \mathbf{K}, w \models_{\text{ML}} \varphi.$$

Here  $\models_{\text{ML}}$  refers to the ordinary satisfaction relation of modal logic defined via pointed Kripke models.

The syntax for *modal dependence logic*  $\text{MDL}(\Phi)$  is obtained by extending the syntax of  $\text{ML}(\Phi)$  by propositional dependence atoms

$$\varphi ::= \text{dep}(p_1, \dots, p_n, q),$$

where  $p_1, \dots, p_n, q \in \Phi$ , whereas the syntax for *extended modal dependence logic*  $\text{EMDL}(\Phi)$  is obtained by extending the syntax of  $\text{ML}(\Phi)$  by *modal dependence atoms*

$$\varphi ::= \text{dep}(\varphi_1, \dots, \varphi_n, \psi),$$

where  $\varphi_1, \dots, \varphi_n, \psi$  are  $\text{ML}(\Phi)$ -formulae.

The intuitive meaning of the modal dependence atom  $\text{dep}(\varphi_1, \dots, \varphi_n, \psi)$  is that the truth value of the formula  $\psi$  is completely determined by the truth values of the formulae  $\varphi_1, \dots, \varphi_n$ . The semantics for these dependence atoms is defined as follows.

$$\begin{aligned} \mathbf{K}, T \models \text{dep}(\varphi_1, \dots, \varphi_n, \psi) \quad \Leftrightarrow \quad & \forall w, v \in T : \bigwedge_{i=1}^n (\mathbf{K}, \{w\} \models \varphi_i \Leftrightarrow \mathbf{K}, \{v\} \models \varphi_i) \\ & \text{implies } (\mathbf{K}, \{w\} \models \psi \Leftrightarrow \mathbf{K}, \{v\} \models \psi). \end{aligned}$$

The following proposition for  $\text{MDL}$  and  $\text{ML}(\otimes)$  is due to [28] and [8], respectively. For  $\text{EMDL}$  it follows by the fact that  $\text{EMDL}$  translates into  $\text{ML}(\otimes)$ , see [6].

**Proposition 2.7** (Downwards closure). *Let  $\varphi$  be a formula of  $\text{ML}(\otimes)$  or  $\text{EMDL}$ , let  $\mathbf{K}$  be a Kripke model and let  $S \subseteq T$  be teams of  $\mathbf{K}$ . Then  $\mathbf{K}, T \models \varphi$  implies  $\mathbf{K}, S \models \varphi$ .*

The standard concept of bisimulation from modal logic can be lifted, in a straightforward manner, to handle team semantics. Below when stating that  $\mathbf{K}, w$  and  $\mathbf{K}, w'$  are bisimilar, we refer to the standard bisimulation of modal logic, for a definition see, e.g., [1].

**Definition 2.8.** *Let  $\mathbf{K}$  and  $\mathbf{K}'$  be Kripke models and let  $T$  and  $T'$  be teams of  $\mathbf{K}$  and  $\mathbf{K}'$ , respectively. We say that  $\mathbf{K}, T$  and  $\mathbf{K}', T'$  are team bisimilar if*

1. *for every  $w \in T$  there exists some  $w' \in T'$  such that  $\mathbf{K}, w$  and  $\mathbf{K}', w'$  are bisimilar, and*
2. *for every  $w' \in T'$  there exists some  $w \in T$  such that  $\mathbf{K}, w$  and  $\mathbf{K}', w'$  are bisimilar.*

**Theorem 2.9** ([12]). *If  $\mathbf{K}, T$  and  $\mathbf{K}', T'$  are team bisimilar, then for every formula  $\varphi \in \text{ML}(\otimes)$  (and also for every  $\varphi \in \text{EMDL}$ )*

$$\mathbf{K}, T \models \varphi \quad \Leftrightarrow \quad \mathbf{K}', T' \models \varphi.$$

The following result is stated in [29]. It also follows by a direct team bisimulation argument.

**Corollary 2.10.** *Truth of  $\text{ML}(\otimes)$ -formulae is preserved under taking disjoint unions, i.e., if  $\mathbf{K}$  and  $\mathbf{K}'$  are Kripke models,  $T$  is a team of  $\mathbf{K}$  and  $\mathbf{K} \uplus \mathbf{K}'$  denotes the disjoint union of  $\mathbf{K}$  and  $\mathbf{K}'$  then*

$$\mathbf{K}, T \models \varphi \quad \Leftrightarrow \quad \mathbf{K} \uplus \mathbf{K}', T \models \varphi,$$

for every  $\varphi \in \text{ML}(\otimes)$ .

### 3 Dependency quantified Boolean formulae

Deciding whether a given quantified Boolean formula is true is a canonical PSPACE-complete problem. Dependency quantified Boolean formulae introduced by Peterson et al. [22] are variants of quantified Boolean formulae for which the corresponding decision problem is NEXPTIME-complete. In this section we give a definition of quantified Boolean formulae and dependency quantified Boolean formulae suitable for our needs.

