Games for Topological Fixpoint Logic

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Topological fixpoint logics are a family of logics that admits topological models and where the fixpoint operators are defined with respect to the topological interpretations. Here we consider a topological fixpoint logic for relational structures based on Stone spaces, where the fixpoint operators are interpreted via clopen sets. We develop a game-theoretic semantics for this logic. First we introduce games characterising clopen fixpoints of monotone operators on Stone spaces. These fixpoint games allow us to characterise the semantics for our topological fixpoint logic using a two-player graph game. Adequacy of this game is the main result of our paper. Finally, we define bisimulations for the topological structures under consideration and use our game semantics to prove that the truth of a formula of our topological fixpoint logic is bisimulation-invariant.

1 Introduction

By *topological fixpoint logics* we mean a family of fixpoint logics that admit topological models and where the fixpoint operator is defined with respect to topological interpretations. In the standard semantics fixpoint operators are interpreted as the least (or greatest) fixpoint of a monotone map in the powerset lattice. In our topological setting we interpret fixpoint operators as the least (or greatest) fixpoint of a monotone map on some (topological) sublattice of the powerset lattice (e.g., clopen subsets, open or closed subsets, regular open or closed subsets etc.). An important motivation for studying such formalisms is that every axiomatic system of the modal μ-calculus is complete with respect to the topological semantics via clopen sets [1]. Moreover, the powerful Sahlqvist completeness and correspondence result from modal logic can be extended to the axiomatic systems of modal μ-calculus for this semantics [6]. We note that completeness results for axiomatic systems of modal μ-calculus with the standard semantics are very rare, and require highly complex machinery [14], [23], see also [18] and [9]. Note also that axiomatic systems of modal conjugated μ-calculus axiomatized by Sahlqvist formulas are closed under Dedekind-MacNeille completions via topological semantics [5]. However, these systems are not closed under Dedekind-MacNeille completions for the standard semantics [17]. Another motivation for studying topological semantics of fixpoint logic is that it provides an alternative view on fixpoints operators with new notions of expressivity and definability. For a comprehensive discussion on the importance of generalized models in logic, including modal fixpoint logic, we refer to [2]. A rather different approach to interpret fixpoint formulas over topological spaces is taken in [11] where formulas are interpreted in the full powerset lattice and where modalities are interpreted via topological operations such as closure and topological derivative.

We illustrate the difference between standard and topological fixpoint operators with an example. Consider the frame $(\mathbb{N} \cup \{\infty\}, R)$ drawn in Figure 1. We assume that the topology on the set is such that clopen sets are finite subsets of $\mathbb{N}$ and cofinite sets containing the point ∞. The denotation of the formula $\Diamond^* p$ is the set of points that "see points in p wrt the transitive closure of the relation R". Therefore $\Diamond^* p$

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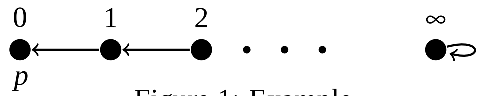

Figure 1: Example

is equal to the set $\mathbb{N}$. Indeed, $\mathbb{N}$ is the least fixed point of the map $S \mapsto \{0\} \cup \Diamond S$, where $\Diamond S = \{s' \mid \exists s \in S.(s', s) \in R\}$. However, if we are looking for a least clopen fixpoint of this map then we see that this will be the set $\mathbb{N} \cup \{\infty\}$. Intuitively, the denotation of the formula $\Diamond^* p$ wrt the clopen semantics is the set of all points that "see points in p wrt the *topological transitive closure* of the relation R". Note that a similar operation was used in [21] for characterising in dual terms subdirectly irreducible modal algebras.

In this paper, we aim to advance the study of topological fixpoint logics by developing a game semantics for them. We will concentrate on a variant of topological fixpoint logic based on interpretations via clopen sets. For clopen sets we consider Stone spaces with a binary relation (*descriptive μ-frames* in the terminology of [1] and [6]). The advantage of clopen sets is that the denotation of modal formulas in clopen sets is the same as in the standard Kripke semantics of modal logic. The negation of a formula is interpreted as the complement, conjuction and disjunction as the intersection and union, respectively, and the modal operators are also interpreted in the standard way. However, clopen sets of an arbitrary Stone space do not form a complete lattice and therefore the fixpoint operators, in general, may not be interpreted in Stone spaces with the clopen semantics. Therefore, we need to restrict to a class of Stone spaces where these operators can be interpreted. We will achieve this by looking at relational structures based on *extremally disconnected spaces* which is a subclass of descriptive μ-frames.

There are several motivations for developing the game semantics for the topological μ-calculus. Firstly, the semantics of a formula can be usually much better understood when formulated in terms of games. This is especially true for formulas with some non-trivial interplay of least and greatest fixpoint operators. Secondly, a game semantics is crucial for the development of automata-theoretic methods of the topological μ-calculus: the game semantics provides an "operational" semantics for the formulas of the logic and the definition of a run of an automata (or of its acceptance game) is entirely based on this operational view on the truth of a formula. Thirdly, the game semantics is an important tool for developing the model-theory of the topological μ-calculus.

The main contribution of this paper is a game semantics for the topological μ-calculus based on clopen sets. Technically, the main result is the proof of adequacy of our game semantics. Finally we are demonstrating how the game semantics can be used in order to obtain model-theoretic results: we prove that the topological μ-calculus is invariant under what we call clopen bisimulations.

We view the results in this paper as first steps towards a full theory of topological fixpoint logics. An ultimate goal is to define game semantics and automata for all descriptive μ-frames (not necessarily based on extremally disconnected spaces). This would enable us to apply the methods of games and automata for tackling problems such as decidability and the finite model property of axiomatic systems of the modal μ-calculus. These systems are complete for descriptive μ-frames, whereas their completeness for the standard Kripke semantics is quite problematic.

2 Preliminaries

2.1 Two Player graph games

Two-player infinite graph games, or *graph games* for short, are defined as follows. For a more comprehensive account of these games, the reader is referred to [12].

A graph game is played on a *board* B, that is, a set of *positions*. Each position $b \in B$ *belongs* to one of the two *players*, $\exists$ (Éloise) and $\forall$ (Abélard). Formally we write $B = B_\exists \cup B_\forall$, and for each position b we use $P(b)$ to denote the player i such that $b \in B_i$. Furthermore, the board is endowed with a binary relation E, so that each position $b \in B$ comes with a set $E[b] \subseteq B$ of *successors*. Note that we do not require the games to be strictly alternating, i.e., successors of positions in $B_\exists$ or $B_\forall$ can lie again in $B_\exists$ or $B_\forall$, respectively. Formally, we say that the *arena* of the game consists of a directed two-sorted graph $\mathbb{B} = (B_\exists, B_\forall, E)$.

A *match* or *play* of the game consists of the two players moving a pebble around the board, starting from some *initial position* b_0. When the pebble arrives at a position $b \in B$, it is player $P(b)$'s turn to move; (s)he can move the pebble to a new position of their liking, but the choice is restricted to a successor of b. Should $E[b]$ be empty then we say that player $P(b)$ *got stuck* at the position. A *match* or *play* of the game thus constitutes a (finite or infinite) sequence of positions $b_0b_1b_2\ldots$ such that b_iEb_{i+1} (for each i such that b_i and b_{i+1} are defined). A *full play* is either (i) an infinite play or (ii) a finite play in which the last player got stuck. A non-full play is called a *partial* play. Each full play of the game has a *winner* and a *loser*. A finite full play is lost by the player who got stuck; the winning condition for infinite games is usually specified using a so-called *parity function*. In our paper, however, we specify the winning conditions on infinite games in more intuitive terms, stating explicitly which infinite plays will be won by which player. Throughout the paper the reader should take it for granted that the winning conditions involved could easily be encoded using suitable parity functions.

A *strategy* for player i tells a player how to play to at a given game position: this can be represented as a *partial* function mapping partial plays $\beta = b_0 \cdots b_n$ with $P(b_n) = i$ to legal next positions, that is, to elements of $E[b_n]$, and that it is undefined if $E[b_n] = \emptyset$. A strategy is *history free* if it only depends on the current position of the match, and not on the history of the match. A strategy is *winning for player i* from position $b \in B$ if it guarantees i to win any match with initial position b, no matter how the adversary plays — note that this definition also applies to positions b for which $P(b) \neq i$. A position $b \in B$ is called a *winning position* for player i, if i has a winning strategy from position b; the set of winning positions for i in a game $\mathscr{G}$ is denoted as $Win_i(\mathscr{G})$. Parity games enjoy *history-free determinacy*, ie., at each position of the game board one of the player has a history free winning strategy (cf. [16, 10]).

