The Optimal Way to Play the Most Difficult Repeated Coordination Games

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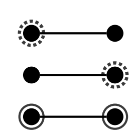
This paper investigates repeated win-lose coordination games (WLC-games). We analyse which protocols are optimal for these games, covering both the worst case and average case scenarios, i,e., optimizing the guaranteed and expected coordination times. We begin by analysing Choice Matching Games (CM-games) which are a simple yet fundamental type of WLC-games, where the goal of the players is to pick the same choice from a finite set of initially indistinguishable choices. We give a fully complete classification of optimal expected and guaranteed coordination times in two-player CM-games and show that the corresponding optimal protocols are unique in every case—except in the CM-game with four choices, which we analyse separately.

Our results on CM-games are essential for proving a more general result on the difficulty of all WLC-games: we provide a complete analysis of least upper bounds for optimal expected coordination times in all two-player WLC-games as a function of game size. We also show that CM-games can be seen as the most difficult games among all two-player WLC-games, as they turn out to have the greatest optimal expected coordination times.

1 Introduction

Pure win-lose coordination games (WLC-games) are simple yet fundamental games where all players receive the same payoffs: 1 (win) or 0 (lose). This paper studies *repeated* WLC-games, where the players make simultaneous choices in discrete rounds until (if ever) succeeding to coordinate on a winning profile. *Choice matching games* (CM-games) are the simplest class of such games. The choice matching game CM_m^n has n players with the goal to choose the same choice among m different indistinguishable choices, with no communication during play. The players can use the history of the game (i.e., the players' choices in previous rounds) for their benefit as the game proceeds. For simplicity, we denote the two-player game CM_m^2 by CM_m.

A paradigmatic real-life scenario with a choice matching game relates to a phenomenon that has humorously been called "pavement tango" or "droitwich" in [1]. Here two people try to pass each other but may end up blocking each other by repeatedly moving sideways into the same direction. For another example of a choice matching game, consider CM_3, the coordination-based variant of the rock-paper-scissors game, pictured on the right.



Here the two players (i.e., columns) coordinate if they succeed choosing an edge from one of the three rows. The players first choose randomly; suppose they select the nodes in dotted circles. Next the players have three options: (1) to repeat their first choice, (2) to try to coordinate with the first choice of the other player, or (3) to select the choice which has not been selected yet. Supposing that the players behave symmetrically, (3) is the only way to guarantee coordination in the second round.

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m	Optimal **expected** coordination time in CM_m	Unique optimal protocol for expected time	Optimal **guaranteed** coordination time in CM_m	Unique optimal protocol for guaranteed time
1	1	(any)	1	(any)
2	2	WM	∞	—
3	$1+\frac{2}{3}$	LA	2	LA
4	$2+\frac{1}{2}$	—	∞	—
5	$2+\frac{1}{3}$	LA	3	LA
6	$2+\frac{2}{3}$	WM	∞	—
7	$2+\frac{5}{7}$	WM	4	LA
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2k$	$3-\frac{1}{k}$	WM	∞	—
$2k+1$	$3-\frac{2}{2k+1}$	WM	k	LA

Figure 1: A complete analysis of two-player choice matching games.

A general n-player WLC-game is a generalization of CM_m^n where the players do not necessarily have to choose from the same row to coordinate, and it may not even suffice to choose from the same row. In classical matrix form representation, two-player choice matching games have ones on the diagonal and zeroes elsewhere, while general two-player WLC-games have general distributions of ones and zeroes; see Definition 2.1 for the full formal details.

In repeated WLC-games, it is natural to try to coordinate as quickly as possible. There are two main scenarios to be investigated: *guaranteeing coordination* (with certainty) in as few rounds as possible and minimizing the *expected number of rounds* for coordination. The former concerns the number of rounds it takes to coordinate in the *worst case* and is measured in terms of *guaranteed coordination times* (GCTs). The latter relates to the *average case* analysis measured in terms of *expected coordination times* (ECTs).

Our contributions. We provide a comprehensive study of upper bounds for coordination in *all* two-player repeated WLC-games, including a classification of related optimal strategies (called *protocols* in this work). WLC-games are represented in a novel way as *relational structures*, which is a key to many of the techniques used in the paper. CM-games are central to our work, being a fundamental class of games and also the most difficult games for coordination—in a sense made precise below.

Two protocols play a central role in our study. We introduce the so-called *loop avoidance* protocol LA (cf. Def. 4.1) that essentially instructs players to play so that the generated history of choices always reduces the symmetries (e.g., automorphisms) of the game structure. We also use the so-called *wait-or-move* protocol WM (cf. Def. 4.5), essentially telling players to randomly alternate between two choices that both coordinate with at least one of the opponent's two choices. We show that WM leads to coordination in *all* two-player WLC-games *very fast*, the ECT being at most $3-2p$, where p is the probability of coordinating in the first round with random choices.

We then provide a complete analysis of the optimal ECTs and GCTs in all choice matching games CM_m. We also identify the protocols giving the optimal ECTs and GCTs and show their *uniqueness*, where possible. The table in Figure 1 on the next page summarizes these results. Our analysis is fully *complete*, as we prove that there exists a continuum of optimal protocols for CM_4 and establish that for all even m, no protocol *guarantees* a win in CM_m.

Concerning the more general class of all WLC-games, we provide the following complete characterization of upper bounds for the optimal ECTs in all two-player WLC-games *as a function of game size* (a game in a classical matrix form is of size m when the maximum of the number of rows and columns is m):

Theorem. *For any m, the greatest optimal* ECT *among two-player* WLC*-games of size m is as follows:*

Game size	$m \in \mathbb{Z}_+ \setminus \{3,5\}$	$m = 5$	$m = 3$
Greatest optimal ECT	$3 - \frac{2}{m}$	$2 + \frac{1}{3}$	$\frac{1+\sqrt{4+\sqrt{17}}}{2}$ (≈ 1.925)

Also, concerning two-player CM-games, we establish that CM_m has the strictly greatest optimal ECT out of all two-player WLC-games of size $m \neq 3$, making CM-games the most difficult WLC-games to coordinating in. We give a separate full analysis of the case $m = 3$.

Two comments are in order. Firstly, in real-life scenarios it can be very inefficient to determine the absolutely optimal protocols taking into account the full game structure. Indeed, this generally requires an analysis of the game structure and thus identifying, e.g., automorphic choices. Computing automorphisms of (even bipartite graphs) is well known to be hard. However, our analysis gives an instant way of finding a fast protocol for any two-player WLC-game G. The ECT given by this protocol in G is better or equal to the greatest optimal ECT of the games of the same size (cf. the theorem above). Secondly, our main analysis concerns only two-player games. However, the arguments for this case are already quite involved, so the n-player case is expected to be highly complex and is thus left for the future. Furthermore, the two-player case is an especially important special case covering, e.g., learning of communication protocols in distributed systems.