A *Boolean variable* is a variable that is assigned either true or false. Let  $\text{BVAR} = \{\gamma_i \mid i \in \mathbb{Z}_+\}$  be the set of exactly all Boolean variables. *Boolean formulae*  $\varphi$  are built from Boolean variables by the following grammar:

$$\varphi ::= \alpha \mid \neg\alpha \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi),$$

where  $\alpha \in \text{BVAR}$ . A formula

$$\psi = Q_1\alpha_1 Q_2\alpha_2 \dots Q_n\alpha_n\varphi,$$

where  $Q_i \in \{\forall, \exists\}$ , for each  $i \leq n$ , is called a *quantified Boolean formula*, if  $\varphi$  is a Boolean formula and  $\psi$  does not have free variables. We let QBF denote the set of all quantified Boolean formulae. Semantics for Boolean formulae and quantified Boolean formulae is defined via assignments  $s : \text{BVAR} \rightarrow \{0, 1\}$  in the obvious way. We define that

$$\text{TQBF} = \{\varphi \in \text{QBF} \mid \varphi \text{ is true}\}.$$

**Theorem 3.1** ([25]). *The membership problem of TQBF is PSPACE-complete.*

We call a formula

$$\psi = \forall\alpha_1 \dots \forall\alpha_n \exists\beta_1 \dots \exists\beta_k \varphi$$

a *simple quantified Boolean formula*, if  $\varphi$  is a Boolean formula,  $\psi$  does not have free variables and each variable quantified in  $\psi$  is quantified exactly once. Let  $P_1, \dots, P_k \subseteq \{\alpha_1, \dots, \alpha_n\}$ . We call the tuple  $(P_1, \dots, P_k)$  a *constraint* for  $\psi$ . If  $P_1 \subseteq P_2 \subseteq \dots \subseteq P_k$ , we call the constraint *simple*. The idea here is that, for each  $i \leq k$ , the value assigned for the existentially quantified Boolean variable  $\beta_i$  may only depend on the values given to the universally quantified Boolean variables in the set  $P_i$ . Thus, the intuition is that the simple quantified Boolean formula

$$\forall\alpha_1 \forall\alpha_2 \exists\beta_1 \exists\beta_2 \theta$$

is true under the constraint  $(\{\alpha_1\}, \{\alpha_2\})$ , if  $\theta$  can be made true such that the dependencies  $\text{dep}(\alpha_1, \beta_1)$  and  $\text{dep}(\alpha_2, \beta_2)$  hold. The formal definition is given below.

**Definition 3.2.** *Let  $\psi = \forall\alpha_1 \dots \forall\alpha_n \exists\beta_1 \dots \exists\beta_k \varphi$  be a simple quantified Boolean formula and  $(P_1, \dots, P_k)$  a constraint for  $\psi$ . We say that  $\psi$  is true under the constraint  $(P_1, \dots, P_k)$ , if there exists a function  $f_i : \{0, 1\}^{|P_i|} \rightarrow \{0, 1\}$ , for each  $i \leq k$ , such that for each assignment  $s : \{\alpha_1, \dots, \alpha_n\} \rightarrow \{0, 1\}$*

$$s' \models \varphi,$$

where  $s'$  is the modified assignment defined as follows:

$$s'(\alpha) := \begin{cases} f_i(s(P_i)) & \text{if } \alpha = \beta_i \text{ and } i \leq k, \\ s(\alpha) & \text{otherwise.} \end{cases}$$

Here  $s(P_i)$  is a shorthand notation for  $(s(\gamma_{i_1}), \dots, s(\gamma_{i_t}))$ , where  $\gamma_{i_1}, \dots, \gamma_{i_t}$  are exactly the Boolean variables in  $P_i$  ordered such that  $i_j < i_{j+1}$ , for each  $j < t$ .

It is easy to see that there is a close connection between quantified Boolean formulae and simple quantified Boolean formulae with simple constraints; there exists a polynomial time computable function  $F$  that associates each quantified Boolean formula to an equivalent simple quantified Boolean formula with a simple constraint, and vice versa. The equivalent quantified Boolean formula is obtained from a simple quantified Boolean formula with a simple constraint by reordering the quantification of variables. The constraint determines the order of quantifiers.

We define that a dependency quantified Boolean formula is a pair  $(\psi, \vec{P})$  where  $\psi$  is a simple quantified Boolean formula and  $\vec{P}$  is a constraint for  $\psi$ . We let DQBF denote the set of all dependency quantified Boolean formulae. We define that

$$\text{TDQBF} = \{(\psi, \vec{P}) \in \text{DQBF} \mid \psi \text{ is true under the constraint } \vec{P}\}.$$

**Theorem 3.3** ([22]). *The membership problem of TDQBF is NEXPTIME-complete.*

## 4 Computational complexity of propositional dependence logics

Computational complexity of the satisfiability problem and the model checking problem for variants of propositional and modal dependence logics have been thoroughly studied, see e.g., [6, 7, 10, 20, 24, 29]. However there is not much research done on the validity problem of these logics. Note that since the logics of dependence are not closed under negation the traditional connection between the satisfiability problem and the validity problem fails.

