

Parametric Interval Temporal Logic over Infinite Words

Laura Bozzelli

University of Napoli “Federico II”, Napoli, Italy

laura.bozzelli@unina.it

Adriano Peron

University of Napoli “Federico II”, Napoli, Italy

adrperon@unina.it

Model checking for Halpern and Shoham’s interval temporal logic HS has been recently investigated in a systematic way, and it is known to be decidable under three distinct semantics. Here, we focus on the *trace-based semantics*, where the infinite execution paths (traces) of the given (finite) Kripke structure are the main semantic entities. In this setting, each finite infix of a trace is interpreted as an interval, and a proposition holds over an interval if and only if it holds over each component state (*homogeneity assumption*). In this paper, we introduce a quantitative extension of HS over traces, called *parametric HS* (PHS). The novel logic allows to express parametric timing constraints on the duration (length) of the intervals. We show that checking the existence of a parameter valuation for which a Kripke structure satisfies a PHS formula (model checking), or a PHS formula admits a trace as a model under the homogeneity assumption (satisfiability) is decidable. Moreover, we identify a fragment of PHS which subsumes parametric LTL and for which model checking and satisfiability are shown to be EXPSpace-complete.

1 Introduction

Interval temporal logic HS. *Point-based* Temporal Logics (PTLs), such as the linear-time temporal logic LTL [33] and the branching-time temporal logics CTL and CTL* [16] provide a standard framework for the specification of the dynamic behavior of reactive systems that makes it possible to describe how a system evolves state-by-state (“point-wise” view). PTLs have been successfully employed in model checking (MC) [15, 35] for the automatic verification of complex finite-state systems modeled as finite propositional Kripke structures. *Interval Temporal Logics* (ITLs) provide an alternative setting for reasoning about time [20, 32, 38]. They assume intervals, instead of points, as their primitive temporal entities allowing one to specify temporal properties that involve, e.g., actions with duration, accomplishments, and temporal aggregations, which are inherently “interval-based”, and thus cannot be naturally expressed by PTLs. ITLs find applications in a variety of computer science fields, including artificial intelligence (reasoning about action and change, qualitative reasoning, planning, and natural language processing), theoretical computer science (specification and verification of programs), and temporal and spatio-temporal databases (see, e.g., [32, 25, 34]).

The most prominent example of ITLs is *Halpern and Shoham’s modal logic of time intervals* (HS) [20] which features one modality for each of the 13 possible ordering relations between pairs of intervals (the so-called Allen’s relations [1]), apart from equality. The satisfiability problem for HS turns out to be highly undecidable for all interesting (classes of) linear orders [20]. The same happens with most of its fragments [13, 24, 28] with some meaningful exceptions like the logic of temporal neighbourhood \overline{AA} , over all relevant (classes of) linear orders [14], and the logic of sub-intervals D, over the class of dense linear orders [31].

Model checking of (finite) Kripke structures against HS has been investigated only recently [25, 26, 27, 29, 30, 6, 7, 4, 9]. The idea is to interpret each finite path of a Kripke structure as an interval, whose labelling is defined on the basis of the labelling of the component states, that is, a proposition letter holds

over an interval if and only if it holds over each component state (*homogeneity assumption* [36]). Most of the results have been obtained by adopting the so-called *state-based semantics* [29]: intervals/paths are “forgetful” of the history leading to their starting state, and time branches both in the future and in the past. In this setting, MC of full HS is decidable: the problem is at least EXPSPACE-hard [5], while the only known upper bound is non-elementary [29]. The known complexity bounds for full HS coincide with those for the linear-time fragment BE of HS which features modalities $\langle B \rangle$ and $\langle E \rangle$ for prefixes and suffixes. These complexity bounds easily transfer to finite satisfiability, that is, satisfiability over finite linear orders, of BE under the homogeneity assumption. Whether or not these problems can be solved elementarily is a difficult open question. On the other hand, in the state-based setting, the exact complexity of MC for many meaningful (linear-time or branching-time) syntactic fragments of HS, which ranges from $\mathbf{co-NP}$ to $\mathbf{P}^{\mathbf{NP}}$, PSPACE, and beyond, has been determined in a series of papers [30, 6, 8, 10, 12, 9].

The expressiveness of HS with the state-based semantics has been studied in [7], together with other two decidable variants: the *computation-tree-based semantics* variant and the *traces-based* one. For the first variant, past is linear: each interval may have several possible futures, but only a unique past. Moreover, past is finite and cumulative, and is never forgotten. The trace-based approach instead relies on a linear-time setting, where the infinite paths (traces) of the given Kripke structure are the main semantic entities. It is known that the computation-tree-based variant of HS is expressively equivalent to finitary CTL* (the variant of CTL* with quantification over finite paths), while the trace-based variant is equivalent to LTL. The state-based variant is more expressive than the computation-tree-based variant and expressively incomparable with both LTL and CTL*. To the best of our knowledge, complexity issues about MC and the satisfiability problem of HS and its syntactic fragments under the trace-based semantics have not been investigated so far.

Parametric extensions of point-based temporal logics. Traditional PTLs such as standard LTL [33] allow only to express *qualitative* requirements on the temporal ordering of events. For example, in expressing a typical request-response temporal requirement, it is not possible to specify a bound on the amount of time for which a request is granted. A simple way to overcome this drawback is to consider quantitative extensions of PTLs where temporal modalities are equipped with timing constraints for allowing the specification of *constant* bounds on the delays among events. A well-known representative of such logics is Metric Temporal Logic (MTL) [22]. However this approach is not practical in the first stages of a design, when not much is known about the system under development, and is useful for designers to use parameters instead of specific constants. Parametric extensions of traditional PTLs, where time bounds can be expressed by means of parameters, have been investigated in many papers. Relevant examples include parametric LTL [2], Prompt LTL [23], and parametric MTL [19].

Our contribution. In this paper we introduce a parametric extension of the interval temporal logic HS under the trace-based semantics, called *parametric HS* (PHS). The extension is obtained by means of inequality constraints on the temporal modalities of HS which allow to specify parametric lower/upper bounds on the duration (length) of the interval selected by the temporal modality. Similarly to parametric LTL [2], we impose that a parameter can be exclusively used either as upper bound or as lower bound in the timing constraints. We address the decision problems of checking the existence of a parameter valuation such that (1) a given PHS formula is satisfiable, and (2) a given Kripke structure satisfies a given PHS formula (MC). By adapting the alternating color technique for Prompt LTL [23] and by exploiting known results on *linear-time hybrid logic HL* [18, 37, 3], we show that the considered problems are decidable. Additionally, we consider the syntactic fragment $P(AB\bar{B})$ of PHS which allows only temporal modalities for the Allen’s relations *meets* \mathcal{R}_A , *started-by* \mathcal{R}_B and its inverse $\mathcal{R}_{\bar{B}}$. We show that $P(AB\bar{B})$

subsumes parametric LTL, and its flat fragment $AB\bar{B}$ is exponentially more succinct than LTL + past. Moreover, we establish that satisfiability and MC of $P(AB\bar{B})$ are EXPSpace-complete, and we provide tight bounds on optimal parameter values for both problems.

2 Preliminaries

We fix the following notation. Let \mathbb{Z} be the set of integers, \mathbb{N} the set of natural numbers, and $\mathbb{N}_+ \stackrel{\text{def}}{=} \mathbb{N} \setminus \{0\}$. Let Σ be an alphabet and w be a non-empty finite or infinite word over Σ . We denote by $|w|$ the length of w ($|w| = \infty$ if w is infinite). For all $i, j \in \mathbb{N}$, with $i \leq j < |w|$, $w(i)$ is the $(i+1)$ -th letter of w , while $w[i, j]$ is the infix of w given by $w(i) \cdots w(j)$.

We fix a finite set AP of atomic propositions. A *trace* is an infinite word over 2^{AP} . For a logic \mathfrak{F} interpreted over traces and a formula $\varphi \in \mathfrak{F}$, $\mathcal{L}(\varphi)$ denotes the set of traces satisfying φ . The *satisfiability problem* for \mathfrak{F} is checking for a given formula $\varphi \in \mathfrak{F}$, whether $\mathcal{L}(\varphi) \neq \emptyset$.

Kripke Structures. In the context of model-checking, finite state systems are usually modelled as finite Kripke structures over a finite set AP of atomic propositions which represent predicates over the states of the system. A (finite) *Kripke structure* over AP is a tuple $\mathcal{K} = (AP, S, E, Lab, s_0)$, where S is a finite set of states, $E \subseteq S \times S$ is a left-total transition relation, $Lab : S \mapsto 2^{AP}$ is a labelling function assigning to each state s the set of propositions that hold over it, and $s_0 \in S$ is the initial state. An infinite path π of \mathcal{K} is an infinite word over S such that $\pi(0) = s_0$ and $(\pi(i), \pi(i+1)) \in E$ for all $i \geq 0$. A finite path of \mathcal{K} is a non-empty infix of some infinite path of \mathcal{K} . An infinite path π induces the trace given by $Lab(\pi(0))Lab(\pi(1)) \dots$. We denote by $\mathcal{L}(\mathcal{K})$ the set of traces associated with the infinite paths of \mathcal{K} . Given a logic \mathfrak{F} interpreted over traces, the (linear-time) *model checking problem against \mathfrak{F}* is checking for a given Kripke structure \mathcal{K} and a formula $\varphi \in \mathfrak{F}$, whether $\mathcal{L}(\mathcal{K}) \subseteq \mathcal{L}(\varphi)$.