2.2 Tarski's fixpoint game

Recall that on any complete lattice the least fixpoint μF and the greatest fixpoint νF of a monotone function F exist and can be obtained as follows: first we define for each ordinal $\alpha \in \mathrm{ORD}$ two sequences $\{F^\mu_\alpha\}_{\alpha\in\mathrm{ORD}}$ and $\{F^\nu_\alpha\}_{\alpha\in\mathrm{ORD}}$ by putting

$$\begin{array}{lllllll} F^\mu_0 &=& \bot, & F^\mu_{\alpha+1} = F(F^\mu_\alpha) & \text{and} & F^\mu_\alpha = \bigvee_{\beta<\alpha} F^\mu_\beta & \text{for } \alpha \text{ a limit ordinal.} \\ F^\nu_0 &=& \top, & F^\nu_{\alpha+1} = F(F^\nu_\alpha) & \text{and} & F^\nu_\alpha = \bigwedge_{\beta<\alpha} F^\nu_\beta & \text{for } \alpha \text{ a limit ordinal.} \end{array}$$

The core of the game-theoretic semantics of the modal μ-calculus is based on Tarski's game-theoretic characterisation of fixpoints. Given a monotone function $F : \mathscr{P}X \to \mathscr{P}X$, the game board of the standard fixpoint game is defined as follows:

Position	Player	Moves
$x \in X$	$\exists$	$\{C \subseteq X \mid x \in F(C)\}$
$C \subseteq X$	$\forall$	C

We will use the above notation in the following to introduce graph games: the table specifies that $B_\exists = X$, $B_\forall = \mathscr{P}X$ and in the third column of the table the successors of each game board position are specified.

The condition on infinite plays in the standard fixpoint game is that all infinite plays of the game are won by $\forall$ in the least fixpoint game and by $\exists$ in the greatest fixpoint game.

It is a standard result in fixpoint theory (cf. e.g. [22]) that the above least and greatest fixpoint games characterise the least and greatest fixpoint of F, respectively. For example, $\exists$ has a winning strategy at a position $x \in X$ in the least fixpoint game iff x is an element of μF. If x is an element of the least fixpoint, we know that there exists an ordinal α such that $x \in F^{\mu}_{\alpha}$. In case that α is a limit ordinal this means that $x \in \bigvee_{\beta<\alpha} F^{\mu}_{\beta} = \bigcup_{\beta<\alpha} F^{\mu}_{\beta} \subseteq F(\bigcup_{\beta<\alpha} F^{\mu}_{\beta})$ where the inclusion is easily verifiable. This means $\exists$ can move from position x to position $\bigcup_{\beta<\alpha} F^{\mu}_{\beta}$ and $\forall$ is forced to move to some $x' \in F^{\mu}_{\beta}$ with $\beta < \alpha$. Similary, if $\alpha = \beta + 1$, $\exists$ can ensure that the play reaches a position in F^{μ}_{β} after one round. In any case, due to the well-foundedness of the ordinals, $\exists$ can ensure that the play moves from $x \in F^{\mu}_{\alpha}$ to some $x \in F^{\mu}_{\beta}$ with $\beta < \alpha$ which implies that $\exists$ has a strategy that forces $\forall$ to get stuck after a finite number of moves.

2.3 Topological preliminaries

We will work with Kripke frames that are endowed with a topology. The most important class of such frames used in the study of modal logic is that of *modal spaces* (aka *descriptive frames*). This is due to the Stone representation theorem for Boolean algebras and Jónsson-Tarski representation theorem for Boolean algebras with operators. A *modal space* is a triple (X, τ, R) such that $\mathbb{X} = (X, \tau)$ is a Stone space and $R \subseteq X \times X$ is a binary relation that is *point-closed* and *clopen*. The latter mean that $R(x) = \{y \in X : xRy\}$ is a closed set for each $x \in X$ and that $\Diamond U \in \mathrm{Clp}(\mathbb{X})$ for each $U \in \mathrm{Clp}(\mathbb{X})$, where $\mathrm{Clp}(\mathbb{X})$ is the set of all clopen subsets of $\mathbb{X}$ and $\Diamond U = \{x \in X \mid \exists y \in U.\, xRy\}$. Every modal algebra can be represented as the algebra $(\mathrm{Clp}(\mathbb{X}), \Diamond)$, where $\mathbb{X}$ is the ultrafilter space. As a result every axiomatic system of modal logic is complete wrt modal spaces. We refer to [8] for more details on completeness of modal logics wrt modal spaces. We also note that modal spaces can be also represented as Vietoris coalgebras on the category of Stone spaces [15]. Throughout this paper we will tacitly assume that all topological Kripke frames are modal spaces.

A Stone space $\mathbb{X} = (X, \tau)$ is called *extremally disconnected* if the closure of any open subset of $\mathbb{X}$ is open. It is well known (see e.g., [19]) that if $\mathbb{X}$ is an extremally disconnected space, then $\mathrm{Clp}(\mathbb{X})$ is a complete Boolean algebra. Moreover, for a set of clopen sets $\{U_i : i \in I\}$ the infinite meets and joins are computed as: $\bigvee\{U_i : i \in I\} = \mathrm{Cl}(\bigcup\{U_i : i \in I\})$ and $\bigwedge\{U_i : i \in I\} = \mathrm{Int}(\bigcap\{U_i : i \in I\})$. We call a modal space (X, τ, R) an *extremally disconnected modal space* if (X, τ) is extremally disconnected.

2.4 Modal μ-calculus on topological spaces: denotational semantics

The complete lattice structure on $\mathrm{Clp}(\mathbb{X})$ of an extremally disconnected space $\mathbb{X} = (X, \tau)$ enables us to define a topological semantics of the modal μ-calculus that is based on clopen sets.

Definition 2.1. *Given a countably infinite set* Prop *of propositional variables (p, q, p_0, q_1, etc), the language $\mathscr{L}_\mu$ of the modal μ-calculus is inductively defined as follows:*

$$\begin{aligned}\mathscr{L}_\mu \ni \varphi \quad ::= \quad & p,\ p \in \mathrm{Prop} \mid \neg p,\ p \in \mathrm{Prop} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \bot \mid \top \mid \Diamond\varphi \mid \Box\varphi \mid \\ & \mu p.\varphi(p, q_1, \ldots, q_n) \mid \nu p.\varphi(p, q_1, \ldots, q_n)\end{aligned}$$

where in formulas of the form $\mu p.\varphi$ and $\nu p.\varphi$ we require that the variable p does not occur under a negation[1]*. The sets $FVar(\varphi)$ and $BVar(\varphi)$ of free and bound variables of a given formula $\varphi \in \mathscr{L}_\mu$ are defined in a standard way.*

[1] Formulas are always in negation normal form, ie., negations only occur in front of propositional variables.

Definition 2.2. *Given an extremally disconnected modal space* $(\mathbb{X}, R)$ *based on a space* $\mathbb{X} = (X, \tau)$ *and a valuation* $V : \mathrm{Prop} \to \mathrm{Clp}(\mathbb{X})$ *we define the semantics* $[\![\varphi]\!]_V^{\mathbb{X}} \in \mathrm{Clp}(\mathbb{X})$ *of a formula* φ *by induction:*

$$\begin{array}{rcl@{\qquad}rcl}
[\![p]\!]_V & := & V(p) & [\![\neg p]\!]_V & := & X \setminus V(p) \\
[\![\psi_1 \wedge \psi_2]\!]_V & := & [\![\psi_1]\!]_V \cap [\![\psi_2]\!]_V & [\![\psi_1 \vee \psi_2]\!]_V & := & [\![\psi_1]\!]_V \cup [\![\psi_2]\!]_V \\
[\![\bot]\!]_V & := & \emptyset & [\![\top]\!]_V & := & X \\
[\![\Diamond\psi]\!]_V & := & \{x \in X \mid R(x) \cap [\![\psi]\!]_V \neq \emptyset\} & [\![\Box\psi]\!]_V & := & \{x \in X \mid R(x) \subseteq [\![\psi]\!]_V\} \\
[\![\mu p.\psi]\!]_V & := & \mathrm{lfp}(\psi_p^V) & [\![\nu p.\psi]\!]_V & := & \mathrm{gfp}(\psi_p^V)
\end{array}$$

where $\psi_p^V : \mathrm{Clp}(\mathbb{X}) \to \mathrm{Clp}(\mathbb{X})$ *is the (monotone) operator defined by* $\psi_p^V(U) := [\![\psi]\!]_{V[p \mapsto U]}$ *for* $U \in \mathrm{Clp}(\mathbb{X})$ *and with*

$$V[p \mapsto U](q) := \begin{cases} U \text{ if } q = p \\ V(q) \text{ otherwise.} \end{cases}$$

We call the triple $\mathbb{M} = (\mathbb{X}, R, V)$ *an extremally disconnected (Kripke) model and write* $\mathbb{M}[p \mapsto U]$ *to denote the model* $\mathbb{M} = (\mathbb{X}, R, V[p \mapsto U])$.