### 4.1 Satisfiability, validity and model checking in team semantics

We start by defining satisfiability and validity in the context of team semantics.

A formula  $\varphi$  of propositional dependence logic is said to be *satisfiable*, if there exists a propositional team  $X$  such that  $X \models \varphi$ . A formula  $\varphi$  of propositional dependence logic is said to be *valid*, if  $X \models \varphi$  holds for all teams  $X$  such that the proposition symbols of  $\varphi$  are in the domain of  $X$ . Analogously, a formula  $\psi$  of EMDL (or  $\text{ML}(\otimes)$ ) is said to be *satisfiable*, if there exists a Kripke model  $K$  and a team  $T$  of  $K$  such that  $K, T \models \psi$ . A formula  $\psi$  of EMDL (or  $\text{ML}(\otimes)$ ) is said to be *valid*, if  $K, T \models \psi$  holds for every Kripke model  $K$  (such that the proposition symbols in  $\psi$  are mapped by the valuation of  $K$ ) and every team  $T$  of  $K$ .

The satisfiability problem and the validity problem for these logics is defined in the obvious manner. Given a binary encoding of a formula of a given logic, decide whether the formula is satisfiable (valid, respectively). The variant of the model checking problem, we are concerned in this article is the following. Given binary encodings of a formula  $\varphi$  of propositional dependence logic and of a (finite) propositional team  $X$ , decide whether  $X \models \varphi$ . The corresponding problem for modal logics is defined as follows. Given binary encodings of a formula  $\psi$  of EMDL (or  $\text{ML}(\otimes)$ ), of a finite Kripke model  $K$  and of a team  $T$  of  $K$ , decide whether  $K, T \models \psi$ .

### 4.2 The validity problem of propositional dependence logic

The complexity of the satisfiability problem for PL and PD is known to coincide; both are NP-complete. The result for PL is due to Cook [5] and Levin [18]. For PD, the NP-hardness follows directly from the result of Cook and Levin, and the inclusion to NP follows from the work of Lohmann and Vollmer [20].

A natural question then arises: Is there a similar connection between the validity problem of PL and that of PD? Since the syntax of propositional logic is closed under taking negations, it follows that the

validity problem for PL is *coNP*-complete. However, since the syntax of propositional dependence logic is not closed under taking negations, the corresponding connection between the satisfiability problem and the validity problem of PD fails. This indicates that there might not be any direct connection between the validity problem of PL and that of PD. In fact, as we will see, the validity problem for PD is much harder than the corresponding problem for PL. Surprisingly, we are able to show that the validity problem for PD is *NEXPTIME*-complete.

We shall first show that the validity problem for PD is in *NEXPTIME*. To that end, we use the following result concerning the model checking problem of PD.

**Theorem 4.1** ([7]). *The model checking problem for PD is NP-complete.*

Let  $D$  be a finite set of proposition symbols. By  $X_{\max D}$  we denote the set of all assignments  $s : D \rightarrow \{0, 1\}$ . The following lemma follows directly from the fact that PD is downward closed, i.e., Proposition 2.3.

**Lemma 4.2.** *Let  $\varphi$  be a formula of PD and let  $D$  be the set of proposition symbols occurring in  $\varphi$ . Then  $\varphi$  is valid if and only if  $X_{\max D} \models \varphi$ .*

**Lemma 4.3.** *The validity problem for PD is in NEXPTIME.*

*Proof.* Let  $\varphi$  be a PD-formula. Let  $D$  be the set of proposition symbols occurring in  $\varphi$ . Now, by Lemma 4.2,  $\varphi$  is valid if and only if  $X_{\max D} \models \varphi$ . The size of  $X_{\max D}$  is  $2^{\|D\|}$  and thus  $\leq 2^{\|\varphi\|}$ . Therefore  $X_{\max D}$  can be clearly constructed from  $\varphi$  in exponential time. By Theorem 4.1, there exists an NP algorithm (with respect to  $\|X_{\max D}\| + \|\varphi\|$ ) for checking whether  $X_{\max D} \models \varphi$ . Clearly this algorithm works in *NEXPTIME* with respect to the size of  $\varphi$ . Therefore, we conclude that the validity problem for PD is in *NEXPTIME*.  $\square$

We will then show that the validity problem for PD is *NEXPTIME*-hard. We give a reduction from TDQBF to the validity problem of PD.

**Lemma 4.4.** *The validity problem for PD is NEXPTIME-hard.*

*Proof.* We will give a reduction from the truth problem of dependency quantified Boolean formulae to the validity problem of PD. Since Boolean variables and proposition symbols in the context of PD are essentially the same, we will in this proof treat Boolean variables as proposition symbols, and vice versa. Consequently, we may treat quantifier free Boolean formulae as formulae of propositional logic, and vice versa.