Büchi nondeterministic automata. A *Büchi nondeterministic finite automaton* over infinite words (Büchi NFA for short) is a tuple $\mathcal{A} = (\Sigma, Q, q_0, \delta, F)$, where Σ is a finite input alphabet, Q is a finite set of states, $q_0 \in Q$ is the initial state, $\delta : Q \times \Sigma \mapsto 2^Q$ is the transition relation, and $F \subseteq Q$ is a set of accepting states. Given an infinite word w over Σ , a run π of \mathcal{A} over w is an infinite sequence π of states such that $\pi(0) = q_0$ and $\pi(i+1) \in \delta(\pi(i), w(i))$ for all $i \geq 0$. The run is accepting if for infinitely many $i \geq 0$, $\pi(i) \in F$. The language $\mathcal{L}(\mathcal{A})$ accepted by \mathcal{A} is the set of infinite words w over Σ such that there is an accepting run of \mathcal{A} over w .

2.1 Allen's relations and Interval Temporal Logic HS

An interval algebra to reason about intervals and their relative orders was proposed by Allen in [1], while a systematic logical study of interval representation and reasoning was done a few years later by Halpern and Shoham, who introduced the interval temporal logic HS featuring one modality for each Allen relation, but equality [20].

Let $\mathbb{U} = (Pt, <)$ be a linear order over the nonempty set $Pt \neq \emptyset$, and \leq be the reflexive closure of $<$. Given two elements $x, y \in Pt$ such that $x \leq y$, we denote by $[x, y]$ the (non-empty closed) *interval* over Pt given by the set of elements $z \in Pt$ such that $x \leq z$ and $z \leq y$. We denote the set of all intervals over \mathbb{U} by $\mathbb{I}(\mathbb{U})$. We now recall the Allen's relations over intervals of the linear order $\mathbb{U} = (Pt, <)$:

1. the *meet* relation \mathcal{R}_A , defined by $[x, y] \mathcal{R}_A [v, z]$ if $y = v$ (i.e., the start-point of the second interval coincides with the end-point of the first interval);
2. the *before* relation \mathcal{R}_L , defined by $[x, y] \mathcal{R}_L [v, z]$ if $y < v$ (i.e., the start-point of the second interval strictly follows the end-point of the first interval);

Table 1: Allen's relations and corresponding HS modalities.

Allen relation	HS	Definition w.r.t. interval structures	Example
MEETS	$\langle A \rangle$	$[x, y] \mathcal{R}_A [v, z] \iff y = v$	
BEFORE	$\langle L \rangle$	$[x, y] \mathcal{R}_L [v, z] \iff y < v$	
STARTED-BY	$\langle B \rangle$	$[x, y] \mathcal{R}_B [v, z] \iff x = v \wedge z < y$	
FINISHED-BY	$\langle E \rangle$	$[x, y] \mathcal{R}_E [v, z] \iff y = z \wedge x < v$	
CONTAINS	$\langle D \rangle$	$[x, y] \mathcal{R}_D [v, z] \iff x < v \wedge z < y$	
OVERLAPS	$\langle O \rangle$	$[x, y] \mathcal{R}_O [v, z] \iff x < v < y < z$	

3. the *started-by* relation \mathcal{R}_B , defined by $[x, y] \mathcal{R}_B [v, z]$ if $x = v$ and $z < y$ (i.e., the second interval is a proper prefix of the first interval);
4. the *finished-by* relation \mathcal{R}_E , defined by $[x, y] \mathcal{R}_E [v, z]$ if $y = z$ and $x < v$ (i.e., the second interval is a proper suffix of the first interval);
5. the *contains* relation \mathcal{R}_D , defined by $[x, y] \mathcal{R}_D [v, z]$ if $x < v$ and $z < y$ (i.e., the second interval is contained in the interval of the first interval);
6. the *overlaps* relation \mathcal{R}_O , defined by $[x, y] \mathcal{R}_O [v, z]$ if $x < v < y < z$ (i.e., the second interval overlaps at the right the first interval);
7. for each $X \in \{A, L, B, E, D, O\}$ the relation $\mathcal{R}_{\bar{X}}$, defined as the inverse of \mathcal{R}_X , i.e. $[x, y] \mathcal{R}_{\bar{X}} [v, z]$ if $[v, z] \mathcal{R}_X [x, y]$.

Table 1 gives a graphical representation of the Allen's relations $\mathcal{R}_A, \mathcal{R}_L, \mathcal{R}_B, \mathcal{R}_E, \mathcal{R}_D$, and \mathcal{R}_O together with the corresponding HS (existential) modalities.

Syntax and semantics of HS. HS formulas φ over AP are defined as follows:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle X \rangle \varphi$$

where $p \in AP$ and $\langle X \rangle$ is the existential temporal modality for the (non-trivial) Allen's relation \mathcal{R}_X , where $X \in \{A, L, B, E, D, O, \bar{A}, \bar{L}, \bar{B}, \bar{E}, \bar{D}, \bar{O}\}$. The size $|\varphi|$ of a formula φ is the number of distinct subformulas of φ . We also exploit the standard logical connectives \vee (disjunction) and \rightarrow (implication) as abbreviations, and for any temporal modality $\langle X \rangle$, the dual universal modality $[X]$ defined as: $[X] \psi \stackrel{\text{def}}{=} \neg \langle X \rangle \neg \psi$. Moreover, we will also use the reflexive closure of the Allen's relation $\mathcal{R}_{\bar{B}}$ (resp., $\mathcal{R}_{\bar{E}}$) and the associated temporal modalities $\langle \bar{B}_w \rangle$ and $[\bar{B}_w]$ (resp., $\langle \bar{E}_w \rangle$ and $[\bar{E}_w]$) where $\langle \bar{B}_w \rangle \varphi$ corresponds to $\varphi \vee \langle \bar{B} \rangle \varphi$ and $\langle \bar{E}_w \rangle \varphi$ corresponds to $\varphi \vee \langle \bar{E} \rangle \varphi$. Given any subset of Allen's relations $\{\mathcal{R}_{X_1}, \dots, \mathcal{R}_{X_n}\}$, we denote by $X_1 \cdots X_n$ the HS fragment featuring temporal modalities for $\mathcal{R}_{X_1}, \dots, \mathcal{R}_{X_n}$ only.

The logic HS is interpreted on *interval structures* $\mathcal{S} = (AP, \mathbb{U}, Lab)$, which are linear orders \mathbb{U} equipped with a labelling function $Lab : \mathbb{I}(\mathbb{U}) \rightarrow 2^{AP}$ assigning to each interval the set of propositions that hold over it. Given an HS formula φ and an interval $I \in \mathbb{I}(\mathbb{U})$, the satisfaction relation $I \models_{\mathcal{S}} \varphi$, meaning that φ holds at the interval I of \mathcal{S} , is inductively defined as follows (we omit the semantics of the Boolean connectives which is standard):

$$\begin{aligned} I \models_{\mathcal{S}} p &\iff p \in Lab(I); \\ I \models_{\mathcal{S}} \langle X \rangle \varphi &\iff \text{there is an interval } J \in \mathbb{I}(\mathbb{U}) \text{ such that } I \mathcal{R}_X J \text{ and } J \models_{\mathcal{S}} \varphi. \end{aligned}$$

It is worth noting that we assume the *non-strict semantics of HS*, which admits intervals consisting of a single point. Under such an assumption, all HS-temporal modalities can be expressed in terms of

$\langle B \rangle$, $\langle E \rangle$, $\langle \bar{B} \rangle$, and $\langle \bar{E} \rangle$ (see [38]). As an example, $\langle D \rangle \varphi$ can be expressed in terms of $\langle B \rangle$ and $\langle E \rangle$ as $\langle B \rangle \langle E \rangle \varphi$, while $\langle A \rangle \varphi$ can be expressed in terms of $\langle E \rangle$ and $\langle \bar{B} \rangle$ as

$$([\bar{E}] \neg \top \wedge (\varphi \vee \langle \bar{B} \rangle \varphi)) \vee \langle E \rangle ([E] \neg \top \wedge (\varphi \vee \langle \bar{B} \rangle \varphi)).$$

Interpretation of HS over traces. In this paper, we focus on interval structures $\mathcal{S} = (AP, (\mathbb{N}, <), Lab)$ over the standard linear order on \mathbb{N} (\mathbb{N} -interval structures for short) satisfying the *homogeneity principle*: a proposition holds over an interval if and only if it holds over all its subintervals. Formally, \mathcal{S} is *homogeneous* if for every interval $[i, j]$ over \mathbb{N} and every $p \in AP$, it holds that $p \in Lab([i, j])$ if and only if $p \in Lab([h, h])$ for every $h \in [i, j]$. Note that homogeneous \mathbb{N} -interval structures over AP correspond to traces where, intuitively, each interval is mapped to an infix of the trace. Formally, each trace w induces the homogeneous \mathbb{N} -interval structure $\mathcal{S}(w)$ whose labeling function Lab_w is defined as follows: for all $i, j \in \mathbb{N}$ with $i \leq j$ and $p \in AP$, $p \in Lab_w([i, j])$ if and only if $p \in w(h)$ for all $h \in [i, j]$. For the given finite set AP of atomic propositions, this mapping from traces to homogeneous \mathbb{N} -interval structures is evidently a bijection. For a trace w , an interval I over \mathbb{N} , and an HS formula φ , we write $I \models_w \varphi$ to mean that $I \models_{\mathcal{S}(w)} \varphi$. The trace w satisfies φ , written $w \models \varphi$, if $[0, 0] \models_w \varphi$.