3 Games for monotone operators on topological spaces

In this section we are going to define topological analogues of the fixpoint game from page 48. We start by looking at fixpoints of a monotone function $F : \mathrm{Clp}(\mathbb{X}) \to \mathrm{Clp}(\mathbb{X})$ on the lattice of clopen subsets $\mathrm{Clp}(\mathbb{X})$ of an extremally disconnected Stone space $\mathbb{X} = (X, \tau)$. This assumption on the topology guarantees the existence of a least and greatest fixpoint of F and these fixpoints can be obtained using the ordinal approximants F^μ_α and F^ν_α, respectively. To understand how the fixpoint game has to be defined we need to inspect how the ordinal approximants F^μ_α and F^ν_α are computed in case α is a limit ordinal:

$$\begin{array}{rcl}
F^\mu_\alpha & = & \bigvee_{\beta<\alpha} F^\mu_\beta = \mathrm{Cl}(\bigcup_{\beta<\alpha} F^\mu_\beta) \\
F^\nu_\alpha & = & \bigwedge_{\beta<\alpha} F^\nu_\beta = \mathrm{Int}(\bigcap_{\beta<\alpha} F^\nu_\beta)
\end{array}$$

Therefore, intuitively speaking, in order to maintain the claim that a given point x is an element of μF it suffices that $\exists$ provides some open set $O \subseteq X$ such that $x \in F(\mathrm{Cl}(O))$, so this will become easier for $\exists$. Likewise, in order to prove that $x \in \nu F$, $\exists$ will now have to provide some closed set C such that $x \in F(\mathrm{Int}(C))$ which is potentially more difficult compared to the standard fixpoint game. Note that in both cases $\mathrm{Cl}(O)$ and $\mathrm{Int}(C)$ are clopen as the closure of an open set and the interior of a closed set are clopen sets in an extremally disconnected Stone space. Our observations form the basis for the following definitions of the fixpoint games:

Definition 3.1. *Let* $\mathbb{X} = (X, \tau)$ *be an extremally disconnected topological space and let* $F : \mathrm{Clp}(\mathbb{X}) \to \mathrm{Clp}(\mathbb{X})$ *be a monotone map. We define two graph games. We start with the game board of the least fixpoint game* $\mathcal{G}^I_\mu(F)$*:*

Position	*Player*	*Moves*
$x \in X$	$\exists$	$\{C \subseteq X \mid x \in F(U) \text{ for all } U \in \mathrm{Clp}(\mathbb{X}) \text{ with } C \subseteq U\}$
$C \subseteq X$	$\forall$	C

ie, at a position $x \in X$, player $\exists$ has to move to some $C \subseteq X$ such that $x \in F(U)$ for all clopen supersets of C and at position $C \subseteq X$ player $\forall$ has to move to some $x' \in C$. Infinite plays are won by $\forall$. The resulting graph game will be called the least clopen fixpoint game and will be denoted by $\mathscr{G}^I_\mu(F)$. The greatest clopen fixpoint game $\mathscr{G}^I_\nu(F)$ is defined similarly with the major difference that an infinite play is won by $\exists$. Also, the game board of $\mathscr{G}^I_\nu(F)$ reflects the aforementioned way of computing meets in $\mathrm{Clp}(\mathbb{X})$*:*

Position	*Player*	*Moves*
$x \in X$	$\exists$	$\{C \subseteq X \mid x \in F(U)$ *for all* $U \in \mathrm{Clp}(\mathbb{X})$ *with* $\mathrm{Int}(C) \subseteq U\}$
$C \subseteq X$	$\forall$	C

With these definitions at hand it is not difficult to prove that $\mathscr{G}^I_\mu(F)$ and $\mathscr{G}^I_\nu(F)$ indeed characterise the least and greatest clopen fixpoints of F, respectively. This is the content to the following proposition.

Proposition 3.2. *Let $\mathbb{X} = (X, \tau)$ be an extremally disconnected space, let $F : \mathrm{Clp}(\mathbb{X}) \to \mathrm{Clp}(\mathbb{X})$ be a monotone operator. Then for any $x \in X$ we have*

1. *$x \in \mu F$ iff $x \in Win_\exists(\mathscr{G}^I_\mu(F))$*
2. *$x \in \nu F$ iff $x \in Win_\exists(\mathscr{G}^I_\nu(F))$*

Proof. We only provide the proof for the greatest fixpoint game $\mathscr{G}^I_\nu(F)$ - the one for the least fixpoint game is very similar. We need to show that $Win_\exists(\mathscr{G}^I_\nu(F)) = \nu F$. Suppose first that $x \in \nu F \in \mathrm{Clp}(\mathbb{X})$. Then $\exists$ has an obvious winning strategy: she is playing the set νF. All $\forall$ can do is choosing another element $x' \in \nu F$ after which $\exists$ can move again to νF and so forth. Note that any such play will be infinite and thus $\exists$ has a strategy to win any play starting at x, ie., $x \in Win_\exists(\mathscr{G}^I_\nu(F))$.

For the converse we show that for all ordinals α we have $X \setminus F^\nu_\alpha \subseteq Win_\forall(\mathscr{G}^I_\nu(F))$ by induction on α.
Case $\alpha = 0$. Then the claim is obvious as $X \setminus F^\nu_0 = X \setminus X = \emptyset$.
Case $\alpha = \beta + 1$. Suppose that in a play starting at position $x \notin F^\nu_\alpha = F(F^\nu_\beta)$ player $\exists$ moves to some $C \subseteq X$ with $x \in F(U)$ for all $U \in \mathrm{Clp}(\mathbb{X})$ with $\mathrm{Int}(C) \subseteq U$. Clearly $C \not\subseteq F^\nu_\beta$ for otherwise $\mathrm{Int}(C) \subseteq F^\nu_\beta$ and thus $x \in F(F^\nu_\beta) = F^\nu_\alpha$. Hence $\forall$ can pick an element $x' \in C \setminus F^\nu_\beta$. Now by I.H. we have that $x' \in Win_\forall(\mathscr{G}^I_\nu(F))$ and thus $\forall$ has a strategy to win the play from now on. This shows that $\forall$ has a winning strategy at position x in $\mathscr{G}_\nu(F)$ as required.
Case α is a limit ordinal. Consider some $x \notin F^\nu_\alpha = \bigwedge F^\nu_\beta$ and let $C \subseteq X$ be chosen by $\exists$ as in the previous case. By our assumption on the topology we have $\bigwedge F^\nu_\beta = \mathrm{Int}(\bigcap F^\nu_\beta)$. It is not difficult to see that $C \not\subseteq \bigcap F^\nu_\beta$ for suppose otherwise: then $\mathrm{Int}(C) \subseteq \mathrm{Int}(\bigcap F^\nu_\beta) = \bigwedge F^\nu_\beta$ and thus $x \in F(\bigwedge F^\nu_\beta) \subseteq \bigwedge F^\nu_\beta$ which contradicts our assumption on x. Therefore there exists a $\beta < \alpha$ such that $C \not\subseteq F^\nu_\beta$, ie., such that there exists $x' \in C$ with $x' \notin F^\nu_\beta$. By the induction hypothesis we know that $x' \in Win_\forall(\mathscr{G}^I_\nu(F))$ and from position x' $\forall$ has a strategy to win the play. Therefore $\forall$ has a winning strategy from position x as required. ☐

This shows that the games $\mathscr{G}^I_\mu$ and $\mathscr{G}^I_\nu$ characterise the least and greatest clopen fixpoint of a monotone operator. We will use these games to prove adequacy of our game semantics for the topological modal μ-calculus: If $\exists$ has a winning strategy in the evaluation game for a formula of the form $\mu p.\varphi$ and $\nu p.\varphi$ then we will construct a winning strategy for her in the corresponding fixpoint games that we just discussed. Vice versa we would like to transform winning strategies in the fixpoint games into winning strategies of the evaluation game for $\mu p.\varphi$ and $\nu p.\varphi$. For this converse direction we will need second - but equivalent - versions of the fixpoint games.