We will associate each DQBF-formula  $\mu$  with a corresponding PD formula  $\varphi_\mu$ . Let

$$\mu = (\forall \alpha_1 \dots \forall \alpha_n \exists \beta_1 \dots \exists \beta_k \psi, (P_1, \dots, P_k))$$

be a DQBF-formula. For each set of Boolean variables  $P_i$ ,  $i \leq k$ , we stipulate that  $P_i = \{\alpha_{i_1}, \dots, \alpha_{i_{n_i}}\}$ . We then denote by  $D_\mu$  the set of Boolean variables in  $\mu$ , i.e.,  $D_\mu := \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_k\}$ . Recall that we treat Boolean variables also as proposition symbols. Let

$$\varphi_\mu := \psi \vee \bigvee_{i \leq k} \text{dep}(\alpha_{i_1}, \dots, \alpha_{i_{n_i}}, \beta_i).$$

We will show that  $\mu$  is true (i.e.,  $\mu \in \text{TDQBF}$ ) if and only if the corresponding PD-formula  $\varphi_\mu$  is valid. Since TDQBF is *NEXPTIME*-complete and  $\varphi_\mu$  is polynomial with respect to  $\mu$ , it follows that the validity problem for PD is *NEXPTIME*-hard. By Lemma 4.2, it is enough to show that  $\mu$  is true if and only if  $X_{\max D_\mu} \models \varphi_\mu$ .



Assume first that  $\mu$  is true, i.e., that  $\forall \alpha_1 \dots \forall \alpha_n \exists \beta_1 \dots \exists \beta_k \psi$  is true under the constraint  $(P_1, \dots, P_k)$ . Therefore, for each  $i \leq k$ , there exists a function  $f_i : \{0, 1\}^{|P_i|} \rightarrow \{0, 1\}$  such that

$$\text{for every assignment } s : \{\alpha_1, \dots, \alpha_n\} \rightarrow \{0, 1\} : s' \models \psi, \quad (1)$$

where  $s'$  is the modified assignment defined as follows:

$$s'(\alpha) := \begin{cases} f_i(s(P_i)) & \text{if } \alpha = \beta_i \text{ and } i \leq k, \\ s(\alpha) & \text{otherwise.} \end{cases}$$

Our goal is to show that

$$X_{\max D_\mu} \models \psi \vee \bigvee_{i \leq k} \text{dep}(\alpha_{i_1}, \dots, \alpha_{i_{n_i}}, \beta_i).$$

It suffices to show that there exist some  $Y, Z_1, \dots, Z_k \subseteq X_{\max D_\mu}$  such that  $Y \cup Z_1 \cup \dots \cup Z_k = X_{\max D_\mu}$ ,  $Y \models \psi$ , and  $Z_i \models \text{dep}(\alpha_{i_1}, \dots, \alpha_{i_{n_i}}, \beta_i)$ , for each  $i \leq k$ . We define the team  $Z_i$ , for each  $i \leq k$ , by using the function  $f_i$ . We let  $Z_i := \{s \in X_{\max D_\mu} \mid s(\beta_i) \neq f_i(s(\alpha_{i_1}), \dots, s(\alpha_{i_{n_i}}))\}$ , for each  $i \leq k$ . Now, since Boolean variables have only 2 possible values, we conclude that, for each  $i \leq k$ ,  $Z_i \models \text{dep}(\alpha_{i_1}, \dots, \alpha_{i_{n_i}}, \beta_i)$ . Thus

$$\bigcup_{1 \leq i \leq k} Z_i \models \bigvee_{i \leq k} \text{dep}(\alpha_{i_1}, \dots, \alpha_{i_{n_i}}, \beta_i). \quad (2)$$

Note that  $s(\beta_i) = f_i(s(\alpha_{i_1}), \dots, s(\alpha_{i_{n_i}}))$  holds for every  $s \in (X_{\max D_\mu} \setminus Z_i)$  and every  $i \leq k$ . Define then that

$$Y := X_{\max D_\mu} \setminus \bigcup_{1 \leq i \leq k} Z_i.$$

Clearly, for every  $s \in Y$  and  $i \leq k$ , it holds that  $s(\beta_i) = f_i(s(\alpha_{i_1}), \dots, s(\alpha_{i_{n_i}}))$ . Thus from (1), it follows that  $s \models \psi$ , for every  $s \in Y$ . Since  $\psi$  is a PL formula, we conclude by Proposition 2.2 that  $Y \models \psi$ . From this together with (2), we conclude that  $X_{\max D_\mu} \models \varphi_\mu$ .