Related work. Coordination games (see, e.g., [4], [3]) are a key topic in game theory, with the early foundations laid, inter alia, in the works of Schelling [17] and Lewis [15]. Repeated games are—likewise—a key topic, see for example [12], [2], [16]. For seminal work on *repeated coordination games*, see for example the articles [6], [5], [14]. Repeated coordination games have a wide range of applications, from learning and social choice theory to symmetry breaking in distributed systems.

However, WLC-games are a simple class of games that have not been extensively studied in the literature. In particular, choice matching games clearly constitute a *highly fundamental class of games*, and it is thus surprising that the analysis of the current paper has not been previously carried out. Thus the related analysis is well justified; it *closes a gap* in the literature. Indeed, while CM-games are simple, they precisely capture the highly fundamental coordination problem of *picking the same choice from a set of choices*.

Our study differs from the classical game-theoretic work on repeated games where the focus is on accumulated payoffs instead of the discrete outcomes of WLC-games. Indeed, repeated WLC-games are based on *reachability objectives*. Especially our worst case (but also the average case) analysis has only superficial overlap with typical work on repeated games.

However, similar work exists, the most notable example being the seminal article [6] that studies a generalization of WLC-games in a framework that has some similarities with our setting. Also the papers [8], [10], [7], [9] have a similar focus, as they also concentrate on coordination games with discrete win-lose outcomes. However, the papers [8], [10], [7], [9] do not investigate optimality of protocols in repeated games, unlike the current paper. The seminal work [6] introduces (what is equivalent to) the two-player CM-games in their final section on general examples. They also essentially identify the optimal ways of playing CM_2 and CM_3 (discussed also in this article) although in a technically somewhat different setting with accumulated payoffs. Furthermore, they observe that a protocol essentially equivalent to WM is the best way to play CM_6, an observation we also make in our setting. However, optimality of WM in CM_6 is not proved in [6]. This would require an extensive analysis proving that the players cannot make beneficial use of asymmetric histories created by non-coordinating choices. Indeed, the main technical difficulty in our corresponding setting is to show *uniqueness* of the optimal protocol, which we do in each case where a unique protocol exist.

Nonetheless, despite the differences, the framework of [6] bears some conceptual similarities to ours, e.g., the authors also identify *structural protocols* (cf. Definition 3.4 below) as the natural notion of strategy for studying their framework. Furthermore, they make extensive use of focal points [17] in analysing how asymmetric histories can be used for coordination.

Relating to uniqueness of protocols, [11] argues that individual rationality considerations not are not sufficient for "learning how to coordinate" in the setting of [6]. We agree with [11] that some conventions are needed if many protocols lead to the optimal result. However—in our framework—as we can prove *uniqueness* of the optimal protocols for CM_m (for $m \neq 4$), then arguably rational players should adopt precisely these protocols in CM-games.

The paper [13] is the preprint of the current submission with full proofs and further examples.

Techniques used. The core of our work relies on an original approach to games based on relational structures, as opposed to using the traditional matrix form representation. This approach enables us to use graph-theoretic ideas in our arguments. Both in the worst-case and in average-case analysis, the main technical work relies heavily on analysis of symmetries—especially the way the groups of automorphisms of games evolve when playing coordination games. The most involved result of the worst-case analysis, Theorem 5.2, is proved by reducing the cardinality of the automorphism group of the WLC-game studied in a maximally fast fashion. In the average-case analysis, Theorems 6.2, 6.3, 6.4 are proved via a combination of analysis of extrema; keeping track of groups of automorphisms; graph-theoretic methods; and focal points [17] for breaking symmetry. The most demanding part here is to show *uniqueness* of the protocols involved. Also in the average-case analysis, Theorem 7.2 relies on earlier theorems and an extensive and exhaustive analysis of certain bipartite graphs.

In principle one could of course reduce our arguments into the setting with games presented in matrix form. However, then it would become harder to apply natural graph-theoretic notions like degrees of nodes; special features such as cycles; general symmetries (automorphisms) et cetera. Such notions are key elements in our analysis.

2 Preliminaries

We define (pure) win-lose coordination games as *relational structures* as follows.

Definition 2.1. An n-player **win-lose coordination game (WLC-game)** is a relational structure $G = (A, C_1, \dots, C_n, W_G)$ where A is a finite domain of **choices**, each C_i is a non-empty unary relation (representing the choices of player i) such that $C_1 \cup \dots \cup C_n = A$, and $W_G \subseteq C_1 \times \dots \times C_n$ is an n-ary **winning relation**. For technical convenience, we assume the players have pairwise disjoint choice sets, i.e., $C_i \cap C_j = \emptyset$ for all i and $j \neq i$. A tuple $\sigma \in C_1 \times \dots \times C_n$ is a **choice profile** for G and the choice profiles in W_G are **winning (choice) profiles**. We assume there are no surely losing choices, i.e., choices $c \in A$ that do not belong to any winning choice profile, as rational players would never select such choices.

We will use the visual representation of WLC games as hypergraphs; two-player games become just bipartite graphs under this scheme. The choices of each player are displayed as columns of nodes, starting from the choices of player 1 on the left and ending with the column of choices of player n. The winning relation consists of lines that represent the winning choice profiles. Thus winning choice profiles are also called **edges**. See Example A.1 in [13] for an illustration.

Consider a WLC-game $G = (A, C_1, \dots, C_n, W_G)$ with n players and m winning choice profiles that do not intersect, i.e., none of the m winning choice profiles share a choice $c \in A$. Such games form a simple yet fundamental class of games, where the goal of the players is simply to pick the same "choice", i.e., to simultaneously pick one of the m winning profiles. These games are called **choice matching games**. We let CM^n_m denote the choice matching game with n players and m choices for each player. In this article, we extensively make use of the two-player choice matching games, CM^2_m. For these games, we will omit the superscript "2" and simply denote them by CM_m. (Recall the example CM_3 pictured in the introduction.)

Interestingly, out of all n-player WLC-games where each of the n players has m choices, the game CM^n_m has the least probability of coordination when each player plays randomly. In this sense these games can be seen the most difficult for coordination. A fully compelling reason for the maximal difficulty of choice matching games is given later on by Theorem 7.3.