Expressiveness completeness and succinctness of the fragment AB over traces. It is known that HS over traces has the same expressiveness as standard LTL [7], where the latter is expressively complete for standard first-order logic FO over traces [21]. In particular, the fragment AB of HS is sufficient for capturing full LTL [7]: given an LTL formula, one can construct in linear-time an equivalent AB formula [7]. Note that when interpreted on infinite words w , modality $\langle B \rangle$ allows to select proper non-empty prefixes of the current infix subword of w , while modality $\langle A \rangle$ allows to select subwords whose first position coincides with the last position of the current interval. Here, we show that AB is exponentially more succinct than LTL + past. For each $k \geq 1$, we denote by len_k the B formula capturing the intervals of length k : $len_k \stackrel{\text{def}}{=} (\underbrace{\langle B \rangle \dots \langle B \rangle \top}_{k-1 \text{ times}}) \wedge (\underbrace{[\bar{B}] \dots [\bar{B}] \neg \top}_{k \text{ times}})$.

For each $n \geq 1$, let $AP_n = \{p_0, \dots, p_n\}$ and \mathcal{L}_n be the ω -language consisting of the infinite words over 2^{AP_n} such that any two positions that agree on the truth value of propositions p_1, \dots, p_n also agree on the truth value of p_0 . It is known that any Büchi NFA accepting \mathcal{L}_n needs at least 2^{2^n} states [17]. Thus, since any formula φ of LTL + past can be translated into an equivalent Büchi NFA with a single exponential blow-up, it follows that any formula of LTL + past capturing \mathcal{L}_n has size at least single exponential in n . On the other hand, the language \mathcal{L}_n is captured by the following AB formula having size linear in n :

$$[A][A] \left(\bigwedge_{i \in [1, n]} \theta(p_i) \rightarrow \theta(p_0) \right) \quad \theta(p) \stackrel{\text{def}}{=} \langle B \rangle (len_1 \wedge p) \leftrightarrow \langle A \rangle (len_1 \wedge p)$$

Hence, we obtain the following result.

Theorem 1. *AB (over traces) is exponentially more succinct than LTL + past.*

3 Parametric Interval Temporal Logic

In this section, we introduce a parametric extension of the interval temporal logic HS over traces, called *parametric HS* (PHS for short). The extension is obtained by means of inequality constraints on the temporal modalities of HS which allow to compare the length of the interval selected by the temporal modality with an integer parameter. Like parametric LTL [2], the parameterized operators are monotone

(either upward or downward) and a parameter is upward (resp., downward) if it is the subscript of some upward (resp., downward) modality.

Syntax and semantics of PHS Let P_U be a finite set of *upward* parameter variables u and P_L be a finite set of *downward* parameter variables ℓ such that P_U and P_L are disjoint. The syntax of PHS formulas φ over AP and the set $P_U \cup P_L$ of parameter variables is given in positive normal form as follows:

$$\varphi ::= \top \mid p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle X \rangle \varphi \mid \langle X \rangle_{\prec u} \varphi \mid \langle X \rangle_{\succ \ell} \varphi \mid [X] \varphi \mid [X]_{\prec \ell} \varphi \mid [X]_{\succ u} \varphi$$

where $p \in AP$, $X \in \{A, L, B, E, D, O, \bar{A}, \bar{L}, \bar{B}, \bar{B}_w, \bar{E}_w, \bar{D}, \bar{O}\}$, $\prec \in \{<, \leq\}$, $\succ \in \{>, \geq\}$, $u \in P_U$, and $\ell \in P_L$. We denote by PromptHS the fragment of PHS where the unique parameterized temporal modalities are of the form $\langle X \rangle_{\prec u}$. Moreover, given any subset of Allen's relations $\{\mathcal{R}_{X_1}, \dots, \mathcal{R}_{X_n}\}$, we denote by $P(X_1 \dots X_n)$ (resp., Prompt($X_1 \dots X_n$)) the PHS (resp., PromptHS) fragment featuring temporal modalities for $\mathcal{R}_{X_1}, \dots, \mathcal{R}_{X_n}$ only. We will focus on PHS and the fragment $P(\text{ABBB}_w)$.

For an interval $I = [i, j]$ over \mathbb{N} , we denote by $|I|$ the length of I , given by $j - i + 1$. The semantics of a PHS formula φ is inductively defined with respect to a trace w , an interval I over \mathbb{N} , and a *parameter valuation* $\alpha : P_U \cup P_L \mapsto \mathbb{N}_+$ assigning to each parameter variable a positive integer. We write $(I, \alpha) \models_w \varphi$ to mean that φ holds at the interval I of w under the valuation α . The interpretation of all temporal operators of HS and connectives is identical to their HS interpretations. The parameterized operators are interpreted as follows, where $\wp \in P_U \cup P_L$ and $\sim \in \{<, \leq, >, \geq\}$:

$$\begin{aligned} (I, \alpha) \models_w \langle X \rangle_{\sim \wp} \varphi &\Leftrightarrow \text{there is some interval } J \text{ such that } I \mathcal{R}_X J, |J| \sim \alpha(\wp), \text{ and } J, \alpha \models_w \varphi; \\ (I, \alpha) \models_w [X]_{\sim \wp} \varphi &\Leftrightarrow \text{for each interval } J \text{ such that } I \mathcal{R}_X J \text{ and } |J| \sim \alpha(\wp): J, \alpha \models_w \varphi. \end{aligned}$$

We say that the trace w is a model of formula φ under the parameter valuation α , written $(w, \alpha) \models \varphi$, if $([0, 0], \alpha) \models_w \varphi$. For a PHS formula φ and a Kripke structure \mathcal{K} over AP , we consider:

- (i) the set $V(\mathcal{K}, \varphi)$ consisting of the parameter valuations α such that for each trace $w \in \mathcal{L}(\mathcal{K})$ of \mathcal{K} , $(w, \alpha) \models \varphi$, and
- (ii) the set $S(\varphi)$ consisting of the valuations α such that $(w, \alpha) \models \varphi$ for some trace w .

The (*linear-time*) *model-checking problem against PHS* is checking for a given Kripke structure \mathcal{K} and PHS formula φ whether $V(\mathcal{K}, \varphi) \neq \emptyset$. The *satisfiability problem against PHS* is checking for a given PHS formula φ whether $S(\varphi) \neq \emptyset$.

Given two valuations α and β , we write $\alpha \leq \beta$ to mean that $\alpha(\wp) \leq \beta(\wp)$ for all $\wp \in P_U \cup P_L$. A parameterized operator Θ is *upward-monotone* (resp., *downward-monotone*) if for all formulas φ , valuations α and β such that $\alpha \leq \beta$, $I, \alpha \models_w \Theta \varphi$ entails that $I, \beta \models_w \Theta \varphi$ (resp., $I, \beta \models_w \Theta \varphi$ entails that $I, \alpha \models_w \Theta \varphi$). By construction, all the parameterized operators are monotone. In particular, being P_L and P_U disjoint, by increasing (resp., decreasing) the values of upward (resp., downward) parameters, the satisfaction relation is preserved.

Proposition 1. • *The operators in PHS parameterized by variables in P_U are upward-closed, while those parameterized by variables in P_L are downward-closed.*

- *Let φ be a PHS formula and let α and β be variable valuations satisfying $\beta(u) \geq \alpha(u)$ for every $u \in P_U$ and $\beta(\ell) \leq \alpha(\ell)$ for every $\ell \in P_L$. Then $(w, \alpha) \models \varphi$ entails that $(w, \beta) \models \varphi$.*

Note that if we also allow for all $\ell \in P_L$ and $u \in P_U$, the parameterized modalities $\langle X \rangle_{\succ u}$, $\langle X \rangle_{\prec \ell}$, $[X]_{\succ \ell}$, and $[X]_{\prec u}$, then the modalities $\langle X \rangle_{\sim \wp}$ and $[X]_{\sim \wp}$, for $\wp \in P_U \cup P_L$ and $\sim \in \{<, \leq, >, \geq\}$ are dual and have opposite kind of monotonicity. It easily follows that the logic is indeed closed under negation.