Assume then that  $X_{\max D_\mu} \not\models \varphi_\mu$ . Therefore

$$Y \not\models \bigvee_{i \leq k} \text{dep}(\alpha_{i_1}, \dots, \alpha_{i_{n_i}}, \beta_i)$$

and  $Z \not\models \psi$ , for some  $Y$  and  $Z$  such that  $Y \cup Z = X_{\max D_\mu}$ . Hence there exist some  $Y_1, \dots, Y_k, Z$  such that  $Y_1 \cup \dots \cup Y_k \cup Z = X_{\max D_\mu}$ ,  $Z \models \psi$ , and  $Y_i \models \text{dep}(\alpha_{i_1}, \dots, \alpha_{i_{n_i}}, \beta_i)$  for each  $i \leq k$ . Assume that we have picked  $Y_1, \dots, Y_k, Z$  such that  $Z$  is minimal. We will show that then  $Z \models \text{dep}(\alpha_{i_1}, \dots, \alpha_{i_{n_i}}, \beta_i)$ , for each  $i \leq k$ . Assume for the sake of contradiction that, for some  $i \leq k$ , there exist  $s, t \in Z$  such that

$$s(\alpha_{i_1}) = t(\alpha_{i_1}), \dots, s(\alpha_{i_{n_i}}) = t(\alpha_{i_{n_i}}) \text{ but } s(\beta_i) \neq t(\beta_i).$$

Now clearly either  $Y_i \cup \{s\} \models \text{dep}(\alpha_{i_1}, \dots, \alpha_{i_{n_i}}, \beta_i)$  or  $Y_i \cup \{t\} \models \text{dep}(\alpha_{i_1}, \dots, \alpha_{i_{n_i}}, \beta_i)$ . This contradicts the fact that  $Z$  was assumed to be minimal.

We will then show that for every  $a_1, \dots, a_n \in \{0, 1\}$  there exists some assignment  $s$  in  $Z$  that expands

$$(\alpha_1, \dots, \alpha_n) \mapsto (a_1, \dots, a_n).$$

Let  $a_1, \dots, a_n \in \{0, 1\}$ . Now, for every  $i \leq k$ , since  $Y_i \models \text{dep}(\alpha_{i_1}, \dots, \alpha_{i_{n_i}}, \beta_i)$ , it follows that for any two  $s', s'' \in Y_i$  that expand  $(\alpha_1, \dots, \alpha_n) \mapsto (a_1, \dots, a_n)$ , it holds that  $s'(\beta_i) = s''(\beta_i)$ . Thus, for each  $i \leq k$ , there exists a truth value  $b_i \in \{0, 1\}$  such that there is no expansions of  $(\alpha_1, \dots, \alpha_n, \beta_i) \mapsto (a_1, \dots, a_n, b_i)$  in  $Y_i$ . Therefore, the assignment  $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_k) \mapsto (a_1, \dots, a_n, b_1, \dots, b_k)$  is not in  $Y_i$ , for any  $i \leq k$ . Thus the assignment  $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_k) \mapsto (a_1, \dots, a_n, b_1, \dots, b_k)$  is in  $Z$ . Hence, for every  $a_1, \dots, a_n \in \{0, 1\}$ , there exists some expansion of  $(\alpha_1, \dots, \alpha_n) \mapsto (a_1, \dots, a_n)$  in  $Z$ .

Now, for each  $i \leq k$ , we define the function  $f_i : \{0, 1\}^{|\beta_i|} \rightarrow \{0, 1\}$  as follows. We define that

$$f_i(b_1, \dots, b_{|\beta_i|}) := s(\beta_i),$$

where  $s$  is an assignment in  $Z$  that expands  $(\alpha_{i_1}, \dots, \alpha_{i_{n_i}}) \mapsto (b_1, \dots, b_{|\beta_i|})$ . Since  $Z \models \text{dep}(\alpha_{i_1}, \dots, \alpha_{i_{n_i}}, \beta_i)$ , for each  $i \leq k$ , the functions  $f_i$  are well defined. Now since  $\psi$  is syntactically a PL formula and since  $Z \models \psi$ , it follows from proposition 2.2 that  $s' \models \psi$ , for each  $s' \in Z$ . Clearly the functions  $f_i$ , for  $i \leq k$ , are as required in 1. Thus we conclude that 1 holds. Thus  $\mu$  is true.

Now since the truth problem for DQBF is NEXPTIME-hard and  $\varphi_\mu$  is clearly polynomial with respect to  $\mu$ , we conclude that the validity problem for PD is NEXPTIME-hard.  $\square$

By Lemmas 4.3 and 4.4, we obtain the following:

**Theorem 4.5.** *The validity problem for PD is NEXPTIME-complete.*

## 5 The validity problem for modal dependence logic

The satisfiability problem for both MDL and EMDL is known to be NEXPTIME-complete. For MDL this was shown by Sevenster [24] and for EMDL Ebbing et al. [6]. In Theorem 4.5 we showed that the validity problem for PD is NEXPTIME-complete and thus much more complex than the corresponding satisfiability problem. This together with the fact that the validity problem for modal logic is known to be PSPACE-complete (Laddner [17]) seems to suggest EXPSPACE as a candidate for the complexity of the validity problem of MDL and EMDL. However, we manage to do a bit better. We establish that the validity problem of MDL and EMDL is in  $\text{NEXPTIME}^{\text{NP}}$ , i.e., in NEXPTIME with access to NP oracles. Thus we obtain that the precise complexity of these problems lie somewhere between and NEXPTIME and  $\text{NEXPTIME}^{\text{NP}}$ , since the NEXPTIME-hardness follows directly from Lemma 4.4.