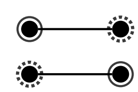
3 Repeated WLC-Games

A **repeated play of a WLC-game** G consists of consecutive (one-step) plays of G. The repeated play is continued until the players successfully coordinate, i.e., select their choices from a winning choice profile. This may lead to infinite plays. We assume that each player can remember the full history of the repeated play and use this information when planning choices. The history of the play after k rounds is encoded in a sequence $\mathscr{H}_k$ defined as follows.

Definition 3.1. Let G be an n-player WLC-game. A pair $(G, \mathscr{H}_k)$ is called a **stage** k **(or** k**th stage) in a repeated play of** G, where the **history** $\mathscr{H}_k$ is a k-sequence of choice profiles in G. More precisely, $\mathscr{H}_k = \big(H_i\big)_{i\in\{1,\dots,k\}}$ where each H_i is an n-ary relation $H_i = \{(c_1, \dots, c_n)\}$ with a single tuple $(c_1, \dots, c_n) \in C_1 \times \cdots \times C_n$. In the case $k = 0$, we define $\mathscr{H}_0 = \emptyset$. The stage $(G, \mathscr{H}_0)$ is the **initial stage** (or the 0th stage). Like G, also $(G, \mathscr{H}_k)$ is a relational structure.

A stage k contains a history specifying precisely k choice profiles chosen in a repeated play. A winning profile of $(G, \mathscr{H}_k)$ is called a **touched edge** if it contains some choice c picked in some round $1, \dots, k$ leading to $(G, \mathscr{H}_k)$. As we assume that the players only need to coordinate once, we consider repeated plays only up to the first stage where some winning choice profile is selected. If coordination occurs in the kth round, the kth stage is called the **final stage** of the repeated play. (But a play can take infinitely long without coordination.)

On the right is a drawing of the stage 2 in a repeated play of CM_2, the "coordination game variant" of the *matching pennies game* (or the "pavement tango" from the introduction). Here the players have failed to coordinate in round 1 (having picked the choices with dotted circles) and then failed again by both swapping their choices in round 2 (solid circles).



A protocol gives a *mixed strategy for all stages in all* WLC*-games and for all player roles i*:

Definition 3.2. A **protocol** π is a function outputting a probability distribution $f : C_i \to [0,1]$ (so $\sum_{c\in C_i} f(c) = 1$) with the input of a player i and a stage $(G, \mathscr{H}_k)$ of a repeated WLC-game.

Since a protocol can depend on the full history of the current stage, it gives a mixed, memory-based strategy for any repeated WLC-game. Thus protocols can informally be regarded as global "behaviour styles" of agents over the class of all repeated WLC-games. It is important to note that all players can see (and remember) the previous choices selected by all the other players—and also the order in which the choices have been made.

In the scenario that we study, it is obvious to require that the protocols should act *independently of the names of choices and the names (or ordering) of player roles i*.[1] In [6], this requirement follows from the "assumption of no common language" (for describing the game), and in [10], such protocols are called *structural*. We shall adopt the terminology of [10]. To extend this concept for repeated games, we first need to define the notion of a *renaming*. The intuitive idea of renamings is to extend *isomorphisms* between game graphs—including the history—to additionally enable *permuting the players* $1,\dots,n$ (see Example A.2 in [13] for an illustration of the definition).

Definition 3.3. A **renaming** between stages $(G,\mathscr{H}_k)$ and $(G',\mathscr{H}_k')$ of n-player WLC-games G and G' is a pair (β,h) where β is a permutation of $\{1,\dots,n\}$ and h a bijection from the domain of G to that of G' such that

- $c \in C_{\beta(i)} \Leftrightarrow h(c) \in C_i'$ for all $i \leq n$ and c in the domain of G,
- $(c_1,\dots,c_n) \in W_G \Leftrightarrow (h(c_{\beta(1)}),\dots,h(c_{\beta(n)})) \in W_{G'}$,
- $(c_1,\dots,c_n) \in H_i \Leftrightarrow (h(c_{\beta(1)}),\dots,h(c_{\beta(n)})) \in H_i'$ for all $i \leq k$.

If $(G,\mathscr{H}_k)$ and $(G',\mathscr{H}_k')$ have the same domain A, we say that (β,h) is a **renaming of** $(G,\mathscr{H}_k)$. Choices $c \in C_i$ and $d \in C_j$ are **structurally equivalent**, denoted by $c \sim d$, if there is a renaming (β,h) of $(G,\mathscr{H}_k)$ s.t. $\beta(i) = j$ and $h(c) = d$. It is easy to see that $\sim$ is an equivalence relation on A. We denote the equivalence class of a choice c by $[c]$.

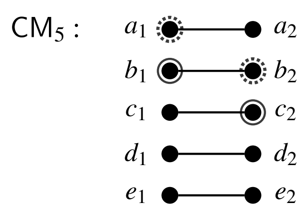
Definition 3.4. A protocol π is **structural** if it is indifferent with respect to renamings, i.e., if $(G,\mathscr{H}_k)$ and $(G',\mathscr{H}_k')$ are stages with a renaming (β,π) between them, then for any i and any $c \in C_i$, we have $f(c) = f'(\pi(c))$, where $f = \pi((G,\mathscr{H}_k),i)$ and $f' = \pi((G',\mathscr{H}_k'),\beta(i))$.

Note that a structural protocol may depend on the full history which records even the order in which the choices have been played. Hereafter we assume all protocols to be structural.

Definition 3.5. Let G be a WLC-game and let S and S' be stages of G. Let $\sim$ (respectively, $\sim'$) be the structural equivalence relation over S (respectively, S'). We say that S and S' are **automorphism-equivalent** if $\sim = \sim'$. The stages S and S' are **structurally similar** if one can be obtained from the other by a chain of renamings and automorphism-equvalences.

A choice c in a stage S is a **focal point** if it is not structurally equivalent to any other choice in that same stage S, with the possible exception of choices c' that *all belong to some single edge with c*. A focal point breaks symmetry and can be used for winning a repeated coordination game. This requires that the players have some (possibly prenegotiated) way to agree on which focal point to use. See the following example for an illustration.

Example 3.6. Consider the first two rounds of the game CM_5, pictured below, where the players fail to coordinate by first selecting the pair (a_1,b_2) and then fail again by selecting the pair (b_1,c_2).



[1] Note that if this assumption is not made, coordination can trivially be guaranteed in a single round in any WLC-game by using a protocol which chooses some winning choice profile with probability 1.

The structural equivalence classes become modified in this scenario as follows:

- Initially all choices are structurally equivalent.
- After the first round, the equivalence classes are $\{a_1, b_2\}$, $\{b_1, a_2\}$ and $\{c_1, d_1, e_1, c_2, d_2, e_2\}$.
- After the second round, the equivalence classes are $\{a_1\}$, $\{a_2\}$, $\{b_1\}$, $\{b_2\}$, $\{c_1\}$, $\{c_2\}$ and $\{d_1, e_1, d_2, e_2\}$.