Proposition 2. *Given a PHS formula φ with upward (resp., downward) parameters in P_U (resp., P_L), one can construct in linear time a PHS formula $\bar{\varphi}$ with upward (resp., downward) parameters in P_L (resp.,*

P_U) corresponding to the negation of φ , i.e. such that for each parameter valuation α and trace w over 2^{AP} , $(w, \alpha) \models \varphi$ iff $(w, \alpha) \not\models \bar{\varphi}$.

We now show that parametric LTL (PLTL) [2] can be easily expressed in P(AB). Recall that PLTL formulas φ over AP and the set of parameters $P_U \cup P_L$ are defined as:

$$\varphi ::= \top \mid p \mid \neg p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid X\varphi \mid \varphi U \psi \mid G\varphi \mid F_{\leq u}\varphi \mid G_{\leq \ell}\varphi$$

where $p \in AP$, $u \in P_U$, $\ell \in P_L$, X , U , and G are the standard next, until, and always modalities, respectively, and $F_{\leq u}$ and $G_{\leq \ell}$ are parameterized versions of the always and eventually modalities. Other parameterized modalities such as $F_{> \ell}$ or $G_{> u}$ can be easily expressed in the considered logic [2]. For a PLTL formula φ , a trace w , a parameter valuation α , and a position $i \geq 0$, the satisfaction relation $(w, i, \alpha) \models \varphi$ is defined by induction as follows (we omit the semantics of LTL constructs which is standard):

$$\begin{aligned} (w, i, \alpha) \models F_{\leq u}\varphi &\Leftrightarrow \text{there is some } k \leq \alpha(u) \text{ such that } (w, i+k, \alpha) \models \varphi; \\ (w, i, \alpha) \models G_{\leq \ell}\varphi &\Leftrightarrow \text{for each } k \leq \alpha(\ell): (w, i+k, \alpha) \models \varphi. \end{aligned}$$

Proposition 3. *For a PLTL formula φ , one can build in linear time a P(AB) formula $f(\varphi)$ such that for all traces w , $i \geq 0$, and parameter valuations α , $(w, i, \alpha) \models \varphi$ iff $([i, i], \alpha) \models_w f(\varphi)$.*

Proof. The mapping $f : \text{PLTL} \mapsto \text{P(AB)}$, homomorphic with respect to atomic propositions and Boolean connectives, is defined as follows:

$$\begin{aligned} f(X\varphi) &\stackrel{\text{def}}{=} \langle A \rangle (\text{len}_2 \wedge \langle A \rangle (\text{len}_1 \wedge \varphi)); \\ f(\varphi U \psi) &\stackrel{\text{def}}{=} \langle A \rangle (\langle A \rangle (\text{len}_1 \wedge f(\psi)) \wedge [B] \langle A \rangle (\text{len}_1 \wedge f(\varphi))); \\ f(G\varphi) &\stackrel{\text{def}}{=} [A] \langle A \rangle (\text{len}_1 \wedge \varphi); \\ f(F_{\leq u}\varphi) &\stackrel{\text{def}}{=} \langle A \rangle_{\leq u} \langle A \rangle (\text{len}_1 \wedge \varphi); \\ f(G_{\leq \ell}\varphi) &\stackrel{\text{def}}{=} [A]_{\leq \ell} \langle A \rangle (\text{len}_1 \wedge \varphi). \end{aligned}$$

□

Note that by Proposition 3 and the results in [2], the relaxation of the assumption $P_U \cap P_L = \emptyset$ or the adding of parameterized operators of the form $\langle X \rangle_{= \varphi}$ would lead to an undecidable model-checking problem already for the parameterized extension of AB by just one parameter.

Expressively complete fragments. Two PHS formulas φ and ψ are *strongly equivalent*, denoted by $\varphi \equiv \psi$, if for all traces, intervals I over \mathbb{N} , and parameter valuations α , we have that $(I, \alpha) \models_w \varphi$ iff $(I, \alpha) \models_w \psi$. We show that the fragment consisting of $\text{P}(\overline{\text{B}}\overline{\text{B}}_w\overline{\text{E}}\overline{\text{E}}_w)$ formulas with no occurrences of parameterized operators $[X]_{>u}$ is sufficient to capture the full logic PHS.

Proposition 4. *Given a PHS formula φ , one can build in linear time a strongly equivalent $\text{P}(\overline{\text{B}}\overline{\text{B}}_w\overline{\text{E}}\overline{\text{E}}_w)$ formula ψ with no occurrences of the parameterized operators $[X]_{>u}$.*

Proof. We first show that the fragment $\text{P}(\overline{\text{B}}\overline{\text{B}}_w\overline{\text{E}}\overline{\text{E}}_w)$ is expressively complete for PHS. The strong equivalences exploited for expressing all the HS modalities in terms of the modalities in the fragment $\overline{\text{B}}\overline{\text{E}}\overline{\text{E}}$ can be trivially adapted to the parameterized setting. Here, we illustrate the equivalences for the

existential parameterized operators where $\sim \in \{<, \leq, >, \geq\}$ and $\wp \in P_U \cup P_L$:

$$\begin{aligned}
\langle A \rangle_{\sim \wp} \varphi &\equiv (\text{len}_1 \wedge \langle \overline{B} \rangle_{\sim \wp} \varphi) \vee \langle E \rangle (\text{len}_1 \wedge \langle \overline{B} \rangle_{\sim \wp} \varphi); \\
\langle \overline{A} \rangle_{\sim \wp} \varphi &\equiv (\text{len}_1 \wedge \langle \overline{E} \rangle_{\sim \wp} \varphi) \vee \langle B \rangle (\text{len}_1 \wedge \langle \overline{E} \rangle_{\sim \wp} \varphi); \\
\langle L \rangle_{\sim \wp} \varphi &\equiv \langle \overline{B} \rangle \langle E \rangle (\text{len}_1 \wedge \langle \overline{B} \rangle_{\sim \wp} \varphi); \\
\langle \overline{L} \rangle_{\sim \wp} \varphi &\equiv \langle \overline{E} \rangle \langle B \rangle (\text{len}_1 \wedge \langle \overline{E} \rangle_{\sim \wp} \varphi); \\
\langle D \rangle_{\sim \wp} \varphi &\equiv \langle B \rangle \langle E \rangle_{\sim \wp} \varphi; \\
\langle \overline{D} \rangle_{\sim \wp} \varphi &\equiv \langle \overline{B} \rangle \langle \overline{E} \rangle_{\sim \wp} \varphi; \\
\langle O \rangle_{\sim \wp} \varphi &\equiv \langle E \rangle (\neg \text{len}_1 \wedge \langle \overline{B} \rangle_{\sim \wp} \varphi); \\
\langle \overline{O} \rangle_{\sim \wp} \varphi &\equiv \langle B \rangle (\neg \text{len}_1 \wedge \langle \overline{E} \rangle_{\sim \wp} \varphi).
\end{aligned}$$

It remains to show that for the fragment $P(\overline{B}\overline{B}\overline{E}_w\overline{E}\overline{E}_w)$, the universal upward parameterized operators can be expressed in terms of the other modalities. One can easily show that the following strong equivalences hold, where \prec is $<$ (resp., \prec is \leq) and \succ is \geq (resp., \succ is $>$). Hence, the result follows.

$$\begin{aligned}
[B]_{\succ u} \varphi &\equiv [B](\varphi \vee \langle \overline{B}_w \rangle_{\prec u} \top); \\
[E]_{\succ u} \varphi &\equiv [E](\varphi \vee \langle \overline{E}_w \rangle_{\prec u} \top); \\
[\overline{B}]_{\succ u} \varphi &\equiv \langle \overline{B} \rangle_{\prec u} ([\overline{B}] \varphi) \vee [\overline{B}] \varphi; \\
[\overline{E}]_{\succ u} \varphi &\equiv \langle \overline{E} \rangle_{\prec u} ([\overline{E}] \varphi) \vee [\overline{E}] \varphi; \\
[\overline{B}_w]_{\succ u} \varphi &\equiv \langle \overline{B}_w \rangle_{\prec u} ([\overline{B}] \varphi) \vee [\overline{B}_w] \varphi; \\
[\overline{E}_w]_{\succ u} \varphi &\equiv \langle \overline{E}_w \rangle_{\prec u} ([\overline{E}] \varphi) \vee [\overline{E}_w] \varphi.
\end{aligned}$$

□

For the logic $P(\overline{A}\overline{B}\overline{B}_w)$, we obtain a similar result.