Definition 3.3. *Let $\mathbb{X}$ be an extremely disconnected space and let $F : \mathrm{Clp}(\mathbb{X}) \to \mathrm{Clp}(\mathbb{X})$ be a monotone map. As elements of $\mathrm{Clp}(\mathbb{X})$ can occur both as position of $\exists$ and $\forall$, we clearly mark the owner of such a position using the set of markers $M = \{\exists, \forall\}$. We define the following two-player game $\mathscr{G}^{II}_\mu(F)$ by putting*

Position	Player	Moves
$x \in X$	$\exists$	$\{(\forall,U) \in M \times \mathrm{Clp}(\mathbb{X}) \mid x \in F(U)\}$
$(\forall,U) \in M \times \mathrm{Clp}(\mathbb{X})$	$\forall$	$\{(\exists,U') \in M \times \mathrm{Clp}(\mathbb{X}) \mid U \cap U' \neq \emptyset\}$
$(\exists,U) \in M \times \mathrm{Clp}(\mathbb{X})$	$\exists$	U

ie, at a position $x \in X$, player $\exists$ has to move to some clopen set $U \subseteq X$ such that $x \in F(U)$, $\forall$ challenges this by playing a element $U' \in \mathrm{Clp}(\mathbb{X})$ with $U \cap U' \neq \emptyset$ and at position $(\exists,U') \in M \times \mathrm{Clp}(\mathbb{X})$ player $\exists$ has to move to some $x' \in U'$. Again $\forall$ wins all infinite plays of the game. Similarly we define the game $\mathscr{G}^{II}_\nu(F)$ by defining the following game board and by stipulating that $\exists$ wins all infinite plays:

Position	Player	Moves
$x \in X$	$\exists$	$\{U \in \mathrm{Clp}(\mathbb{X}) \mid x \in F(U)\}$
$U \in \mathrm{Clp}(\mathbb{X})$	$\forall$	U

Remark 3.4. The reader familiar with fixpoint games might be surprised and slightly worried as there is an unexpected asymmetry between the games $\mathscr{G}^{II}_\mu(F)$ and $\mathscr{G}^{II}_\nu(F)$. Both games have in fact been derived from two completely symmetric games with the following game boards (omitting the markers in M) and the usual winning conditions for infinite least and greatest fixpoint games:

$\mathscr{G}_\mu$ Position	Pl.	Moves
$x \in X$	$\exists$	$\{U \in \mathrm{Clp}(\mathbb{X}) \mid x \in F(U)\}$
$U \in \mathrm{Clp}(\mathbb{X})$	$\forall$	U
$x' \in X$	$\forall$	$\{U' \in \mathrm{Clp}\mathbb{X} \mid x' \in U'\}$
$U' \in \mathrm{Clp}(\mathbb{X})$	$\exists$	U'

$\mathscr{G}_\nu$ Position	Pl.	Moves
$x \in X$	$\exists$	$\{U \in \mathrm{Clp}(\mathbb{X}) \mid x \in F(U)\}$
$U \in \mathrm{Clp}(\mathbb{X})$	$\forall$	U
$x' \in X$	$\exists$	$\{U' \in \mathrm{Clp}\mathbb{X} \mid x' \in U'\}$
$U' \in \mathrm{Clp}(\mathbb{X})$	$\forall$	U'

It is not difficult to see, however, that both games can be simplified to the games $\mathscr{G}^{II}_\mu(F)$ and $\mathscr{G}^{II}_\nu(F)$.

We will now show that games for μ and ν characterise the least and greatest clopen fixpoint.

Proposition 3.5. *Let $\mathbb{X}$ be an extremally disconnected space, let $F : \mathrm{Clp}(\mathbb{X}) \to \mathrm{Clp}(\mathbb{X})$ be a monotone operator. Then for any $x \in X$ we have*

1. *$x \in \mu F$ iff $x \in Win_\exists(\mathscr{G}^{II}_\mu(F))$.*
2. *$x \in \nu F$ iff $x \in Win_\exists(\mathscr{G}^{II}_\nu(F))$*

Proof. We first focus on the least fixpoint operator. Suppose that $x \in \mu F$ for some $x \in X$. Then there is a least ordinal α such that $x \in F^\mu_\alpha$, we call this the μ-depth of x. We will show that $\exists$ has a winning strategy in $\mathscr{G}^{II}_\mu(F)$ at x by describing a strategy for $\exists$ that ensures that either $\forall$ gets stuck within the next round or that the play reaches a position $x' \in F^\mu_{\alpha'}$ with $\alpha' < \alpha$. Both facts entail that $\exists$ has a strategy such that all plays compliant with her strategy are finite and that $\forall$ is the player who will eventually get stuck.
Case $\alpha = \beta + 1$. Then $x \in F^\mu_{\beta+1} = F(F^\mu_\beta)$ and $\exists$'s strategy is to move from x to $(\forall, F^\mu_\beta)$. Player $\forall$ either gets stuck (if $F^\mu_\beta = \emptyset$) or responds by moving to some $(\exists,U')$ with $U' \in \mathrm{Clp}(\mathbb{X})$ such that $U' \cap F^\mu_\beta \neq \emptyset$. Now $\exists$ picks an arbitrary $x' \in U' \cap F^\mu_\beta$ and the round finished on a position $x' \in F^\mu_\beta$ with strictly smaller μ-depth as required.
Case α is a limit ordinal. Then $\exists$'s strategy is to move from x to $(\forall, \bigvee_{\beta<\alpha} F^\mu_\beta) = (\forall, \mathrm{Cl}(\bigcup_{\beta<\alpha} F^\mu_\beta))$ which is a legal move as $x \in \bigvee_{\beta<\alpha} F^\mu_\beta \subseteq F(\bigvee_{\beta<\alpha} F^\mu_\beta)$. Unless $\forall$ gets stuck, he will move to some position $(\exists,U')$ where $U' \in \mathrm{Clp}(\mathbb{X})$ with $U' \cap \bigvee_{\beta<\alpha} F^\mu_\beta \neq \emptyset$. In other words, the clopen subset U' has a non empty intersection with the closure of $\bigcup_{\beta<\alpha} F^\mu_\beta$ which implies $U' \cap \bigcup_{\beta<\alpha} F^\mu_\beta \neq \emptyset$. Therefore $\exists$ can pick a suitable element $x' \in \bigcup_{\beta<\alpha} F^\mu_\beta$ such that the round finishes in a position x' of smaller μ-depth.

We now show that the game $\mathscr{G}^\nu_{II}(F)$ characterises the greatest clopen fixpoint. Suppose that $x \in \nu F \in \mathrm{Clp}(\mathbb{X})$. Then, as in the proof for the game $\mathscr{G}^I_\nu(F)$, $\exists$ has a simple winning strategy by always moving to $\nu F \in \mathrm{Clp}(\mathbb{X})$. For the converse we show that for all ordinals α we have $X \setminus F^{\nu}_{\alpha} \subseteq Win_{\forall}(\mathscr{G}^{II}_{\nu}(F))$ by induction on α. The cases $\alpha = 0$ and $\alpha = \beta + 1$ follow easily from the inductive hypothesis. Suppose α is a limit ordinal and consider some $x \notin F^{\nu}_{\alpha} = \bigwedge_{\beta<\alpha} F^{\nu}_{\beta}$ and suppose that $\exists$ moves to some $U \in \mathrm{Clp}(\mathbb{X})$ such that $x \in F(U)$. Then it is easy to see that $U \not\subseteq \bigcap_{\beta<\alpha} F^{\nu}_{\beta}$, for otherwise $U \subseteq \mathrm{Int}(\bigcap_{\beta<\alpha} F^{\nu}_{\beta}) = \bigwedge_{\beta<\alpha} F^{\nu}_{\beta}$ and hence

$$x \in F(U) \subseteq F(\bigwedge_{\beta<\alpha} F^{\nu}_{\beta}) \subseteq \bigwedge_{\beta<\alpha} F^{\nu}_{\beta}.$$

Therefore $\forall$ can pick some $x' \notin \bigcap_{\beta<\alpha} F^{\nu}_{\beta}$, ie., $x' \notin F^{\nu}_{\beta}$ for some $\beta < \alpha$. By I.H. we know that $\forall$ has a winning strategy from position x' and hence - as $\exists$'s move to U was arbitrary - we showed that $\forall$ has a winning strategy from position x. This finishes the proof of $X \setminus F^{\nu}_{\alpha} \subseteq Win_{\forall}(\mathscr{G}^{II}_{\nu}(F))$ which is equivalent to $F^{\nu}_{\alpha} \subseteq X \setminus Win_{\forall}(\mathscr{G}^{II}_{\nu}(F)) = Win_{\exists}(\mathscr{G}^{II}_{\nu}(F))$ for all $\alpha \in \mathrm{ORD}$. The latter implies $Win_{\exists}(\mathscr{G}^{II}_{\nu}(F)) \subseteq \nu F$. $\square$

We conclude our discussion of fixpoint games on extremally disconnected spaces. The reader might wonder why we introduced two games $\mathscr{G}^{I}_{\mu}(F)$, $\mathscr{G}^{II}_{\mu}(F)$ for the least fixpoint of F and two games for the greatest fixpoint. Do we really need both variants of the μ- and ν-games? The reason why both variants seem necessary for proving our adequacy theorem is based on the following observation[2]: The games $\mathscr{G}^{I}_{\mu}$ and $\mathscr{G}^{II}_{\mu}$ characterise both the same least fixpoints and have therefore the same winning regions within the set of states X. It is, however, in general not possible to transform strategies of $\exists$ in the first variant of the μ-game into corresponding strategies for $\exists$ in the second game. To see this, suppose that $\exists$ has a strategy f in $\mathscr{G}^{I} = \mathscr{G}^{I}_{\mu}(F)$ at position x and suppose $f(x) = C$. We would like to equip $\exists$ with a corresponding strategy g in $\mathscr{G}^{II} = \mathscr{G}^{II}_{\mu}(F)$ at position x such that for the next "round" $xUU'y$ of $\mathscr{G}^{II}$ that is compliant with g, there is a corresponding round xCy of $\mathscr{G}^{I}$ compliant with f (and by re-using that argument round-by-round, one could ensure that f is a winning strategy for $\exists$ in $\mathscr{G}^{I}$ iff g is a winning stratgey for $\exists$ in $\mathscr{G}^{II}$).