There are no focal points in the initial stage S_0 and the same is true for the next stage S_1. However, in the stage S_2, all the choices $a_1, b_1, c_1, a_2, b_2, c_2$ become focal points, and the players can thus immediately guarantee coordination in the third round by selecting any winning pair of focal points, i.e., any of the pairs (a_1, a_2), (b_1, b_2), (c_1, c_2). (We note that, from the point of view of the general study of rational choice, it may not be obvious which of these pairs should be selected, so a convention may be needed to fix which protocol to use.) For another type of example on focal points, see Example A.3 in [13].

In repeated coordination games, it is natural to try to *coordinate as quickly as possible*. There are two principal scenarios related to optimizing coordination times: the *average case* and the *worst case*. The former concerns the expected number rounds for coordination and the latter the maximum number in which coordination can be guaranteed with certainty.

Definition 3.7. Let $(G, \mathscr{H}_k)$ be a stage and let π be a protocol. The **one-shot coordination probability (OSCP) from $(G, \mathscr{H}_k)$ with π** is the probability of coordinating in a single round from $(G, \mathscr{H}_k)$ when each player follows π. The **expected coordination time (ECT) from $(G, \mathscr{H}_k)$ with π** is the expected value for the number of rounds until coordination from $(G, \mathscr{H}_k)$ when all players follow π. The **guaranteed coordination time (GCT) from $(G, \mathscr{H}_k)$ with π** is the number n such that the players are *guaranteed* to coordinate from $(G, \mathscr{H}_k)$ in n rounds, but not in $n-1$ rounds, when all players follow π, if such a number exists. Otherwise this value is ∞.

The OSCP, ECT and GCT from the initial stage $(G, \emptyset)$ with π are referred to as the OSCP, ECT and GCT in G with π. We say that π is **ECT-optimal** for G if π gives the minimum ECT in G, i.e., the ECT given by any protocol π' is at least as large as the one given by π. **GCT-optimality** of π for G is defined analogously.

It is possible that there are several different protocols giving the optimal ECT (or GCT) for a given WLC-game. If two protocols π_1 and π_2 are both optimal, it may be that the optimal value is nevertheless *not* obtained when some of the players follow π_1 and the others π_2. This leads to a meta-coordination problem about choosing the same optimal protocol to follow. However, such a problem will be avoided if there exists a unique optimal protocol.

Definition 3.8. Let π be a protocol and G a WLC-game. We say that π is **uniquely ECT-optimal** for G if π is ECT-optimal for G and the following holds for all other protocols π' that are ECT-optimal for G: for any stage S in G that is reachable with π, we have $\pi'(S) = \pi(S)$. **Unique GCT-optimality** of π for G is defined analogously.[2]

The next lemma states that two structurally similar stages are essentially the same stage with respect to different ECTs and GCTs. The proof is straightforward.

Lemma 3.9. *Assume stages S and S' of G are structurally similar. Now, for any protocol π, there exists a protocol π' which gives the same* ECT *and* GCT *from S' as π gives from S.*

[2]Note that if two different protocols are uniquely ECT-optimal for G (and similarly for unique GCT-optimality), then their behaviour on G can differ only on stages that are not reachable in the first place by the protcols. Also, their behaviour can of course differ on games other than G.

4 Protocols for Repeated WLC-Games

In this section we introduce two special protocols, the *loop avoidance protocol* LA and the *wait-or-move protocol* WM. Informally, LA asserts that in every round, every player i should avoid—if possible—all choices c that could possibly make the resulting stage automorphism-equivalent (cf. Def. 3.5) to the current stage, i.e., the stage just before selecting c.

Definition 4.1. The **loop avoidance protocol** (LA) asserts that in every round, every player i should avoid—if possible—all choices c for which the following condition holds: if the player i selects c, then there exist choices for the other players so that the resulting stage is automorphism-equivalent to the current stage. If this condition holds for all choices of the player i, then i makes a random choice. Moreover, uniform probability is used among all the possible choices of i.

It is easy to see that LA avoids, when possible, all such stages that are structurally similar to *any* earlier stage in the repeated play. As structurally similar stages are essentially identical (cf. Lemma 3.9), repetition of such stages can be seen as a "loop" in the repeated play. When trying to *guarantee* coordination as quickly as possible, such loops should be avoided. In addition to this heuristic justification, Theorems 5.1 and 5.2 give a fully compelling justification for LA when considering guaranteed coordination in two-player CM-games. For now, we present the following results (for the proofs, see the correspondingly numbered Propositions 4.2 and 4.3 in [13]); see also Example 4.4 below for an illustration of the use of LA.

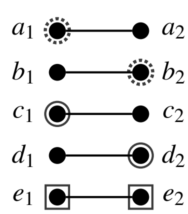
Proposition 4.2. LA *is uniquely* ECT*-optimal and uniquely* GCT*-optimal in* CM_3.

Proposition 4.3. LA *guarantees coordination in games* CM_m *in* $\lceil m/2 \rceil$ *rounds when m is odd, but* LA *does not guarantee coordination in* CM_m *for any even m.*

Example 4.4. We illustrate the use of the LA protocol in the game CM_5, pictured below. Suppose that coordination fails in the first round. By symmetry, we may assume that the players selected a_1 and b_2. Now, in the resulting stage S_1, the structural equivalence classes are $\{a_1,b_2\}$, $\{b_1,a_2\}$ and $\{c_1,d_1,e_1,c_2,d_2,e_2\}$.

If the pair (b_1,a_2) is selected in the next round, then the structural equivalence classes do not change and thus the resulting next stage is automorphism-equivalent to S_1. Hence, by following LA, player 1 should avoid selecting b_1 and player 2 should avoid selecting a_2. For the same reason, the players should also avoid selecting the choices a_1 and b_2.

CM_5 :



Hence, by following LA in S_1, the players will select among the set $\{c_1,d_1,e_1,c_2,d_2,e_2\}$ with the uniform probability distribution. Supposing that they fail again in coordination, we may assume by symmetry that they selected the pair (c_1,d_2). The equivalence classes in the resulting stage S_2 are $\{a_1,b_2\}$, $\{b_1,a_2\}$, $\{c_1,d_2\}$, $\{d_1,c_2\}$ and $\{e_1,e_2\}$. Now, selecting any of the pairs (a_1,b_2), (b_1,a_2), (c_1,d_2) and (d_1,c_2) leads to a next stage which is automorphism-equivalent to S_2. Thus, by following LA in S_2, the players will select the pair (e_1,e_2). This leads to guaranteed coordination in the third round.