Proposition 5. *Given a $P(\overline{A}\overline{B}\overline{B}_w)$ formula φ , one can build in linear time a strongly equivalent $P(\overline{A}\overline{B}\overline{B}_w)$ formula ψ with no occurrences of the parameterized operators $[X]_{\succ u}$.*

Proof. The result directly follows from the strong equivalences provided in the proof of Proposition 4 and the following one, where \prec is $<$ (resp., \prec is \leq) and \succ is \geq (resp., \succ is $>$): $[A]_{\succ u} \varphi \equiv \langle A \rangle_{\prec u} [\overline{B}] \varphi$. □

By the monotonicity of the parameterized modalities and Propositions 4 and 5, we can eliminate all the parameterized modalities, but the existential upward ones, for solving the model-checking and satisfiability problems against PHS (resp., $P(\overline{A}\overline{B}\overline{B}_w)$).

Lemma 1. *Model checking PHS (resp., $P(\overline{A}\overline{B}\overline{B}_w)$) can be reduced in linear time to model checking PromptHS (resp., Prompt($\overline{A}\overline{B}\overline{B}_w$)). Similarly, satisfiability of PHS (resp., $P(\overline{A}\overline{B}\overline{B}_w)$) can be reduced in linear time to satisfiability of PromptHS (resp., Prompt($\overline{A}\overline{B}\overline{B}_w$)).*

Proof. Let φ be a PHS (resp., $P(\overline{A}\overline{B}\overline{B}_w)$) formula. By Propositions 4 and 5, we can assume that φ does not contain occurrences of parameterized operators of the form $[X]_{\succ u}$. Let $f(\varphi)$ be the PromptHS (resp., Prompt($\overline{A}\overline{B}\overline{B}_w$)) formula intuitively obtained from φ by replacing each occurrence of a downward parameter ℓ with the constant 1. Formally, $f(\varphi)$ is homomorphic w.r.t. all the constructs but the downward parameterized modalities and:

- $f(\langle X \rangle_{\geq \ell} \varphi) \stackrel{\text{def}}{=} \langle X \rangle f(\varphi)$;
- $f(\langle X \rangle_{> \ell} \varphi) \stackrel{\text{def}}{=} \langle X \rangle (\neg \text{len}_1 \wedge f(\varphi))$;
- $f([X]_{\leq \ell} \varphi) \stackrel{\text{def}}{=} [X] (\neg \text{len}_1 \vee f(\varphi))$;
- $f([X]_{< \ell} \varphi) \stackrel{\text{def}}{=} \top$.

As for the model checking problem, we show that $V(\mathcal{K}, \varphi) \neq \emptyset$ iff $V(\mathcal{K}, f(\varphi)) \neq \emptyset$ for each Kripke structure \mathcal{K} . Let α_1 be a parameter valuation such that $\alpha_1(\ell) = 1$ for each downward parameter $\ell \in L$. By construction, for all traces w , $(w, \alpha_1) \models \varphi$ iff $(w, \alpha_1) \models f(\varphi)$. Hence, $V(\mathcal{K}, f(\varphi)) \neq \emptyset$ implies that $V(\mathcal{K}, \varphi) \neq \emptyset$. On the other hand, if $V(\mathcal{K}, \varphi) \neq \emptyset$, there is a parameter valuation α such that for each trace w of \mathcal{K} , $(w, \alpha) \models \varphi$. Let α_1 be defined as: $\alpha_1(u) = \alpha(u)$ for each $u \in U$, and $\alpha_1(\ell) = 1$ for each $\ell \in L$. By Proposition 1, it follows that for each trace w of \mathcal{K} , $(w, \alpha_1) \models \varphi$. Thus, we obtain that $V(\mathcal{K}, f(\varphi)) \neq \emptyset$ as well, and the result for the model-checking problem follows. The result for the satisfiability problem is similar. \square

4 Decision procedures for PHS

In this section, we first provide a translation of HS formulas into equivalent Büchi NFA (asymptotically optimal for $AB\overline{B}\overline{B}_w$ formulas), by exploiting as an intermediate step a translation of HS formulas into equivalent formulas of *linear-time hybrid logic HL* [18, 37, 3] (Subsection 4.1). Then, in Subsection 4.2, we apply the results of Subsection 4.1 and the alternating color technique for Prompt LTL [23] in order to solve satisfiability and model checking against PHS and $P(AB\overline{B}\overline{B}_w)$. In particular, for the logic $P(AB\overline{B}\overline{B}_w)$, we show that the considered problems are EXPSPACE-complete.

4.1 Translation of HS in linear-time Hybrid Logic

In this section, we recall the linear-time hybrid logic HL [18, 37, 3], which extends standard LTL + past by first-order concepts. We show that while HS can be translated into the two-variable fragment of HL, for the logic $AB\overline{B}\overline{B}_w$, it suffices to consider the one-variable fragment HL_1 of HL. Thus, by exploiting known results on HL_1 [37, 3], we obtain an asymptotically optimal automata-theoretic approach for $AB\overline{B}\overline{B}_w$ of elementary complexity.

Syntax and semantics of HL. Given a set X of (position) variables, the set of HL formulas φ over AP and X is defined by the following syntax:

$$\varphi \stackrel{\text{def}}{=} \top \mid p \mid x \mid \neg \varphi \mid \varphi \wedge \varphi \mid X\varphi \mid Y\varphi \mid F\varphi \mid P\varphi \mid \downarrow x.\varphi$$

$p \in AP$, $x \in X$, Y and P are the past counterparts of the next modality X and the eventually modality F , respectively, and $\downarrow x$ is the *downarrow binder* operator which assigns the variable name x to the current position. We denote by HL_1 (resp., HL_2) the one-variable (resp., two-variable) fragment of HL. An HL *sentence* is a formula where each variable x is not free (i.e., occurs in the scope of a binder modality $\downarrow x$). The size $|\varphi|$ of an HL formula φ is the number of distinct subformulas of φ .

HL is interpreted over traces w . A *valuation* g is a mapping assigning to each variable a position $i \geq 0$. The satisfaction relation $(w, i, g) \models \varphi$, meaning that φ holds at position i along w w.r.t. the valuation g , is inductively defined as follows (we omit the semantics of LTL constructs which is standard):

$$\begin{aligned} (w, i, g) \models x &\iff i = g(x) \\ (w, i, g) \models \downarrow x.\varphi &\iff (w, i, g[x \mapsto i]) \models \varphi \end{aligned}$$

where $g[x \mapsto i](x) = i$ and $g[x \mapsto i](y) = g(y)$ for $y \neq x$. Thus, $\downarrow x$ binds the variable x to the current position. Note that the satisfaction relation depends only on the values assigned to the variables occurring free in the given formula φ . We write $(w, i) \models \varphi$ to mean that $(w, i, g_0) \models \varphi$, where g_0 maps each variable to position 0, and $w \models \varphi$ to mean that $(w, 0) \models \varphi$. Note that HL formulas can be trivially translated into

equivalent formulas of first-order logic FO over traces and LTL formulas can be trivially translated into equivalent HL formulas. Thus, by the first-order expressiveness completeness of LTL, HL and LTL have the same expressiveness [18].

Translation of HS into HL. We establish the following result.

Proposition 6. *Given an HS (resp., $AB\overline{B}\overline{B}_w$) formula φ , one can construct in linear-time a two-variable (resp., one-variable) sentence HL φ' such that $\mathcal{L}(\varphi) = \mathcal{L}(\varphi')$.*

Proof. We first consider full HS. We can restrict ourselves to consider the fragment $B\overline{B}\overline{E}\overline{E}$ of HS since all temporal modalities in HS can be expressed in $B\overline{B}\overline{E}\overline{E}$ by a linear-time translation. Fix two distinct variables x_L and x_R . We define a mapping $f : B\overline{B}\overline{E}\overline{E} \mapsto HL_2$ assigning to each $B\overline{B}\overline{E}\overline{E}$ formula φ a HL₂ formula $f(\varphi)$ with variables x_L and x_R which occur free in $f(\varphi)$. Intuitively, in the translation, x_L and x_R refer to the left and right endpoints of the current interval in \mathbb{N} , while the current position corresponds to the left endpoint of the current interval. Formally, the mapping f is homomorphic w.r.t. the Boolean connectives and is inductively defined as follows:

- $f(p) \stackrel{\text{def}}{=} G(Fx_r \rightarrow p)$;
- $f(\langle B \rangle \varphi) \stackrel{\text{def}}{=} F(XFx_R \wedge \downarrow x_R. P(x_L \wedge f(\varphi)))$;
- $f(\langle \overline{B} \rangle \varphi) \stackrel{\text{def}}{=} F(x_R \wedge XF\downarrow x_R. P(x_L \wedge f(\varphi)))$;
- $f(\langle E \rangle \varphi) \stackrel{\text{def}}{=} XF(Fx_R \wedge \downarrow x_L. f(\varphi))$;
- $f(\langle \overline{E} \rangle \varphi) \stackrel{\text{def}}{=} XP\downarrow x_L. f(\varphi)$.