To achieve this, we have to define $\exists$'s strategy g such that she moves from x in $\mathscr{G}^{II}$ to some suitable clopen set U. Suppose $U \subseteq \mathrm{Cl}(C)$. Then $\forall$ can respond with some $U' \in \mathrm{Clp}(\mathbb{X})$ such that $U \cap U' \neq \emptyset$. This implies $U' \cap \mathrm{Cl}(C) \neq \emptyset$ and thus - as U' is clopen - that $U' \cap C \neq \emptyset$. Hence, $\exists$ can continue the play by picking an element y of $U' \cap C$ which overall results in the partial $\mathscr{G}^{II}$-play $xUU'y$. Clearly, the sequence xCy is also an f-compliant $\mathscr{G}^{I}$-play and therefore can act as the corresponding play for the $\mathscr{G}^{II}$-play $xUU'y$. Similarly one can show that in any play where $\exists$ moves from position x to some U with $U \not\subseteq \mathrm{Cl}(C)$, $\forall$ can ensure that the next state y that is reached in the play will be an element of $X \setminus C$ and therefore that the resulting $\mathscr{G}^{II}$-play is no longer linked to any corresponding f-compliant G^{I}-play.

Therefore we can construct a corresponding strategy for $\exists$ in $\mathscr{G}^{II}$ iff there is a legitimate move U for $\exists$ at x with $U \subseteq \mathrm{Cl}(C)$. In general, however, there is no suitable clopen set $U \subseteq \mathrm{Cl}(C)$ with $x \in F(U)$ - and this property is required for a legitimate move in $\mathscr{G}^{II}$. This is demonstrated by the following example.

Example 3.6. Consider the Stone-Čech compactification $\beta(\mathbb{N})$ of the natural numbers[3], let $C \subseteq \beta(\mathbb{N})$ be the collection of non-principal ultrafilters over $\mathbb{N}$ and consider the (trivially monotone) operator

$$F = id_{\mathrm{Clp}(\beta(\mathbb{N}))} : \mathrm{Clp}(\beta(\mathbb{N})) \to \mathrm{Clp}(\beta(\mathbb{N})).$$

For any clopen $U \in \mathrm{Clp}(\beta(\mathbb{N}))$ we have $U = \hat{S} = \{u \in \beta(\mathbb{N}) \mid S \in u\}$ for some suitable set $S \subseteq \mathbb{N}$. With this in mind, it is easy to see that for all clopens U we have $U \subseteq C$ implies $U = \emptyset$.

Consider now an arbitrary $x \in C$. We have that $x \in F(U)$ for all $U \in \mathrm{Clp}(\beta\mathbb{N})$ such that $C \subseteq U$ (in particular, C would be a legitimate move in $\mathscr{G}^{I}_{\mu}(F)$ at x). On the other hand, for $U \in \mathrm{Clp}(\beta\mathbb{N})$ we have

[2] We state this observation for μ, but it equally applies to ν.

[3] Which is extremally disconnected, see eg [19].

that $U \subseteq \mathrm{Cl}(C) = C$ implies $U = \emptyset$ and thus $x \notin F(U)$ for all these U (which shows that there is no suitable move for $\exists$ in $\mathscr{G}^{II}_{\mu}(F)$ at x that correponds to her move from x to C).

4 Game semantics for the μ-calculus on topological spaces

We are now ready to define the game characterisation of the clopen semantics of the modal μ-calculus. Our presentation follows the presentation of the standard game semantics of the modal μ-calculus that can be found e.g. in [22]. In the following we assume that we are dealing with *"clean"* formulas in $\mathscr{L}_{\mu}$:

Definition 4.1. *A formula $\varphi \in \mathscr{L}_{\mu}$ is called clean if no two distinct occurrences of fixpoint operators in φ bind the same propositional variable and if a variable occurs either free or bound in φ (but not both bound and free). For any bound variable $p \in \mathrm{Prop}$ that occurs within a clean formula φ we denote by $\varphi@p = \eta p.\psi$ the unique subformula of φ where p is bound by the fixpoint operator $\eta \in \{\mu, \nu\}$.*

The restriction to clean formulas is standard practice in the modal literature. It will simplify the game definition. Furthermore it allows us to give a concise definition of when the unfolding of one fixpoint variable depends on the unfolding of another one.

Definition 4.2. *For a clean formula $\varphi \in \mathscr{L}_{\mu}$ and bound variables $x,y \in \mathrm{Prop}$ occurring in φ we say $x \leq_{\varphi} y$ if $\varphi@x$ is a subformula of $\varphi@y$.*

Definition 4.3. *Let $\varphi \in \mathscr{L}_{\mu}$ be a formula and let $\mathbb{M} = (\mathbb{X}, R, V)$ be an extremally disconnected Kripke model together with valuation $V : \mathrm{Prop} \to \mathrm{Clp}(\mathbb{X})$. The game board of the evaluation game $\mathscr{E}(\varphi, \mathbb{M})$ is specified in the table in Figure 2.*

As usually a finite full play of $\mathscr{E}(\varphi, \mathbb{M})$ is lost by the player who got stuck at the end of the play. In order to specify the winning condition on infinite plays π we need the following notation:

$$Inf(\pi) := \{p \in BVar(\varphi) \mid p \text{ occurs infinitely often in } \pi\}.$$

A standard argument shows that for any infinite play π of $\mathscr{E}(\varphi, \mathbb{M})$ the set $Inf(\pi)$ is nonempty, finite and upwards directed with respect to the dependency order $\leq_{\varphi}$. Therefore the maximal element $\max(Inf(\pi))$ of $Inf(\pi)$ wrt $\leq_{\varphi}$ is well-defined and we declare $\exists$ to be the winner of an infinite play π of $\mathscr{E}(\varphi, \mathbb{M})$ iff $\max(Inf(\pi))$ is a ν-variable, ie., a variable bound by a greatest fixpoint operator.

After our discussion of fixpoint games, the reader should have little problems with understanding the intuition behind the winning condition: an infinite play during which the highest infinitely often "unfolded" fixpoint variable is a ν-variable corresponds to an infinite play of a greatest fixpoint game. Therefore $\exists$ wins such a play. Similarly all infinite plays in which the highest infinitely often unfolded variable is a μ-variable are won by $\forall$. We now turn to the formulation and proof of the main theorem of this section. First we need to introduce some terminology and an auxiliary lemma.

Definition 4.4. *Consider a two-player graph game $\mathscr{G}$ with set of positions B. For a set $Y \subseteq B$ we say a $\mathscr{G}$-play π is Y-full if either π is a full play or $\pi = b_0 \dots b_n$ is a partial play with $b_0, \dots, b_{n-1} \notin Y$ and $b_n \in Y$, i.e., b_n is the first position of the play occurring in Y.*

Lemma 4.5. *Let $\mathbb{M} = (\mathbb{X}, R, V)$ be an extremally disconnected model, let $\varphi = \eta p.\delta$ with $\eta \in \{\mu, \nu\}$ be a fixpoint formula and consider the games $\mathscr{G}_{\eta} = \mathscr{E}(\eta p.\delta, \mathbb{M})$ and $\mathscr{G}_U = \mathscr{E}(\delta, \mathbb{M}[p \mapsto U])$ with $U \in \mathrm{Clp}(\mathbb{X})$. Furthermore we let $\mathrm{unfold}_p = \{(p, x') \mid x' \in X\}$.*

(i) Any strategy f_{η} for $\exists$ in $\mathscr{G}_{\eta}$ at (δ, x) corresponds to a strategy f_U for $\exists$ in $\mathscr{G}_U$ at (δ, x) such that any unfold_p-full, f_{η}-compliant $\mathscr{G}_{\eta}$-play starting at (δ, x) is an f_U-compliant, full $\mathscr{G}_U$-play.