**Corollary 5.1.** *The validity problem for MDL and EMDL is NEXPTIME-hard.*

The rest of this section is devoted on showing that the validity problem for EMDL is in  $\text{NEXPTIME}^{\text{NP}}$ .

Let  $\varphi$  be a formula of EMDL or  $\text{ML}(\odot)$ . The set  $\text{nbSubf}(\varphi)$  of *non-Boolean subformulas* of  $\varphi$  is defined recursively as follows.

$$\begin{aligned} \text{nbSubf}(\neg p) &:= \text{nbSubf}(p) := \{p\}, & \text{nbSubf}(\Delta \varphi) &:= \{\Delta \varphi\} \cup \text{nbSubf}(\varphi) \quad \text{for } \Delta \in \{\diamond, \square\}, \\ \text{nbSubf}(\varphi \circ \psi) &:= \text{nbSubf}(\varphi) \cup \text{nbSubf}(\psi) \quad \text{for } \circ \in \{\odot, \vee, \wedge\}, \\ \text{nbSubf}(\text{dep}(\varphi_1, \dots, \varphi_n, \psi)) &:= \text{nbSubf}(\varphi_1) \cup \dots \cup \text{nbSubf}(\varphi_n) \cup \text{nbSubf}(\psi). \end{aligned}$$

The following lemma follows directly from [24, Claim 15].

**Lemma 5.2.** *Let  $\varphi \in \text{ML}$  and let  $k = |\text{nbSubf}(\varphi)|$ . Then,  $\varphi$  is valid if and only if  $\mathbf{K}, w \models \varphi$  holds for every Kripke model  $\mathbf{K} = (W, R, V)$  and  $w \in W$  such that  $|W| \leq 2^k$ .*

The following proposition for EMDL is based on a similar result for MDL that essentially combines the ideas of [28], [24] and [19].

**Proposition 5.3.** *For every formula  $\varphi \in \text{EMDL}$  there exists an equivalent formula*

$$\varphi^* = \bigvee_{i \in I} \varphi_i,$$

where  $I$  is a finite set of indices and  $\varphi_i \in \text{ML}$ , for each  $i \in I$ . Furthermore, for each  $i \in I$ , the size of  $\varphi_i$  is only exponential in the size of  $\varphi$  and  $|\text{nbSubf}(\varphi_i)| \leq 3 \times |\varphi|$ .

*Proof.* We will first recall an exponential translation  $\varphi \mapsto \varphi^+$  from EMDL to  $\text{ML}(\otimes)$  given in [6, Theorem 2]. The cases for proposition symbols, Boolean connectives and modalities are trivial, i.e.,  $p \mapsto p$ ,  $\neg p \mapsto \neg p$ ,  $(\varphi \wedge \psi) \mapsto (\varphi^+ \wedge \psi^+)$ ,  $(\varphi \vee \psi) \mapsto (\varphi^+ \vee \psi^+)$ ,  $\diamond \varphi \mapsto \diamond \varphi^+$ ,  $\square \varphi \mapsto \square \varphi^+$ . The only interesting case is the case for the dependence atom. We define that

$$\text{dep}(\varphi_1, \dots, \varphi_n, \psi) \mapsto \bigvee_{a_1, \dots, a_n \in \{\perp, \top\}} \left( \bigwedge_{i \leq n} \varphi_i^{a_i} \wedge (\psi \otimes \psi^\perp) \right),$$

where  $\varphi^\top$  denotes  $\varphi$  and  $\varphi^\perp$  denotes the ML formula obtained from  $\neg \varphi$  by pulling all negations to the atomic level. Notice that the size of  $\varphi^+$  is  $\leq c \times |\varphi| \times 2^{|\varphi|}$ , for some constant  $c$ . Thus the size of  $\varphi^+$  is at most exponential with respect to the size of  $\varphi$ . From  $\varphi^+$  it is easy to obtain an equivalent  $\text{ML}(\otimes)$ -formula  $\varphi^*$  of the form

$$\bigvee_{i \in I} \varphi_i,$$

where  $I$  is a finite index set and  $\varphi_i$ , for  $i \in I$ , is an ML-formula. Let  $F$  be the set of all selection functions  $f$  that select, separately for each occurrence, either the left disjunct  $\psi$  or the right disjunct  $\theta$  of each subformula of the form  $(\psi \otimes \theta)$  of  $\varphi^+$ . Now let  $\varphi_f^+$  denote the formula obtained from  $\varphi^+$  by substituting each occurrence of a subformula of type  $(\psi \otimes \theta)$  by  $f((\psi \otimes \theta))$ . We then define that