By a straightforward induction on φ , we obtain that given a trace w , an interval $[i, j]$, a valuation g such that $g(x_L) = i$ and $g(x_R) = j$, it holds that $[i, j] \models_w \varphi$ if and only if $(w, i, g) \models f(\varphi)$. The desired HL₂ sentence φ' equivalent to φ is then defined as follows: $\varphi' \stackrel{\text{def}}{=} \downarrow x_L. \downarrow x_R. f(\varphi)$.

We now consider the logic $AB\overline{B}\overline{B}_w$. We can restrict ourselves to consider the fragment $AB\overline{B}$ of HS since the modality $\langle \overline{B}_w \rangle$ can be trivially expressed in terms of $\langle \overline{B} \rangle$. Fix a variable x . We define a mapping $h : AB\overline{B} \mapsto HL_1$ assigning to each $AB\overline{B}$ formula φ an HL₁ formula $h(\varphi)$ with one variable x , which occurs free in $h(\varphi)$. Intuitively, in the translation, x refers to the left endpoint of the current interval in \mathbb{N} , while the current position corresponds to the right endpoint of the current interval. Formally, the mapping h is homomorphic w.r.t. the Boolean connectives and is inductively defined as follows:

- $h(p) \stackrel{\text{def}}{=} \neg P(Px \wedge \neg p)$;
- $h(\langle A \rangle) \stackrel{\text{def}}{=} \downarrow x. Fh(\varphi)$;
- $h(\langle B \rangle \varphi) \stackrel{\text{def}}{=} YP(h(\varphi) \wedge Px)$;
- $h(\langle \overline{B} \rangle \varphi) \stackrel{\text{def}}{=} XFh(\varphi)$.

By a straightforward induction on φ , we can prove that given a trace w , an interval $[i, j]$, a valuation g such that $g(x) = i$, it holds that $[i, j] \models_w \varphi$ if and only if $(w, j, g) \models h(\varphi)$. The desired HL₁ sentence φ' equivalent to φ is then defined as follows: $\varphi' \stackrel{\text{def}}{=} \downarrow x. h(\varphi)$. \square

It is known that HL₂ is already non-elementarily decidable [37] and for an HL formula φ , one can construct a Büchi NFA accepting $\mathcal{L}(\varphi)$ whose size is a tower of exponentials having height equal to the nesting depth of the binder modality plus one [3]. For the one-variable fragment HL₁ of HL, one can do much better [3]: the size of the Büchi NFA equivalent to a HL₁ formula φ has size doubly exponential in the size of φ . Hence, by Proposition 6, we obtain the following result.

Proposition 7. *Given an HS formula φ , one can build a Büchi NFA \mathcal{A}_φ accepting $\mathcal{L}(\varphi)$. Moreover, if φ is a $AB\overline{B}\overline{B}_w$ formula, then \mathcal{A}_φ has size doubly exponential in the size of φ .*

Note that by [3], the Büchi NFA equivalent to a HL_1 formula can be built on the fly. Recall that non-emptiness of Büchi NFA is NLOGSPACE-complete, and the standard model checking algorithm consists in checking emptiness of the Büchi NFA resulting from the synchronous product of the given finite Kripke structure with the Büchi NFA associated with the negation of the fixed formula. Thus, by Proposition 7, we obtain algorithms for satisfiability and model-checking of $AB\overline{B}\overline{B}_w$ which run in non-deterministic single exponential space. In [11], it is shown that satisfiability and model checking of AB over *finite words* is already EXPSPACE-hard. The EXPSPACE-hardness proof in [11] can be trivially adapted to handle AB over infinite words. Thus, since $EXPSPACE = NEXPSPACE$, we obtain the following result.

Corollary 1. *Model checking and satisfiability problems for $AB\overline{B}\overline{B}_w$ are both EXPSPACE-complete.*

4.2 Solving satisfiability and model checking of PHS

In this section, we provide an automata-theoretic approach for solving satisfiability and model checking of PromptHS and $\text{Prompt}(AB\overline{B}\overline{B}_w)$ based on Proposition 7 and the alternating color technique for Prompt LTL [23]. By Lemma 1, we devise algorithms for solving satisfiability and model checking against PHS and $P(AB\overline{B}\overline{B}_w)$ as well, which for the case of $P(AB\overline{B}\overline{B}_w)$ are asymptotically optimal.

Alternating color technique [23]. We fix a fresh proposition $c \notin AP$. Let us consider a trace w . A *c-coloring* of w is a trace w' over $AP \cup \{c\}$ such that w and w' agree at every position on all the truth values of the propositions in AP , i.e. $w'(i) \cap AP = w(i)$ for all $i \geq 0$. A position $i \geq 0$ is a *c-change point* in w' if *either* $i = 0$, *or* the colors of i and $i - 1$ are different, i.e. $c \in w'(i)$ iff $c \notin w'(i - 1)$. A *c-block* of w' is a maximal interval $[i, j]$ which has exactly one *c-change point* in w' , and this change point is at the first position i of $[i, j]$. Given $k \geq 1$, we say that w' is *k-bounded* if each *c-block* of w' has length at most k , which implies that w' has infinitely many *c-change points*. Dually, we say that w' is *k-spaced* if w' has infinitely many *c-change points* and every *c-block* has length at least k .

We apply the alternating color technique [23] for replacing a parameterized modality $\langle X \rangle_{\prec u} \psi$ in PromptHS with a non-parameterized one requiring that the selected interval where ψ holds has at most one *c-change point*. Formally, let $rel_c : \text{PromptHS} \mapsto \text{HS}$ be the mapping associating to each PromptHS formula a HS formula, homomorphic w.r.t. propositions, connectives, and non-parameterized modalities, and defined as follows on parameterized formulas $\langle X \rangle_{\prec u} \psi$:

$$rel_c(\langle X \rangle_{\prec u} \psi) \stackrel{\text{def}}{=} \langle X \rangle (rel_c(\psi) \wedge (\theta_c \vee \theta_{\neg c})).$$

where for each $d \in \{c, \neg c\}$, θ_d is an AB formula requiring that the current interval has at most one *c-change point* and the right endpoint is a *d-colored position*:

$$\theta_d \stackrel{\text{def}}{=} \langle A \rangle (\text{len}_1 \wedge d) \wedge [B] (\langle A \rangle (\text{len}_1 \wedge \neg d) \rightarrow [B] \langle A \rangle (\text{len}_1 \wedge \neg d)).$$

For a PromptHS formula φ , let $c(\varphi)$ be the HS formula defined as follows:

$$c(\varphi) \stackrel{\text{def}}{=} rel_c(\varphi) \wedge alt_c \quad alt_c \stackrel{\text{def}}{=} [A] \langle A \rangle \langle A \rangle (\text{len}_1 \wedge c) \wedge [A] \langle A \rangle \langle A \rangle (\text{len}_1 \wedge \neg c)$$

Note that $c(\varphi)$ is a $AB\overline{B}\overline{B}_w$ formula if φ is a $\text{Prompt}(AB\overline{B}\overline{B}_w)$ formula. Moreover, the AB formula alt_c requires that there are infinitely many *c-change points*. Thus, $c(\varphi)$ forces a *c-coloring* of the given trace w to be partitioned into infinitely many blocks such that each parameterized modality selects an interval with at most one *c-change point*. Like Prompt LTL [23], there is a weak equivalence between φ and $c(\varphi)$ on *k-bounded* and *k-spaced c-coloring* of w . The following lemma rephrases Lemma 2.1 in [23] and can be proved in a similar way.

Lemma 2. *Let φ be a PromptHS formula and w be a trace.*

1. *If $(w, \alpha) \models \varphi$, then $w' \models c(\varphi)$ for each k -spaced c -coloring w' of w with $k = \max_{u \in P_U} \alpha(u)$.*
2. *Let $k \geq 1$. If w' is a k -bounded c -coloring of w with $w' \models c(\varphi)$, then $(w, \alpha) \models \varphi$, where $\alpha(u) = 2k$ for each $u \in P_U$.*

Solving satisfiability. Let φ be a PromptHS formula and \mathcal{A}_c be the Büchi NFA of Proposition 7 accepting the models of the HS formula $c(\varphi)$. By Lemma 2, we deduce that $S(\varphi) \neq \emptyset$ if and only if there is $k \geq 1$ and some k -bounded c -coloring w' accepted by \mathcal{A}_c . Indeed, if $S(\varphi) \neq \emptyset$, then there is a parameter valuation α and a trace w such that $(w, \alpha) \models \varphi$. Let $k = \max_{u \in P_U} \alpha(u)$ and w' be the c -coloring of w whose c -blocks have length exactly k . Note that w' is both k -spaced and k -bounded. By Lemma 2(1), $w' \models c(\varphi)$, hence, w' is accepted by \mathcal{A}_c . Vice versa, if there is a trace w and a k -bounded c -coloring w' of w accepted by \mathcal{A}_c , then, by Lemma 2(2), $(w, \alpha) \models \varphi$, where $\alpha(u) = 2k$ for each $u \in P_U$. Hence, $S(\varphi) \neq \emptyset$.