We next present the *wait-or-move protocol* WM, which naturally appears in numerous real-life two-player coordination scenarios. Informally, both players alternate (with equal probability) between two choices: the players own initial choice and another choice that coordinates with the initial choice of the other player.

Definition 4.5. The **wait-or-move protocol** (WM) for repeated two-player WLC-games goes as follows: first randomly select any choice c, and thereafter choose, with equal probability, c or a choice c' that coordinates with the initial choice of the other player (thereby never picking other choices than c and c').

Definition A.5 in [13] specifies WM in yet further detail. The following theorem shows that WM is very fast in relation to ECTs. This holds for *all* two-player WLC-games, not only choice matching games CM_m. (The claim is validated by the proof of Proposition 4.5 in [13].)

Theorem 4.6. *Let G be a* WLC*-game with one-shot coordination probability p when both players make their first choice randomly. Then the expected coordination time by* WM *is at most* $3-2p$*. Thus the* ECT *with* WM *is strictly less than* 3 *in every two-player* WLC*-game.*

It follows from the proof of Theorem 4.6 that the ECT with WM is *exactly* $3-\frac{2}{m}$ in all choice matching games CM_m. Thus the last claim of the theorem *cannot be improved*, as the ECTs of the games CM_m grow asymptotically closer to the strict upper bound 3 when m is increased. In the particular case of CM_2, the ECT with WM is $3-\frac{2}{2}=2$. Thus the following clearly holds.

Lemma 4.7. *When* $S=(\mathsf{CM}_m,\mathscr{H}_k)$ *is a non-final stage with exactly two touched edges, then the* ECT *from S with* WM *is 2. Moreover, in any* WLC*-game G, if* $S'=(G,\mathscr{H}_k)$ *is a non-final stage that is reachable by using* WM*, then the* ECT *from* S' *with* WM *is at most 2.*

WM eventually leads to coordination with asymptotic probability 1 in all two-player WLC-games. But it does not guarantee (with certainty) coordination in any number of rounds in WLC-games where the winning relation is not the total relation. In a typical real-life scenario, eternal non-coordination is of course impossible by WM, but it is conceivable, e.g., that two computing units using the very same pseudorandom number generator will never coordinate due to being synchronized to swap their choices in precisely the same rounds.

It is easy to show that WM is the unique protocol which gives the optimal ECT (namely, 2 rounds) in the "droitwich-scenario" of the game CM_2 (see the proof of Proposition 4.8 in [13]).

Proposition 4.8. WM *is uniquely* ECT*-optimal in* CM_2.

Next we compare the pros and cons of LA and WM in two-player CM-games. Recall that WM does not guarantee coordination in CM_m (when $m\neq 1$), while LA does guarantee coordination in CM_m if and only if m is *odd*. Concerning *expected* coordination times, it is easy to prove that WM gives a smaller ECT than LA in CM_m for all even m (except for the case $m=2$, where WM and LA behave identically). Thus we now restrict attention to the games CM_m with odd m. Then, the probability of coordinating in the ℓ-th round of CM_m using LA, with $\ell\leq\lceil m/2\rceil$, can relatively easily be seen to be calculable by the formula $P_{\ell,m}$ defined below (where the product is 1 when $\ell=1$). And using the formula for $P_{\ell,m}$, we also get a formula for the expected coordination time E_m in CM_m with LA:

$$P_{\ell,m}=\frac{1}{m-2\ell+2}\prod_{k=0}^{k=\ell-2}\frac{m-2k-1}{m-2k},\qquad E_m=\sum_{\ell=1}^{\ell=\lceil m/2\rceil}\ell\cdot P_{\ell,m}.$$

Using this and Theorem 4.6, we can compare the ECTs in CM_m with LA and WM for odd m (see the following table).

m	ECT in CM_m with WM	ECT in CM_m with LA
1	1	1
3	$2+\frac{1}{3}$	$1+\frac{2}{3}$
5	$2+\frac{3}{5}$	$2+\frac{1}{3}$
7	$2+\frac{5}{7}$	3
9	$2+\frac{7}{9}$	$3+\frac{2}{3}$

Especially the case $m = 7$ is interesting, as the ECT with LA is exactly 3 which is precisely the strict upper bound for the ECTs with WM for the class of all two-player choice matching games CM_m. Furthermore, $m = 7$ is the case where WM becomes faster than LA in relation to ECTs. Thus WM clearly stays faster than LA for all $m \geq 7$, including even values of m.

5 Optimizing Guaranteed Coordination Times

In this section we investigate when coordination can be guaranteed in two-player CM-games and which protocols give the optimal GCT for them. The following result is a direct consequence of the symmetries of CM_m for even m (see the proof of Proposition 5.1 in [13]):

Theorem 5.1. *For all even $m \geq 2$, there is no protocol guaranteeing coordination in* CM_m*.*

We then consider choice matching games CM_m with an odd m. Proposition 4.3 showed that the GCT with LA in these games is $\lceil m/2 \rceil$. The next theorem shows that this is the optimal GCT for CM_m, and moreover, LA is the unique protocol giving this GCT. The proof of Theorem 5.2 is an interesting exercise in the optimally fast elimination of automorphisms.

Theorem 5.2. *For any odd $m \geq 1$,* LA *is uniquely* GCT*-optimal for* CM_m*.*

Proof. Let m be odd. Recall that, by Proposition 4.3, the GCT in CM_m with LA is $\lceil m/2 \rceil$ rounds. We assume, for contradiction, that there is some protocol $\pi \neq \mathsf{LA}$ that guarantees coordination in CM_m in at most $\lceil m/2 \rceil$ of rounds, possibly less. As $\pi \neq \mathsf{LA}$, there exists some play of CM_m where both players follow π, and in some round, at least one of the players chooses a node on a touched edge. (Recall from the proof of Proposition 4.3 that LA never chooses from a touched edge in CM-game with an odd number of edges.) Now, let $S_\ell = (\mathsf{CM}_m, \mathscr{H}_\ell)$ be the first stage of that play when this happens—so if (c, c') is the most recently recorded pair of choices in S_ℓ, then at least one of c and c' is part of an edge that has already been touched in some earlier round. And furthermore, in all stages $S_{\ell'}$ with $\ell' < \ell$, the most recently chosen pair does not contain a choice belonging to an edge that was touched in some yet earlier round $\ell'' < \ell'$.