$$\varphi^* := \bigvee_{f \in F} \varphi_f^+.$$

It is straightforward to prove that  $\varphi^*$  is equivalent to  $\varphi^+$  and hence to  $\varphi$ . Since, for each  $f \in F$ ,  $\varphi_f^+$  is obtained from  $\varphi^+$  by substituting subformulae of type  $(\psi \otimes \theta)$  with either  $\psi$  or  $\theta$ , it is clear that the size of  $\varphi_f^+$  is bounded above by the size of  $\varphi^+$ . Recall that the size of  $\varphi^+$  is at most exponential with respect to the size of  $\varphi$ . Therefore, for each  $f \in F$ , the size of  $\varphi_f^+$  is at most exponential with respect to the size of  $\varphi$ .

We say that the modal operator  $\diamond$  in  $\diamond \theta$  *dominates* an intuitionistic disjunction if  $\otimes$  occurs in  $\theta$ . To see that  $|\text{nbSubf}(\varphi_f^+)| \leq 3 \times |\varphi|$ , for each  $f \in F$ , notice first that in the translation  $\varphi \mapsto \varphi^+$  the only case that can increase the number of non-Boolean subformulae is the case for the dependence atom. Each  $\varphi_i^+$  and  $\psi^\perp$  may introduce new non-Boolean subformulae. Thus it is straightforward to see that  $|\text{nbSubf}(\varphi^+)| \leq 2 \times |\text{nbSubf}(\varphi)|$ . Furthermore, notice that the number of modal operators that dominate an intuitionistic disjunction in  $\varphi^+$  is less or equal to the number of modal operators in  $\varphi$ . Let  $k$  denote the number of modal operators in  $\varphi$ . It is easy to see that  $|\text{nbSubf}(\varphi_f^+)| \leq |\text{nbSubf}(\varphi^+)| + k$ , for each  $f \in F$ . Now since  $k \leq |\varphi|$  and  $|\text{nbSubf}(\varphi)| \leq |\varphi|$ , we obtain that  $|\text{nbSubf}(\varphi_f^+)| \leq 3 \times |\varphi|$ , for each  $f \in F$ . With a more careful bookkeeping, we would obtain that  $|\text{nbSubf}(\varphi_f^+)| \leq 2 \times |\varphi|$ .  $\square$

We say that a formula  $\varphi \in \text{ML}$  is *valid in small models* if  $\mathbf{K}, w \models \varphi$  holds for every Kripke model  $\mathbf{K} = (W, R, V)$  and  $w \in W$  such that  $|W| \leq |\varphi|$ .

**Lemma 5.4.** *The decision problem whether a given formula of ML is valid in small models is in coNP.*

*Proof.* If a formula  $\varphi \in \text{ML}$  is not valid in small models, then there is some  $k \leq |\varphi|$  and a pointed Kripke model  $\mathbb{K}, w$  of size  $k$  such that  $\mathbb{K}, w \not\models \varphi$ . The size of  $\mathbb{K}, w$  is clearly polynomial in  $|\varphi|$ , and thus it can be guessed nondeterministically in polynomial time with respect to  $|\varphi|$ . The model checking problem for modal logic is in P ([4]), and thus  $\mathbb{K}, w \not\models \varphi$  can be verified in polynomial time with respect to  $|\mathbb{K}| + |\varphi|$  and thus in polynomial time with respect to  $|\varphi|$ .  $\square$

**Proposition 5.5.**  *$\text{ML}(\otimes)$  has the  $\otimes$ -disjunction property, i.e., for every  $\varphi, \psi \in \text{ML}(\otimes)$  it holds that  $(\varphi \otimes \psi)$  is valid if and only if either  $\varphi$  is valid or  $\psi$  is valid.*

*Proof.* The direction from right to left is trivial. We will prove here the direction from left to right. Assume that  $(\varphi \otimes \psi)$  is valid. For the sake of contradiction, assume then that neither  $\varphi$  nor  $\psi$  is valid. Thus there exist Kripke models  $\mathbb{K}$  and  $\mathbb{K}'$ , and teams  $T$  and  $T'$  of  $\mathbb{K}$  and  $\mathbb{K}'$ , respectively, such that  $\mathbb{K}, T \not\models \varphi$  and  $\mathbb{K}', T' \not\models \psi$ . From Corollary 2.10 it follows that  $\mathbb{K} \uplus \mathbb{K}', T \not\models \varphi$  and  $\mathbb{K} \uplus \mathbb{K}', T' \not\models \psi$ , where  $\mathbb{K} \uplus \mathbb{K}'$  denotes the disjoint union of  $\mathbb{K}$  and  $\mathbb{K}'$ . Since the formulae of  $\text{ML}(\otimes)$  are downwards closed (Proposition 2.7), we conclude that  $\mathbb{K} \uplus \mathbb{K}', T \cup T' \not\models \varphi$  and  $\mathbb{K} \uplus \mathbb{K}', T \cup T' \not\models \psi$ . Thus  $\mathbb{K} \uplus \mathbb{K}', T \cup T' \not\models (\varphi \otimes \psi)$ . This contradicts the fact that  $(\varphi \otimes \psi)$  is valid.  $\square$