Let N_c be the number of \mathcal{A}_c states. Assume that there is k -bounded c -coloring w' accepted by \mathcal{A}_c for some $k \geq 1$. We claim that there is also a $2N_c + 1$ -bounded c -coloring accepted by \mathcal{A}_c . If $k \leq 2N_c + 1$, the result is obvious. Otherwise, let π be an accepting run of \mathcal{A}_c over w' , and let us consider the infixes ν of π associated with the c -blocks of w' greater than $2N_c + 1$. We replace ν with an infix of length at most $2N_c + 1$ as follows:

- If ν does not visits accepting states, we remove from ν the maximal cycles (but the first states of such cycles) by obtaining a finite path of length at most N_c .
- If ν visits some accepting state, then ν can be written in the form $\nu = \nu_1 \cdot q_a \cdot \nu_2$, where q_a is an accepting state. We remove the maximal cycles from ν_1 and ν_2 (but the first states of such cycles) by obtaining a finite path of length at most $2N_c + 1$.

In this way, we obtain an accepting run of \mathcal{A}_c over a $2N_c + 1$ -bounded c -coloring, and the result follows. Then, starting from \mathcal{A}_c , one can easily construct in time polynomial in the size of \mathcal{A}_c , a Büchi NFA \mathcal{A}'_c accepting the $2N_c + 1$ -bounded colorings which are accepted by \mathcal{A}_c . \mathcal{A}'_c keeps track in its state of the current state of \mathcal{A}_c and the binary encoding of the value of a counter modulo $2N_c + 1$, where the latter is reset whenever a c -change point occurs. Note that if φ is a Prompt(\overline{ABBB}_w) formula, then $c(\varphi)$ is a \overline{ABBB}_w formula, and by Proposition 7, the size of \mathcal{A}_c is doubly exponential in the size of φ . By the previous observations, it follows that if $S(\varphi) \neq \emptyset$, then there is a parameter valuation $\alpha \in S(\varphi)$ which is bounded doubly exponentially in $|\varphi|$. Thus, since non-emptiness of Büchi NFA is NLOGSPACE-complete, by Lemma 1 and Proposition 7, we obtain the following result.

Theorem 2. *Satisfiability of PHS is decidable. Moreover, satisfiability of $P(\overline{ABBB}_w)$ is EXPSPACE-complete and given a $P(\overline{ABBB}_w)$ formula φ , in case $S(\varphi) \neq \emptyset$, there is a parameter valuation in $S(\varphi)$ which is bounded doubly exponentially in $|\varphi|$.*

We now show that the double exponential upper bound on the values of the parameters in Theorem 2 for satisfiable $P(\overline{ABBB}_w)$ formulas cannot be in general improved. Indeed, we provide a matching lower bound by defining for each $n \geq 1$, a $P(\overline{AB})$ formula of size polynomial in n which encodes a yardstick of length $(n + 1) * 2^n * 2^{2^n}$. This is done by using a 2^n -bit counter for expressing integers in the range $[0, 2^{2^n} - 1]$ and an n -bit counter for keeping track of the position (index) $i \in [0, 2^n - 1]$ of the $(i + 1)^{th}$ -bit of each valuation ν of the 2^n -bit counter. In particular, such a valuation $\nu \in [0, 2^{2^n} - 1]$ is encoded by a sequence, called n -block, of 2^n sub-blocks of length $n + 1$ where for each $i \in [0, 2^n - 1]$, the $(i + 1)^{th}$ sub-block encodes both the value and the index of the $(i + 1)^{th}$ -bit in the binary representation of ν .

Formally, let $AP \stackrel{\text{def}}{=} \{\#, \#, \$, 0, 1\}$. Fix $n \geq 1$. An n -sub-block is a finite word ν over 2^{AP} of length $n + 1$ of the form $\nu = \{\#, p, bit\} \{bit_1\}, \dots, \{bit_n\}$ where $bit, bit_1, \dots, bit_n \in \{0, 1\}$ and $p \in \{\#, \#\}$. If

$p = \#_2$, we say that v is marked. The *content* of v is *bit*, and the *index* of v is the number in $[0, 2^n - 1]$ whose binary code is bit_1, \dots, bit_n . An *n-block* is a finite word v of length $(n + 1) * 2^n$ of the form $v = v_0 \dots v_{2^n - 1}$, where v_0 is a marked *n-sub-block* of index 0, and for each $i \in [1, 2^n - 1]$, v_i is an unmarked *n-sub-block* having index i . The index of i is the natural number in $[0, 2^{2^n} - 1]$ whose binary code is $bit_0, \dots, bit_{2^n - 1}$, where bit_i is the content of the sub-block v_i for each $i \in [0, 2^n - 1]$. The yardstick of length $(n + 1) * 2^n * 2^{2^n}$ is then encoded by the trace, called *n-trace*, given by $bl_0 \cdot \dots \cdot bl_{2^{2^n} - 1} \cdot \{\$\}^\omega$ where bl_i is the *n-block* having index i for each $i \in [0, 2^{2^n} - 1]$. We first show the following result.

Lemma 3. *For each $n \geq 1$, one can construct in time polynomial in n a satisfiable AB formula ψ_n whose unique model is the *n-trace*.*

Proof. Fix $n \geq 1$. The AB formula ψ_n is defined as $\psi_n \stackrel{\text{def}}{=} \psi_{bl} \wedge \psi_{inc}$. The conjunct ψ_{bl} captures the traces w over $AP = \{\#_1, \#_2, \$, 0, 1\}$ having the form $bl_0 \cdot \dots \cdot bl_k \cdot \{\$\}^\omega$, for some $k \geq 1$, such that the following conditions are satisfied:

- bl_0, \dots, bl_k are *n-blocks*;
- bl_0 is the *n-block* of index 0 (i.e., each *n-sub-block* of bl_0 has content 0);
- bl_k is the *n-block* of index $2^{2^n} - 1$ (i.e., each *n-sub-block* of bl_k has content 1).

One can easily construct an LTL formula of size polynomial in n characterizing the traces satisfying the previous requirements. Thus, since an LTL formula can be translated in linear time into an equivalent AB formula [7], we omit the details of the construction of the AB formula ψ_{bl} .

The conjunct ψ_{inc} additionally ensures that $k = 2^{2^n} - 1$ and for each $i \in [1, 2^{2^n} - 2]$, bl_i is the *n-block* of index i . To this purpose, it suffices to guarantee that in moving from a non-last *n-block* bl to the next one bl' , the 2^n -counter is incremented. This is equivalent to require that there is an *n-sub-block* sbl_0 of bl whose content is 0 such that for each *n-sub-block* sbl of bl , denoted by sbl' the *n-sub-block* of bl' having the same index as sbl , the following holds: (i) if sbl precedes sbl_0 , then the content of sbl (resp., sbl') is 1 (resp., 0), (ii) if sbl corresponds to sbl_0 , then the content of sbl' is 1, and (iii) if sbl follows sbl_0 , then there is $b \in \{0, 1\}$ such that the content of both sbl and sbl' is b . In order to express these conditions, we define auxiliary AB formulas. Recall that proposition $\#_1$ marks the first position of an *n-sub-block*, while $\#_2$ marks the first position of an *n-block* bl (corresponding to the first position of the first *n-sub-block* of bl). For each AB formula φ , the AB formula $right(\varphi)$ requires that φ holds at the singleton interval corresponding to the right endpoint of the current interval:

$$right(\varphi) \stackrel{\text{def}}{=} \langle A \rangle (\text{len}_1 \wedge \varphi)$$

The AB formula $\psi_{one}(\#_2)$ ensures that proposition $\#_2$ occurs exactly once in the current interval, while $\psi_{not}(\#_2)$ ensures that $\#_2$ does not occur in the current interval. We focus on the definition of $\psi_{one}(\#_2)$ (the definition of $\psi_{not}(\#_2)$ being similar).

$$\psi_{one}(\#_2) \stackrel{\text{def}}{=} [right(\#_2) \vee \langle B \rangle right(\#_2)] \wedge \neg \langle B \rangle [right(\#_2) \wedge \langle B \rangle right(\#_2)] \wedge \neg [right(\#_2) \wedge \langle B \rangle right(\#_2)]$$

Moreover, we define the AB formulas $\psi_{=}(b, b')$ where $b, b' \in \{0, 1\}$. Formula $\psi_{=}(b, b')$ holds at a singleton interval $[h, h]$ (along the given trace) iff whenever h corresponds to the beginning of a *n-sub-block* sbl of an *n-block* bl , then (i) the content of sbl is b , (ii) the *n-block* bl is followed by an *n-block* bl' , and (iii) the *n-sub-block* of bl' having the same index as sbl has content b' .