Position	Player	Possible Moves
(p,x), $p \in FVar(\varphi)$ and $x \notin V(p)$	$\exists$	$\emptyset$
(p,x), $p \in FVar(\varphi)$ and $x \in V(p)$	$\forall$	$\emptyset$
$(\neg p,x)$, $p \in FVar(\varphi)$ and $x \notin V(p)$	$\forall$	$\emptyset$
$(\neg p,x)$, $p \in FVar(\varphi)$ and $x \in V(p)$	$\exists$	$\emptyset$
$(\psi_1 \wedge \psi_2, x)$	$\forall$	$\{(\psi_1,x),(\psi_2,x)\}$
$(\psi_1 \vee \psi_2, x)$	$\exists$	$\{(\psi_1,x),(\psi_2,x)\}$
$(\Diamond\psi,x)$	$\exists$	$\{(\psi,x') \mid Rxx'\}$
$(\Box\psi,x)$	$\forall$	$\{(\psi,x') \mid Rxx'\}$
$(\eta p.\psi,x)$, $\eta \in \{\mu,\nu\}$	$\exists/\forall$	(ψ,x)
(p,x), $p \in BVar(\varphi)$, $\varphi@p = \mu p.\psi$	$\forall$	$\{(p,U) \mid U \in \mathrm{Clp}(\mathbb{X}),\ x \in U\}$
(p,x), $p \in BVar(\varphi)$, $\varphi@p = \nu p.\psi$	$\exists$	$\{(p,U) \mid U \in \mathrm{Clp}(\mathbb{X}),\ x \in U\}$
(p,U), $p \in BVar(\varphi)$, $\varphi@p = \mu p.\psi$	$\exists$	$\{(\psi,x') \mid x' \in U\}$
(p,U), $p \in BVar(\varphi)$, $\varphi@p = \nu p.\psi$	$\forall$	$\{(\psi,x') \mid x' \in U\}$

where x,x' denote elements of X and U denotes a clopen subset of $\mathbb{X} = (X,\tau)$.

Figure 2: Game board of the evaluation game $\mathscr{E}(\varphi,\mathbb{M})$

(ii) Any strategy f_U of $\exists$ in $\mathscr{G}_U$ at (δ,x) corresponds to a strategy f_η for $\exists$ in $\mathscr{G}_\eta$ at (δ,x) such that for any full f_U-compliant $\mathscr{G}_U$-play starting at (δ,x) is an f_η-compliant, unfold$_p$*-full $\mathscr{G}_\eta$-play.*

Proof. The lemma follows from the fact that a sequence of the form $\pi = (\delta,x)b_1 \dots b_j \dots$ is an unfold$_p$-full $\mathscr{G}_\eta$-play iff it is a full $\mathscr{G}_U$-play. □

Theorem 4.6 (Adequacy). *Let $\mathbb{M} = (\mathbb{X},R,V)$ be an extremally disconnected model with valuation V :* Prop $\to$ Clp$(\mathbb{X})$. *For every formula $\varphi \in \mathscr{L}_\mu$ and every $x \in X$ the following are equivalent:*

(i) $x \in [\![\varphi]\!]_V$, and

(ii) $\exists$ has a winning strategy at position (φ,x) in $\mathscr{E}(\varphi,\mathbb{M})$.

(Sketch). The proof goes by induction on φ. We only will sketch the induction step for the case that $\varphi = \mu p.\delta$ - the full proof of the theorem is quite lengthy and most of the details are similar to the adequacy proof of the standard game semantics for the modal μ-calculus. We put $\mathscr{G} = \mathscr{E}(\varphi,\mathbb{M})$ and for any clopen subset $U \in \mathrm{Clp}(\mathbb{X})$ we put $\mathscr{G}_U = \mathscr{E}(\delta,\mathbb{M}[p \mapsto U])$.

By the induction hypothesis on δ and because $[\![\delta]\!]_{V[p\mapsto U]} = \delta_p^V(U)$ we have for all $U \in \mathrm{Clp}(\mathbb{X})$ that

$$x \in \delta_p^V(U) \text{ iff } (\delta,x) \in Win_\exists(\mathscr{G}_U). \tag{1}$$

In order to prove the theorem for $\varphi = \mu p.\delta$ it suffices to show that the following are equivalent:

$$x \in Win_\exists\left(\mathscr{G}_\mu^I(\delta_p^V)\right) \tag{2}$$

$$x \in Win_\exists\left(\mathscr{G}_\mu^{II}(\delta_p^V)\right) \tag{3}$$

$$(\varphi,x) \in Win_\exists(\mathscr{G}). \tag{4}$$

We proved the equivalence of (2) and (3) in the previous section. To prove all of the equivalences, we will now show that (3) implies (4) which in turn implies (2). For the implication from (3) to (4) consider some state $x \in Win_{\exists}(\mathscr{G}^{II}_{\mu}(\delta^V_p))$, ie., $\exists$ has a history-free winning strategy at position x in $\mathscr{G}^{II}_{\mu}(\delta^V_p))$ represented by two (possibly partial) functions

$$U : X \to \mathrm{Clp}(\mathbb{X}) \qquad \text{and} \qquad N : \mathrm{Clp}(\mathbb{X}) \to X.$$

W.l.o.g. we can assume that $\langle U, N\rangle$ is winning for $\exists$ from all positions in $Win_{\exists}(\mathscr{G}^{II}_{\mu}(\delta^V_p))$ (in particular, U and N are defined at those positions). As the strategy U is winning (and thus legitimate) at all $x \in Win_{\exists}(\mathscr{G}^{II}_{\mu}(\delta^V_p))$ we have that for all such x that $U(x)$ is a legitimate move at x. Hence $x \in \delta^V_p(U(x))$ and thus, by (1), $(\delta, x) \in Win_{\exists}(\mathscr{G}_{U(x)})$. Therefore, for each $x \in Win_{\exists}(\mathscr{G}^{II}_{\mu}(\delta^V_p))$, we can assume

(a) that there is a winning strategy $f_{U(x)}$ for $\exists$ in the game $\mathscr{G}_{U(x)}$ at position (δ, x) and

(b) that $(\forall, U(x)) \in Win_{\exists}(\mathscr{G}^{II}_{\mu}(\delta^V_p))$.

As seen in Lemma 4.5, the winning strategy $f_{U(x)}$ can be (trivially) turned into a valid strategy $f_{\mu,x}$ for $\exists$ in $\mathscr{G}$ at (δ, x) that can be followed until another position of the form (p, x') is reached or until $\exists$ wins the game. This observation is important for defining $\exists$'s strategy in $\mathscr{G}$ starting from position (φ, x):

- starting from (φ, x), the play proceeds to (δ, x) and after that $\exists$ plays strategy $f_{\mu,x}$.
- if the $f_{\mu,x}$-compliant play never reaches a position of the form (p, x') then $\exists$ continues playing according to $f_{\mu,x}$ and wins: the resulting $f_{\mu,x}$-compliant, full $\mathscr{G}$-play contains a $f_{U(x)}$-compliant full $\mathscr{G}_U$-play (by Lemma 4.5) starting at (δ, x) which is won by $\exists$ as $f_{U(x)}$ is a winning strategy for $\exists$ in $\mathscr{G}_{U(x)}$ at (δ, x).
- Suppose an $f_{\mu,x}$-compliant play reaches a position of the form (p, x'). Until now - by Lemma 4.5 - the play corresponds to a $f_{U(x)}$-compliant play of $\mathscr{G}_{U(x)}$. As $f_{U(x)}$ is a winning strategy for $\exists$ in $\mathscr{G}_{U(x)}$ this entails that $x' \in U(x)$. It is now $\forall$'s turn to move in $\mathscr{G}$ to a position (p, U') with $x' \in U'$. As $x' \in U(x) \cap U'$ (by the definition of $\mathscr{G}$), we have $U(x) \cap U' \neq \emptyset$, i.e., the move to $(\exists, U')$ is a legal move for $\forall$ in $\mathscr{G}^{II}_{\mu}(\delta^V_p)$ at position $(\forall, U(x))$. As the latter is an element of $Win_{\exists}(\mathscr{G}^{II}_{\mu}(\delta^V_p))$, we also have that $(\exists, U') \in Win_{\exists}(\mathscr{G}^{II}_{\mu}(\delta^V_p))$. Hence $\exists$'s winning strategy N in $Win_{\exists}(\mathscr{G}^{II}_{\mu}(\delta^V_p))$ specifies a well-defined, legitimate move at U' that follows $\exists$'s winning strategy in $Win_{\exists}(\mathscr{G}^{II}_{\mu}(\delta^V_p))$.
  Therefore, in $\mathscr{G}$, $\exists$ answers $\forall$'s move to (p, U') by moving to (δ, y) with $y = N(U')$ and continues from there according to strategy $f_{\mu,y}$.