**Proposition 5.6.** *The validity problem for EMDL is in  $\text{NEXPTIME}^{\text{NP}}$ .*

*Proof.* For deciding whether a given EMDL formula is valid, we give a nondeterministic exponential time algorithm that has an access to an NP oracle that decides whether a given ML formula is valid in small models. For each  $\varphi \in \text{EMDL}$  let  $\varphi^+$  denote the equivalent exponential size  $\text{ML}(\otimes)$ -formula from the proof of Proposition 5.3. Clearly  $\varphi^+$  is computable from  $\varphi$  in exponential time. Furthermore let  $\varphi^*$  denote the  $\text{ML}(\otimes)$ -formula of the form  $\bigvee_{f \in F} \varphi_f^+$  of Proposition 5.3. Moreover let  $g : \mathbb{N} \rightarrow \mathbb{N}$  be some exponential function such that  $|\varphi_f^+| \leq g(|\varphi|)$ , for every  $\varphi \in \text{EMDL}$  and  $f \in F$ . By Proposition 5.3 there exists such a function.

We are now ready to give a  $\text{NEXPTIME}^{\text{NP}}$  algorithm for the validity problem of EMDL. Let  $\varphi$  be an EMDL formula. First guess nondeterministically an ML formula  $\psi$  of the same vocabulary as  $\varphi$  of size at most  $g(|\varphi|)$ . Then compute  $\varphi^+$  from  $\varphi$  and check whether  $\psi$  is among the disjuncts  $\varphi_f^+, f \in F$ , of  $\varphi^*$ . Clearly the checking can be done in polynomial time with respect to  $|\varphi^+| + |\psi|$  and thus in exponential time with respect to the size of  $\varphi$ . If  $\psi$  is not among the disjuncts the algorithm outputs “No”, otherwise the algorithm continues. We then give

$$\psi^- := \left( \bigwedge_{i \leq 2^{3 \times |\varphi|}} (p \vee \neg p) \right) \wedge \psi$$

as an input to an NP oracle that decides whether the ML formula  $\psi^-$  is valid in small models. Clearly  $\psi^-$  is computable from  $\psi$  in exponential time with respect to the size of  $\varphi$ . The algorithm outputs “No” if the oracle outputs “No” and “Yes” if the oracle outputs “Yes”. Clearly this algorithm is in  $\text{NEXPTIME}^{\text{NP}}$ .

Now by Proposition 5.3,  $\varphi$  is valid if and only if  $\varphi^*$  is valid, and furthermore, by Proposition 5.5,  $\varphi^*$  is valid if and only if  $\varphi_f^+$  is valid for some  $f \in F$ . By Proposition 5.3,  $|\text{nbSubf}(\varphi_f^+)| \leq 3 \times |\varphi|$ , for every  $f \in F$ . Thus by Lemma 5.2, for every  $f \in F$ ,  $\varphi_f^+$  is valid if and only if  $\varphi_f^+$  is true on all pointed models of size at most  $2^{3 \times |\varphi|}$ . Now clearly, for every  $f \in F$ ,  $\varphi_f^+$  is valid if and only if the formula

$$\varphi_f^- := \left( \bigwedge_{i \leq 2^{3 \times |\varphi|}} (p \vee \neg p) \right) \wedge \varphi_f^+$$

is valid. Thus, for every  $f \in F$ ,  $\varphi_f^+$  is valid if and only if  $\varphi_f^-$  is valid in small models. Therefore, and since  $\psi^- = \varphi_f^-$ , for some  $f \in F$ , the algorithm decides the validity problem of EMDL.  $\square$

**Corollary 5.7.** *The validity problem for MDL is in  $\text{NEXPTIME}^{\text{NP}}$ .*

## 6 Conclusion

In this article we studied the validity problem of propositional dependence logic, modal dependence logic, and extended modal dependence logic. We established that the validity problem for propositional dependence logic is  $\text{NEXPTIME}$ -complete. In addition we showed that the corresponding problem for modal dependence logic and extended modal dependence logic is  $\text{NEXPTIME}$ -hard and contained in  $\text{NEXPTIME}^{\text{NP}}$ . The exact complexity of the validity problem for MDL and EMDL remain open. We conjecture that both of these problems are harder than  $\text{NEXPTIME}$ . We also believe that the complexity of MDL and EMDL coincide. In addition to resolving the precise complexity of the validity problem of MDL and EMDL, we are interested in the complexity of the entailment problem of PD, MDL, and EMDL. Note that in the context of dependence logic the entailment problem cannot be reduced directly to the validity problem. However the validity problem can be reduced to the entailment problem. Hence the entailment problem of PD, MDL, and EMDL is at least as hard as the corresponding validity problem.

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