In the stage $S_{\ell-1}$ it therefore holds that for every choice profile (c_i, d_i), chosen in some round $i \leq (\ell-1)$, the nodes c_i and d_i are structurally equivalent. Of course also the nodes of $S_{\ell-1}$ on so far untouched edges are structurally equivalent to each other. Furthermore, the number of already touched edges in $S_{\ell-1}$ is the *even* number $m' = 2(\ell-1)$.

We will now show that π does *not* guarantee a win in $\lceil m/2 \rceil - (\ell-1)$ rounds when starting from the stage $S_{\ell-1}$. This completes the proof, contradicting the assumption that π guarantees a win in CM_m in at most $\lceil m/2 \rceil$ rounds.

Now, recall the stage S_ℓ from above where (c, c') contained a choice from an already touched edge. By symmetry, we may assume that c is such a choice. Starting from the stage $S_{\ell-1}$, consider a newly defined stage S'_ℓ where the first player again makes the choice c but the other player this time *makes a structurally equivalent choice* $c^* \sim c$. This is possible as π is a structural protocol. Now note that the choice profile (c, c^*) is not winning since c and c^* are structurally equivalent choices from already touched edges, and thus either (c,c^*) is a choice profile that has already been chosen in some earlier round $j < \ell$, or the nodes d,d^* adjacent in CM_m to c^*,c (respectively) form a choice profile (d,d^*) chosen in some earlier round $j < \ell$.

Therefore, in the freshly defined stage S'_ℓ, the players have in every stage (including the stage S'_ℓ itself) selected a choice profile that consists of two structurally equivalent choices. Both choices in the most recently selected choice profile in S'_ℓ have been picked from edges that have become touched even earlier. It now suffices to show that it can still take $\lceil m/2 \rceil - (\ell - 1)$ rounds to finish the game. To see that this is the case, we shall next consider a play from the stage S'_ℓ onwards where in each remaining round, the choice profile (e,e^*) picked by the players consists of structurally equivalent choices; such a play exists since π is structural.

Due to picking only structurally equivalent choices in the remaining play, when choosing a profile from the already touched part, the players will clearly never coordinate. And when choosing from the untouched part, immediate coordination is guaranteed if and only if there is only one untouched edge left. Therefore the players coordinate exactly when they ultimately select from the last untouched edge. As the stage S'_ℓ has precisely $m-2(\ell-1)$ untouched edges, winning in this play takes at least $\lceil \frac{m-2(\ell-1)}{2} \rceil = \lceil m/2 \rceil - (\ell - 1)$ rounds to win from S'_ℓ. □

6 Optimizing Expected Coordination Times

In this section we investigate which protocols give the best ECTs for two-player choice matching games. We also investigate when the best ECT is obtained by a unique protocol. We already know by Propositions 4.8 and 4.2 that the optimal ECTs for CM_2 and CM_3 are uniquely given by WM and LA, respectively. Thus it remains to consider the games CM_m with $m \geq 4$. We first cover the case $m \geq 6$ and show that then WM is the unique protocol giving the best ECT. The remaining special cases $m = 4$ and $m = 5$ will then be examined. The following auxiliary lemma (proven in [13]) will be used in the proofs.

Lemma 6.1. *The* ECT *from* $(\mathsf{CM}_m, \mathscr{H}_k)$ *with no focal point is at least* $\frac{3}{2}$ *with any protocol.*

We are now ready to cover the case for choice matching games CM_m with $m \geq 6$.

Theorem 6.2. WM *is uniquely* ECT*-optimal for each* CM_m *with* $m \geq 6$.

Proof. We first present a formula for estimating the best ECTs in stages of CM_m (with any $m \geq 1$). Let $S := (\mathsf{CM}_m, \mathscr{H}_k)$ be a non-final stage with exactly two touched edges. Thus there are $n := m-2$ untouched edges. Suppose the players use a protocol π behaving as follows in round $k+1$. Both players pick a choice from some touched edge with probability p and from an untouched edge with probability $(1-p)$. A uniform distribution is used on choices in both classes: probability $\frac{p}{2}$ for both choices on touched edges (which makes sense by Lemma B.2 of [13]) and probability $\frac{1-p}{n}$ for each choice on untouched edges (which is necessary with a structural protocol). If one player selects a choice c from a touched edge and the other one a choice c' from an untouched edge, the players win in the next round by choosing the edge with c'. Note that c' is a focal point, so the winning edge can be chosen by a structural protocol with probability 1. (Also other focal points arise which could alternatively be used; cf. Example 3.6):

Suppose then that E_1 is the ECT with π from a stage $(\mathsf{CM}_m, \mathscr{H}_{k+1})$ where both players have chosen a touched edge in round $k+1$ but failed to coordinate. Two different such stages $(\mathsf{CM}_m, \mathscr{H}_{k+1})$ exist, but they are automorphism-equivalent, so π can give the same ECT from both of them by Lemma 3.9.

(Indeed, if π gave two different ECTs, it would make sense to adjust it to give the smaller one.) Similarly, suppose E_2 is the ECT with π from a stage $(\mathsf{CM}_m, \mathscr{H}'_{k+1})$ where both players have chosen an untouched edge in round $k+1$ but failed to coordinate. Note that all possible such stages $(\mathsf{CM}_m, \mathscr{H}'_{k+1})$ are renamings of each other, so π gives the same ECT from each one. We next establish that the expected coordination time from $(\mathsf{CM}_m, \mathscr{H}_k)$ with π is now given by the following formula (called formula (E) below):

$$p^2\Big(\frac{1}{2}+\frac{1}{2}(1+E_1)\Big) \;+\; 2p(1-p)\cdot 2 \;+\; (1-p)^2\left(\frac{1}{n}+\frac{n-1}{n}\Big(1+E_2\Big)\right) \qquad \text{(E)}$$

Indeed, both players choose a touched edge in round $k+1$ with probability p^2. In that case the ECT from $(\mathsf{CM}_m, \mathscr{H}_k)$ is $\frac{1}{2}+\frac{1}{2}(1+E_1)$, the first occurrence of $\frac{1}{2}$ corresponding to direct coordination and the remaining term covering the case where coordination fails at first. Both players choose an untouched edge in round $k+1$ with probability $(1-p)^2$, and then the ECT from $(\mathsf{CM}_m, \mathscr{H}_k)$ is $\frac{1}{n}+\frac{n-1}{n}(1+E_2)$. The remaining term $2p(1-p)\cdot 2$ is the contribution of the case where one player chooses a touched edge and the other player an untouched one. The probability for this is $2p(1-p)$, and the remaining factor 2 indicates that coordination immediately happens in the subsequent round $k+2$ using the focal point created in round $k+1$.