It is not difficult to check, that this describes indeed a winning strategy for $\exists$ in $\mathscr{G}$ from position (φ, x). The key observation is that for any $\mathscr{G}$-play of the form

$$\pi = (\varphi, x) \ldots (p, x_1)(\delta, U_1)(\delta, y_1) \ldots (p, x_2)(\delta, U_2)(\delta, y_2) \ldots (p, x_i)(\delta, U_i)(\delta, y_i) \ldots$$

there is a corresponding infinite play of $\mathscr{G}^{II}_{\mu}(\delta^V_p)$ of the form

$$\pi' = x\ (\forall, U(x))\ (\exists, U_1)\ y_1\ (\forall, U(y_1))\ (\exists, U_2)\ y_2 \ldots (\forall, U(y_{i-1}))\ (\exists, U_i)\ y_i \ldots$$

which is compliant with $\exists$'s winning strategy in $\mathscr{G}^{II}_{\mu}(\delta^V_p)$ and where the number of fixpoint unfoldings in π' is equal to the number of occurrences of positions of the form (p, x') in π. As π' is won by $\exists$, the play π' must end after finitely many moves. Hence there are only finitely many occurrences of positions of the form (p, x') in π, i.e., from a certain position (δ, x') on the play follows $\exists$'s strategy $f_{\mu,x'}$ in $\mathscr{G}$ at (δ, x'). In other words, such a play is won by $\exists$ as - modulo a finite prefix - it corresponds by our construction to a $f_{U(x')}$-compliant $\mathscr{G}_{U(x')}$-play from position (δ, x') and $f_{U(x')}$ is a winning strategy for $\exists$ at (δ, x').

We now turn to the proof of the implication from (4) to (2). Consider a strategy f for $\exists$ in $\mathscr{G}$ such that f is winning for all positions in $Win_{\exists}(\mathscr{G})$ and let $\Delta := \{x \in X \mid (\delta, x) \in Win_{\exists}(\mathscr{G})\}$. To prove our claim it suffices to show that $\Delta \subseteq Win_{\exists}(\mathscr{G}^I_\mu(\delta^V_p))$ by equipping $\exists$ with a suitable strategy in $\mathscr{G}^I_\mu(\delta^V_p)$ that is winning at all positions in Δ. As before, we let $\mathrm{unfold}_p = \{(p,x) \mid x \in X\}$ and for all $x \in \Delta$ we put

$$
\begin{aligned}
C(x) \quad := \quad \{z_y \in X \mid \exists y \in X.(p,y) & \text{ is reachable in an unfold}_p\text{-full } \mathscr{G}\text{-play } \pi \text{ from } (\delta,x) \text{ such that} \\
& \pi \text{ is compliant with } \exists\text{'s strategy } f, \\
& \forall \text{ can move from } (p,y) \text{ to position } (p,U_y) \\
& \text{to which } \exists\text{'s reply according to her strategy } f \text{ is to move to } (\delta,z_y) \text{ with } z_y \in U_y\}
\end{aligned}
$$

Let $x \in \Delta$ and let $U \subseteq X$ be clopen with $C(x) \subseteq U$. With our definition of $C(x)$, it can be easily seen that $\exists$ has a winning strategy at (δ,x) in $\mathscr{G}_U$: Firstly, by Lemma 4.5, for each $U \subseteq X$ we know that $\exists$ has a strategy f_U in $\mathscr{G}_U$ at (δ,x) such that every unfold_p-full $\mathscr{G}$-play π compliant with f starting at (δ,x) corresponds to a full, f_U-compliant $\mathscr{G}_U$-play.

Suppose now for a contradiction that there is some $U' \in \mathrm{Clp}(\mathbb{X})$ with $C(x) \subseteq U'$ for which $(\delta,x) \notin Win_{\exists}(\mathscr{G}_{U'})$. This implies that the strategy $f_{U'}$ cannot be winning for $\exists$ in $\mathscr{G}_{U'}$ at (δ,x) and thus there exists some state (p,y) with $y \notin U'$ and with the property that (p,y) is reachable from (δ,x) in an full $\mathscr{G}$-play π compliant $\exists$'s strategy $f_{U'}$. By definition of $f_{U'}$, there exists a unfold_p-full $\mathscr{G}$-play π from (δ,x) to (p,y) that is compliant with f. This leads to a contradiction: at position (p,y) in $\mathscr{G}$ - as $y \in X \setminus U'$ by assumption - $\forall$ could move to $(p, X \setminus U')$ and $\exists$ could choose an element $z_y \in X \setminus U'$ and move to (δ, z_y) according to her strategy f. By definition of C, we get $z_y \in C(x) \subseteq U'$ and hence $z_y \in U'$ which is a contradiction.

This finishes the proof of the fact that $\exists$ has a winning strategy at (δ,x) in $\mathscr{G}_U$ for any clopen set $U \subseteq X$ with $C(x) \subseteq U$. Consequently, by (1), we have $x \in \delta^V_p(U)$ for all $U \in \mathrm{Clp}(\mathbb{X})$ with $C(x) \subseteq U$. This means that for each $x \in \Delta$, $\exists$ can move from position x to position $C(x)$ in $\mathscr{G}^I_\mu(\delta^V_x)$, i.e., C encodes a legitimate strategy for $\exists$ in all positions $x \in \Delta$. We are now going to prove that for any play

$$x\, C(x)\, x_1\, C(x_1)\, x_2\, C(x_2) \ldots x_n\, C(x_n)$$

of $\mathscr{G}^I_\mu(\delta^V_p)$ starting in x and compliant with $\exists$ strategy C it is possible to construct a "shadow" play of $\mathscr{G}$ starting at (φ,x) that is compliant with $\exists$'s winning strategy in $\mathscr{G}$ and that is of the form

$$(\varphi,x) \ldots (\delta,x_1) \ldots (\delta,x_2) \ldots (\delta,x_n).$$

It suffices to see how a round $x_i\, C(x_i)\, x_{i+1}$ in $\mathscr{G}^I_\mu(\delta^V_p)$ is mirrored in $\mathscr{G}$. To this aim note that $x_{i+1} \in C(x_i)$. Hence there exists some $U \in \mathrm{Clp}(\mathbb{X})$ with $x_{i+1} \in U$ such that (p,U) is reachable from (δ,x_i) via a $\mathscr{G}$-play π compliant with $\exists$'s winning strategy that is continued by $\exists$ by moving to position (δ,x_{i+1}). Clearly the play π followed by $\exists$'s move to (δ,x_{i+1}) constitutes the required shadow play of $\mathscr{G}$. □

Example 4.7. We will give an example of an extremally disconnected modal space $(\mathbb{X},R)$ with $\mathbb{X} = (X,\tau)$, a clopen valuation V and a modal formula $\varphi(q,p)$ such that the standard semantics of $\mu q.\varphi$ and the topological semantics of $\mu q.\varphi$ differ. Let $\mathbb{Z}$ be the set of integers with the discrete topology. Let $X = \beta(\mathbb{Z})$ be the Stone–Čech compactification of $\mathbb{Z}$. Then $\beta(\mathbb{Z})$ is extremally disconnected, see eg [19]. We define a relation R on X by zRy iff ($z,y \in \mathbb{Z}$ and $y = z+1$ or $y = z-1$ or $z \in X$ and $y \in \beta(\mathbb{Z}) \setminus \mathbb{Z}$). Now we define a clopen valuation $V(p) = \{0\}$. Consider the formula $\varphi(q,p) = p \vee \Diamond\Diamond q$. The standard semantics of $\mu q.\varphi$ is equal to the set of all even and negative even numbers. The topological semantics, in contrast, is equal to the whole space X.

5 Bisimulations

We are now going to describe bisimulations for our topological setting. The definition is essentially the standard one with an additional topological condition.

Definition 5.1. *Let $\mathbb{M}_1 = (\mathbb{X}_1, R_1, V)$ and $\mathbb{M}_2 = (\mathbb{X}_2, R_2, V)$ be extremally disconnected Kripke models based on the spaces $\mathbb{X}_1 = (X_1, \tau_1)$ and $\mathbb{X}_2 = (X_2, \tau_2)$. A relation $Z \subseteq X_1 \times X_2$ is called a* clopen bisimulation *iff $Z \subseteq X_1 \times X_2$ is a (standard) Kripke bisimulation and for any clopen subsets $U_1 \in \mathrm{Clp}(\mathbb{X}_1)$ and $U_2 \in \mathrm{Clp}(\mathbb{X}_2)$ we have $Z[U_1] = \{x' \in X_2 \mid \exists x \in U_1.(x,x') \in Z\} \in \mathrm{Clp}(\mathbb{X}_2)$ and $Z^{-1}[U_2] = \{x \in X_1 \mid \exists x' \in U_2.(x,x') \in Z\} \in \mathrm{Clp}(\mathbb{X}_1)$.*

The justification for the notion of clopen bisimulations is provided by the following proposition.

Proposition 5.2. *Let Z be a clopen bisimulation between extremally disconnected Kripke models $\mathbb{M}_1 = (\mathbb{X}_1, R_1, V)$ and $\mathbb{M}_2 = (\mathbb{X}_2, R_2, V)$. Then for any formula $\varphi \in \mathscr{L}_\mu$ of the modal μ-calculus and any states $x \in X_1$ and $x' \in X_2$ such that $(x,x') \in Z$, we have $x \in [\![\varphi]\!]$ iff $x' \in [\![\varphi]\!]$.*

Proof. Suppose that $(x,x') \in Z$ and that $x \in [\![\varphi]\!]$ for some formula φ. This implies by our adequacy theorem that $(\varphi, x) \in \mathit{Win}_\exists(\mathscr{E}(\varphi, \mathbb{M}_1))$. We are now going to transform $\exists$'s winning strategy in $\mathscr{G}_1 = \mathscr{E}(\varphi, \mathbb{M}_1)$ at position (φ, x) into a winning strategy for $\exists$ in $\mathscr{G}_2 = \mathscr{E}(\varphi, \mathbb{M}_2)$ at position (φ, x').