We then present an informal argument sketch for the case CM_m with $m \geq 6$. We may assume that $E_1 \leq 2$ and $E_2 \geq \frac{3}{2}$ by Lemmas 4.7 and 6.1. Figure 2 below illustrates the graph of (E) with $E_1 = 2$, $E_2 = \frac{3}{2}$, $n = 4$, so then (E) has a unique minimum at $p = 1$ when $p \in [0,1]$. This suggests that—under these parameter values—the players should always choose a touched edge in stages with exactly two touched edges. Clearly, lowering E_1, raising E_2 or raising n should make it even more beneficial to choose a touched edge. As we indeed can assume that $E_1 \leq 2$ and $E_2 \geq \frac{3}{2}$ in CM_m for $m \geq 6$, this informally justifies that WM is uniquely ECT-optimal in CM_m.

Next we formalize the argument above. Let $S := (\mathsf{CM}_m, \mathscr{H}_k)$, $m \geq 6$, be a non-final stage with precisely two touched edges and S' a stage extending S by one round where the players both choose an untouched edge but fail to coordinate. Let r_1 (respectively, r_2) be the infimum of all possible ECTs from S (respectively, S') with different protocols. Note that by Lemma 3.9, r_1 and r_2 are independent of which particular representative stages we choose, as long as the stages satisfy the given constraints. Let $\varepsilon > 0$ and fix some numbers E_1 and E_2 such that $|E_1 - r_1| < \varepsilon$ and $|E_2 - r_2| < \varepsilon$. We assume $E_1 \leq 2$ and $E_2 \geq \frac{3}{2}$ by Lemmas 4.7 and 6.1. It is easy to show that with such E_1 and E_2, the minimum value of the formula (E) with $p \in [0,1]$ is obtained at $p = 1$ (for any $n = m-2 \geq 4$).

Thus, after the necessarily random choice in round one, the above reasoning shows the players should choose a touched edge with probability $p = 1$ in each round. Indeed, assume that the earliest occasion that a protocol π_k assigns $p \neq 1$ occurs in round k. Then, as shown above, the ECT of π_k can be strictly improved by letting $p = 1$ in that round. By Lemma B.2 of [13], a uniform probability over the touched choices should be used. □

We then cover the case for CM_5. The argument is similar to the case for CM_m with $m \geq 6$, but this time leads to the use of LA instead of WM.

Theorem 6.3. *For* CM_5*,* LA *is uniquely* ECT*-optimal.*

Proof. Recall the formula (E) from the proof of Theorem 6.2. Let $S := (\mathsf{CM}_5, \mathscr{H}_k)$ be a non-final stage with precisely two touched edges and S' a stage extending S by one round where the players both choose an untouched edge but fail to coordinate. The ECT-optimal protocol from S' chooses the unique winning pair of focal points in round $k+2$, so we now have $E_2 = 1$. Let r_1 be the infimum of all possible ECTs

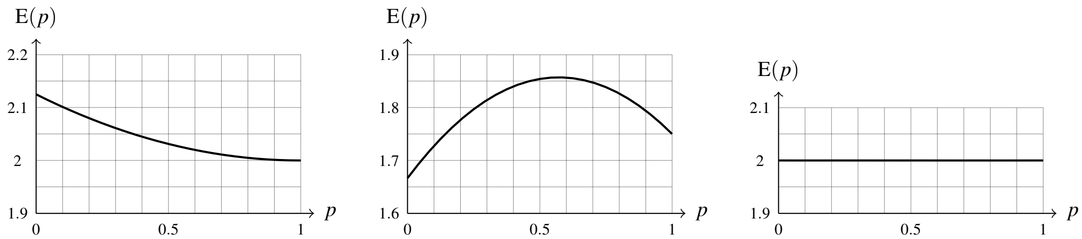


Figure 2: Graph of (E) with (i) $n=4, E_1=2, E_2=\frac{3}{2}$; (ii) $n=3, E_1=\frac{3}{2}, E_2=1$; (iii) $n=2, E_1=E_2=2$.

from S with different protocols. Let $\varepsilon > 0$ and fix some real number E_1 such that $|E_1 - r_1| < \varepsilon$, assuming $E_1 \geq \frac{3}{2}$ (cf. Lemma 6.1). It is straightforward to show that with these values, and with $n = 3$, the minimum of (E) when $p \in [0,1]$ is obtained at $p = 0$. (See also Figure 2 for the graph of (E) when $E_1 = \frac{3}{2}$ for an illustration. Even then the figure suggests to choose an untouched edge.)

Thus, after the necessarily random choice in round one, the above reasoning shows that the players should choose an untouched edge with probability 1 in the second round, thereby following LA. Coordination is guaranteed (latest) in the third round. □

In the last case $m = 4$, WM is ECT-optimal, but not uniquely, as there exist infinitely many other ECT-optimal protocols. The reason for this is that—as shown in Figure 2—the graph of (E) becomes the constant line with the value 2 in the special case where $E_1 = E_2 = 2$, and then any $p \in [0,1]$ gives the optimal value for (E). A complete proof is given in [13].

Theorem 6.4. WM *is* ECT*-optimal for* CM_4*, but there are continuum many other protocols that are also* ECT*-optimal.*

We have thus given a *complete analysis* of optimal ECTs and GCTs in two-player CM-games, summarized in Figure 1. Appendix D of [13] contains reflections on the results. We note that the cases for CM_m with small m are exceptionally important from the point of view of applications, as such cases tend to occur more frequently in real-life scenarios.

7 The Most Difficult Two-Player WLC-Games

In this section we give a complete characterization of the upper bounds of optimal ECTs in WLC-games as a function of game size. For any $m \geq 1$, an ***m*-choice game** refers to any two-player WLC-game $G = (A, C_1, C_2, W_G)$ where $m = \max\{|C_1|, |C_2|\}$. Note that with the classical matrix representation of an m-choice game, the parameter m corresponds to the largest dimension of the matrix. In this section we will also show that CM_m can be seen as the *uniquely most difficult* m-choice game for all $m \neq 3$, see Theorem 7.3.

Our first theorem shows that the wait-or-move protocol is reasonably "safe" to use in any m-choice game with $m \notin \{3,5\}$ as it always guarantees an ECT which is at most equal to the upper bound of optimal ECTs of all m-choice games for the particular m.

Theorem 7.1. *Let $m \notin \{1,3,5\}$ and consider an m-choice game $G = (A, C_1, C_2, W_G) \neq \mathsf{CM}_m$. Then the* ECT *in G with* WM *is strictly smaller than the optimal* ECT *in* CM_m*.*

Proof. By Theorems 6.2, 6.4 and Proposition 4.8, the optimal ECT in CM_m is given by WM. We saw in Section 4 that the ECT with WM is $3 - \frac{2}{m}$ in CM_m and at most $3 - 2p$ in G, where p is the one-shot coordination probability when choosing randomly in G. Since G is an m-choice game, $|W_G| \geq m$. If

$|W_G| > m$, then $p > \frac{m}{m^2} = \frac{1}{m}$. And if $|W_G| = m$, we have $p = \frac{m}{mn} = \frac{1}{n} > \frac{1}{m}$ where $n := \min\{|C_1|, |C_2|\} < m$ since $G \neq \mathsf{CM}_m$. In both cases, we have $3 - 2p < 3 - \frac{2}{m}$. □

The **greatest optimal** ECT among a class $\mathscr{G}$ of WLC-games is the value r such that (1) r is the optimal ECT for some $G \in \mathscr{G}$; and (2) for every $G \in \mathscr{G}$, there is a protocol which gives it an ECT $\leq r$. By Theorem 7.1, the greatest optimal ECT among m-choice games is given by WM in CM_m for $m \notin \{1,3,5\}$. The case $m = 1$ is trivial, but for the special cases $m = 3$ and $m = 5$, we need to perform a systematic graph-theoretic analysis of all such m-choice games and their ECTs to identify the greatest optimal ECT among the class. This analysis is done in Appendix C of [13], but we sketch below the main ideas starting from the case $m = 3$.

Since the optimal ECT for all 3-choice games whose graph has a node with degree 3 is trivially 1 round, we may leave them out of the analysis. All the remaining 3-choice games (up to structural equivalence) are pictured in Figure 3. Except for CM_3 and the last two games on the right, all of these games have a focal point and thus their optimal ECT is 1. The optimal ECT for the second game from the right is $1 + \frac{1}{2}$ which is obtained by the protocol making a uniformly random choice every round. The optimal ECT of the rightmost game is

$$\frac{1}{2}\left(1 + \sqrt{4 + \sqrt{17}}\right)$$

which is indeed the greatest optimal ECT among all 3-choice games. (Moreover, this ECT is obtained by a protocol using highly nontrivial probability distributions for selecting choices.) See Appendix C.1 of [13] for the full details in each case.

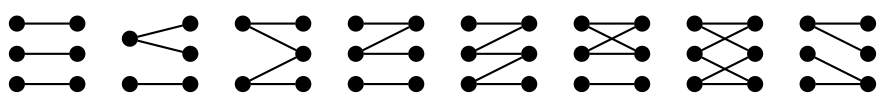
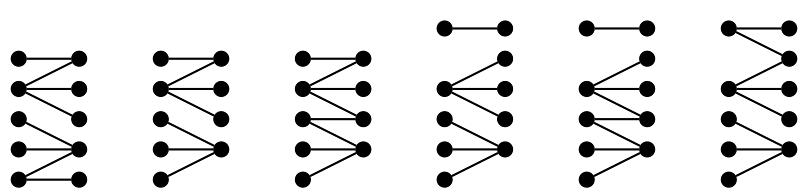


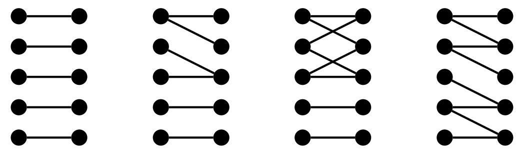
Figure 3: Game graphs of all (nontrivial) 3-choice games. The game on the right has the greatest optimal ECT.

A similar—yet much more complex—analysis can be done for the case $m = 5$ to show that CM_5 has the greatest optimal ECT ($2 + \frac{1}{3}$ rounds) among all 5-choice games. (See Appendix C.2 of [13] for the full details of the analysis sketched here.) We first observe that if $|W_G| > 8$, then the ECT obtained by following WM in G is $2 + \frac{7}{25} < 2 + \frac{1}{3}$. Next we analyze games G with $|W_G| \leq 8$ based on the degrees of choices. If at least one of the players has a choice of degree at least 4, we can prove that either there exists a focal point in G or it has the special form where two choices have degree 4 and all the others have degree 1. In this special case we can obtain the ECT of 2 rounds by giving the probability $\frac{1}{2}$ for the choices with degree 4 in the first round and following WM thereafter.

Suppose next that the degree of all choices in G is at most 3. If at least one of the players has a choice with degree 3, we can use graph-theoretic reasoning to show that there is a way to immediately guarantee coordination—unless both players have *exactly one* choice of degree 3, denoted by c and c'. In this special case, if there is an edge between c and c', they are focal points. Else, G has to be one of the following six games (up to structural equivalence).



Among these games, the leftmost one does not have a focal point unlike all the others, but for that game we can formulate a protocol which gives the ECT of 2 rounds. Finally, supposing that the degrees of all choices in G are at most 2, the graph of G must consist of *disjoint paths and cycles*. Recalling the assumption $|W_G| \leq 8$, there is a total of 28 such games (listed systematically in Appendix C.2 of [13]). Among those, only the following four do not have a focal point.



However, for the last three games above, we can formulate a protocol which gives them an ECT of 2 rounds. Hence CM_5 indeed has the greatest optimal ECT among all 5-choice games.

We argue that these graph-theoretic analyses are interesting for their own sake and demonstrate the usefulness of representing two-player WLC-games as bipartite graphs. By Theorem 7.1 and the analyses for the 3 and 5-choice cases, we obtain the following results.

Theorem 7.2. *For any m, the greatest optimal* ECT *among m-choice games is given below:*

Game size	$m \in \mathbb{Z}_+ \setminus \{3,5\}$	$m = 5$	$m = 3$
Greatest optimal ECT	$3 - \frac{2}{m}$	$2 + \frac{1}{3}$	$\frac{1+\sqrt{4+\sqrt{17}}}{2}$ (≈ 1.925)

Theorem 7.3. *For* $m \neq 3$*, the greatest optimal* ECT *among m-choice games is uniquely realized by* CM_m.

Hence choice matching games can indeed be seen as *the most difficult* two-player WLC-games—excluding the interesting and important special case of 3-choice games.

8 Conclusion

In this paper we gave a complete analysis for two-player CM-games with respect to both GCTs and ECTs, including uniqueness proofs for the related protocols. We also found optimal upper bounds for optimal ECTs for all two-player WLC-games when determined according to game size only. Moreover, our arguments demonstrate the usefulness of representing WLC-games as hypergraphs. The current paper concentrated on the two-player case as this already turned out a challenging question. A complete characterization of the n-player case remains. This is expected to be a highly difficult task that is likely to require sophisticated arguments.

Acknowledgements

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