As a preparation we need to define when we consider positions of $\mathscr{G}_1$ and $\mathscr{G}_2$ to be equivalent: we say $(\psi_1, x_1) \in \mathscr{L}_\mu \times X_1$ and $(\psi_2, x_2) \in \mathscr{L}_\mu \times X_1$ are Z-equivalent if $\psi_1 = \psi_2$ and $(x_1, x_2) \in Z$. Furthermore we write $(p, U_1) \leq_Z (q, U_2)$ for $(p, U_1) \in \mathscr{L}_\mu \times \mathrm{Clp}(\mathbb{X}_1)$ and $(p, U_2) \in \mathscr{L}_\mu \times \mathrm{Clp}(\mathbb{X}_2)$ if $p = q$ and if for all $x \in U_1$ there exists $x' \in U_2$ such that $(x,x') \in Z$. Similarly we define $(p, U_1) \geq_Z (q, U_2)$. Consider two (possibly partial) plays $\pi_1 = b_1 \ldots b_k$ and $\pi_2 = b'_1 \ldots b'_l$ of $\mathscr{G}_1$ and $\mathscr{G}_2$, respectively. We say π_1 and π_2 are Z-equivalent iff $k = l$ and for all $i = 1, \ldots, k$ we have

- b_i and b'_i are of the form $b_i = (\psi, x)$ and $b'_i = (\psi, x')$ and both positions are Z-equivalent, or
- $b_i = (p, U_1)$, $b'_i = (p, U_2)$, p is bound by μ and $(p, U_1) \leq_Z (p, U_2)$, or
- $b_i = (p, U_2)$, $b'_i = (p, U_2)$, p is bound by ν and $(p, U_1) \geq_Z (p, U_2)$.

Let π_1 be a play of $\mathscr{G}_1$ that starts in $\exists$'s winning position (φ, x) and that is played according to $\exists$'s winning strategy. We are going to show that if π_2 is a Z-equivalent play of $\mathscr{G}_2$ that starts at position (φ, x'), then either

- both plays π_1 and π_2 are full (and thus won by $\exists$) or
- it is $\exists$'s turn and $\exists$ has a strategy to extend π_2 to a play $\pi_2 b'$ that is Z-equivalent to an extension $\pi_1 b$ of π_1 such that $\pi_1 b$ is a $\mathscr{G}_1$-play compliant with $\exists$'s winning strategy, or
- it is $\forall$'s turn and for all of $\forall$'s moves that extend π_2 to $\pi_2 b'$ there is a move of $\forall$ in $\mathscr{G}_1$ such that the resulting play $\pi_1 b$ of $\mathscr{G}_1$ is Z-equivalent to $\pi_2 b'$.

Clearly this claim will imply that $\exists$ has a winning strategy in $\mathscr{G}_2$ at position (φ, x') as required. The claim is proven by a case distinction on the last state of π_2. Due to space reasons we only discuss the cases of the modal diamond and the (least) fixpoint cases.

Case: $\pi_2 = b'_1 \ldots b'_n(\Diamond\psi, x_2)$. By assumption there exists a Z-equivalent play $\pi_1 = b_1 \ldots b_n(\Diamond\psi, x_1)$ which in particular implies that $(x_1, x_2) \in Z$. Clearly it is $\exists$'s turn and she can prolong the $\mathscr{G}_1$-play by moving according to her strategy to (ψ, y) for some $y \in X_1$ with $(x_1, y) \in R_1$. As Z is a bisimulation we know that there must be $y' \in X_2$ such that $(x_2, y') \in R_2$ and $(y.y') \in Z$. Hence $\forall$ can prolong the π_2-play by moving to (ψ_1, y') and the resulting plays $\pi_1 = b_1 \ldots b_n(\Diamond\psi, x_2)(\psi, y)$ and $\pi_2 = b'_1 \ldots b'_n(\Diamond\psi, x_1)(\psi, y')$ are Z-equivalent.

Case: $\pi_2 = b'_1 \ldots b'_n(p, x_2)$ for some $p \in \mathit{BVar}(\varphi)$ that is bound by a μ-operator. In this case $\pi_1 = b_1 \ldots b_n(p, x_1)$ and its $\forall$'s turn to continue both plays. Let $\forall$'s move in $\mathscr{G}_2$ be to (p, U_2) for some clopen subset $U \in \mathrm{Clp}(\mathbb{X}_2)$ with $x_2 \in U_2$. Because $(x_1, x_2) \in Z$ and by the definition of a clopen bisimulation we have that $U_1 := Z^{-1}[U_2]$ is a clopen neighbourhood of x_1. Therefore $\forall$ could extend the $\mathscr{G}_1$-play by moving to (p, U_1). The resulting plays $\pi_1 = b_1 \dots b_n(p, x_1)(p, U_1)$ and $\pi_2 = b'_1 \dots b'_n(p, x_2)(p, U_2)$ are clearly Z-equivalent because all elements U_1 have their Z-correspondant in U_2 and hence we have $(p, U_1) \leq_P (p, U_2)$ as required.

Case: $\pi_2 = b'_1 \dots b'_n(p, U_2)$ for some $p \in BVar(\varphi)$ that is bound by a μ-operator. By assumption we have a Z-equivalent $\mathscr{G}_1$-play $\pi_1 = b_1 \dots b_n(p, U_1)$ with the property that $(p, U_1) \leq_Z (p, U_2)$. By the definition of the game it is clear that in both plays $\exists$ has to move. She continues π_1 by moving according to her winning strategy to some (ψ, y) with $y \in U_1$. By definition of $\leq_Z$ there exists a $y' \in U_2$ such that $(y, y') \in Z$ and hence $\exists$ can extend the play π_2 by moving to (ψ, y'). Again the resulting plays $\pi_1 = b_1 \dots b_n(p, U_1)(\psi, y)$ and $\pi_2 = b_1 \dots b_n(p, U_2)(\psi, y')$ are obviously Z-equivalent. The other cases of the induction can be dealt with in a similar fashion. This shows that from $x \in [\![\varphi]\!]$ and $(x, x') \in Z$ we are able to deduce $x' \in [\![\varphi]\!]$. The implication in the opposite direction can be proven in a completely symmetrical way. As φ was arbitrary we conclude that clopen bisimilarity implies equivalence with respect to the topological modal μ-calculus. □

Remark 5.3. We leave it open whether the converse of Proposition 5.2 also holds, i.e., whether we have a Hennessy-Milner property wrt our notion of clopen bisimulation. A closely related question is how our clopen bisimulations compare to the Vietoris bisimulations of [4]. It is obvious that the topological closure of a clopen bisimulation is a Vietoris bisimulation and hence that clopen bisimilarity implies Vietoris bisimilarity. Proving the converse would yield the Hennessy-Milner property with regard to clopen bisimilarity as a corollary of [4, Cor. 3.10].

6 Conclusions and future work

In this paper we developed game semantics for topological fixpoint logic on extremally disconnected modal spaces. These results can be seen as first steps towards the theory of topological fixpoint logic in general and towards admissible game semantics of μ-calculus in particular. As next steps we intend to extend this framework to other classes of descriptive μ-frames and to devise automata that operate on Kripke frames over topological spaces. This will provide a deeper understanding of these structures as well as of axiomatic systems of the modal μ-calculus, since axiomatic systems of the μ-calculus are complete wrt descriptive μ-frames. Other important questions concern the finite model property, decidability and computational complexity and other key properties of topological fixpoint logics.

A further interesting research direction is to investigate modal fixpoint logic of Kripke frames based on compact Hausdorff spaces and beyond. However, instead of clopen sets we will have to work with regular open sets in this setting. This means we will enter the realm of modal compact Hausdorff spaces introduced in [3]. These are exactly the spaces that correspond to coalgebras for the Vietoris functor on the category of compact Hausdorff spaces. Sahlqvist fixpoint correspondence for such spaces has been developed already in [7]. I This approach could pave the way for an expressive and decidable fixpoint logic for the verification of continuous systems or, more generally, systems that combine discrete and continuous systems such as hybrid automata [13].

Finally, we want to clarify the connection of our work to topological games à la Banach-Mazur [20]. These games are similar to our fixpoint games as players move by choosing e.g. open subsets - the fundamental differences are i) they characterise properties of the topology rather than properties of a relational structure over a topological space and ii) our parity winning condition that ensures determinacy.

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