$$\psi_{=}(b, b') \stackrel{\text{def}}{=} \#_1 \rightarrow [b \wedge \langle A \rangle (\text{len}_2 \wedge \langle A \rangle (\psi_{one}(\#_2) \wedge \theta_{=} \wedge right(b' \wedge \#_1)))]$$

$$\theta_{=} \stackrel{\text{def}}{=} \bigwedge_{h=1}^n \bigvee_{b \in \{0, 1\}} [\langle B \rangle (\text{len}_h \wedge right(b)) \wedge \langle A \rangle (\text{len}_{h+1} \wedge right(b))]$$

Finally, the conjunct ψ_{inc} in the definition of ψ_n is given by

$$\begin{aligned}\psi_{inc} &\stackrel{\text{def}}{=} [A]([\text{right}(\#_2) \wedge \langle A \rangle (\neg \text{len}_1 \wedge \text{right}(\#_2))] \longrightarrow \langle A \rangle [\psi_{one}(\#_2) \wedge \text{right}(\#_1 \wedge \psi_{=(0,1)}) \wedge \psi_L \wedge \psi_R]) \\ \psi_L &\stackrel{\text{def}}{=} [B](\text{right}(\#_1) \rightarrow \text{right}(\psi_{=(1,0)})) \\ \psi_R &\stackrel{\text{def}}{=} \langle A \rangle (\text{len}_2 \wedge [A]([\psi_{not}(\#_2) \wedge \text{right}(\#_1)] \rightarrow \bigvee_{b \in \{0,1\}} \text{right}(\psi_{=(b,b)})))\end{aligned}$$

This concludes the proof of Lemma 3. \square

Fix $n \geq 1$ and let ψ_n be the AB formula in Lemma 3. We consider the P(AB) formula φ_n with just one parameter given by $\varphi_n \stackrel{\text{def}}{=} \psi_n \wedge \langle A \rangle_{\leq u} \langle A \rangle (\text{len}_1 \wedge \$)$. By Lemma 3, the smallest value for parameter u for which φ_n has a model is greater than $(n+1) * 2^n * 2^{2^n}$. Hence, we obtain the following result.

Proposition 8. *There is a finite set AP of atomic propositions and a family $\{\varphi_n\}_{n \geq 1}$ of satisfiable P(AB) formulas over AP with just one parameter such that for each $n \geq 1$, φ_n has size polynomial in n and the smallest parameter valuation in $S(\varphi_n)$ is doubly exponential in n .*

Solving model checking. A fair Kripke structure \mathcal{K}_f is a Kripke structure equipped with a set S_f of accepting states. An infinite path of \mathcal{K}_f is *fair* if it visits infinitely many times states in S_f . Assume that \mathcal{K}_f is over the set of atomic propositions given by $AP \cup \{c\}$, and let Lab_c be the associated propositional labeling. A *c-pumpable fair path* of \mathcal{K}_f is a fair infinite path π of \mathcal{K}_f such that each infix of π associated to a c -block of the trace $Lab_c(\pi)$ visits some state at least twice. Let $\mathcal{K} = (AP, S, E, Lab, s_0)$ be a Kripke structure over AP, φ a PromptHS formula, and $\mathcal{A}_{-c} = (2^{AP \cup \{c\}}, Q, q_0, \delta, F)$ be the Büchi NFA of Proposition 7 accepting the models of the HS formula $\neg rel_c(\varphi) \wedge alt_c$ (note that we consider the negation of $rel_c(\varphi)$). We define the fair Kripke structure

$$\mathcal{K} \times \mathcal{A}_{-c} = (AP \cup \{c\}, S \times Q \times 2^{\{c\}}, (s_0, q_0, \emptyset), E_c, Lab_c, S \times F \times 2^{\{c\}})$$

where (i) $((s, q, C), (s', q', C')) \in E_c$ iff $(s, s') \in E$ and $q' \in \delta(q, C \cup Lab(s))$, and (ii) $Lab_c(s, q, C) = Lab(s) \cup C$. By construction, the traces associated to the fair infinite paths of $\mathcal{K} \times \mathcal{A}_{-c}$ correspond to the c -colorings w' of the traces of \mathcal{K} which are accepted by \mathcal{A}_{-c} such that $c \notin w'(0)$. The following lemma is similar to Lemma 4.2 in [23] and provides a characterization of emptiness of the set $V(\mathcal{K}, \varphi)$ of parameter valuations.

Lemma 4. \mathcal{K} does not satisfy φ (i.e., $V(\mathcal{K}, \varphi) = \emptyset$) iff $\mathcal{K} \times \mathcal{A}_{-c}$ has a c -pumpable fair path.

Proof. For the right implication, assume that $V(\mathcal{K}, \varphi) = \emptyset$. We need to show that $\mathcal{K} \times \mathcal{A}_{-c}$ has a c -pumpable fair path. Let $k = |Q||S| + 1$ and α be the parameter valuation defined by $\alpha(u) = 2k$ for each $u \in U$. Since $V(\mathcal{K}, \varphi) = \emptyset$, there is a trace w of \mathcal{K} such that $(w, \alpha) \not\models \varphi$. Let w' be the k -bounded c -coloring of w such that each c -block of w' has length exactly k and $c \notin w'(0)$. Since $w' \models alt_c$ and $c(\varphi) = rel_c(\varphi) \wedge alt_c$, by Lemma 2(2), it follows that $w' \models \neg rel_c(\varphi) \wedge alt_c$. Hence, w' is accepted by the Büchi automaton \mathcal{A}_{-c} , and by construction there is a fair path π of $\mathcal{K} \times \mathcal{A}_{-c}$ whose trace is w' . Now, each infix of π associated to a c -block of w' has length $k = |Q||S| + 1$. Moreover, by construction, the third component C of the states $(s, q, C) \in S \times Q \times 2^c$ along such an infix does not change. It follows that such an infix visits one state at least twice. Thus, π is a c -pumpable fair path of $\mathcal{K} \times \mathcal{A}_{-c}$.

For the left implication, assume that $\mathcal{K} \times \mathcal{A}_{-c}$ has a c -pumpable fair path ρ . Let α be an arbitrary parameter valuation and $k = \max_{u \in U} \alpha(u)$. We need to show that there is a trace w of \mathcal{K} such that

$(w, \alpha) \not\models \varphi$. Since ρ is a c -pumpable fair path, each infix of ρ associated to a c -block of the trace $Lab_c(\rho)$ visits some state at least twice. The corresponding cycle in the infix can be pumped k -times. It follows that there is a c -pumpable fair path ρ' of $\mathcal{K} \times \mathcal{A}_{-c}$ such that the c -blocks of the associated trace w' have length at least k . By construction, w' is the c -coloring of some trace w of \mathcal{K} and w' is accepted by \mathcal{A}_{-c} , i.e. $w' \models \neg rel_c(\varphi) \wedge alt_c$. Hence, w' is a k -spaced coloring of w and $w' \not\models c(\varphi)$. By Lemma 2(1), it follows that $(w, \alpha) \not\models \varphi$, and the result follows. \square

By Lemma 4, we deduce that if $V(\mathcal{K}, \varphi) \neq \emptyset$, then for the parameter valuation α such that $\alpha(u) = 2(|Q||S| + 1)$ for each $u \in P_U$, it holds that $\alpha \in V(\mathcal{K}, \varphi)$. Indeed if $\alpha \notin V(\mathcal{K}, \varphi)$, by the first part of the proof of Lemma 4, there is a c -pumpable fair path of $\mathcal{K} \times \mathcal{A}_{-c}$, which leads to the contradiction $V(\mathcal{K}, \varphi) = \emptyset$. It is known that checking the existence of a c -pumpable fair path in a fair Kripke structure is NLOGSPACE-complete [23]. Recall that if φ is a Prompt($AB\overline{B}\overline{B}_w$) formula, then $c(\varphi)$ is a $AB\overline{B}\overline{B}_w$ formula, and by Proposition 7, the size of \mathcal{A}_{-c} is doubly exponential in the size of φ . Thus, since both \mathcal{A}_{-c} and $\mathcal{K} \times \mathcal{A}_{-c}$ can be built on the fly, by Lemma 1, Proposition 7, and Lemma 4, we obtain the following result.

Theorem 3. *Model checking against PHS is decidable. Moreover, model checking a Kripke structure \mathcal{K} against a P($AB\overline{B}\overline{B}_w$) formula φ is EXPSPACE-complete and, in case $V(\mathcal{K}, \varphi) \neq \emptyset$, there is a parameter valuation in $V(\mathcal{K}, \varphi)$ which is bounded doubly exponentially in $|\varphi|$ and linearly in the number of \mathcal{K} -states.*

Similarly to the satisfiability problem for P($AB\overline{B}\overline{B}_w$), for each $n \geq 1$, we provide a lower bound of 2^{2^n} on the minimal parameter valuation for which a fixed Kripke structure satisfies a P(AB) formula by using a P(AB) formula of size polynomial in n . For each $n \geq 1$, let ψ_n be the AB formula over $AP = \{\#_1, \#_2, \$, 0, 1\}$ in Lemma 3 whose unique model is the n -trace. One can trivially define a Kripke structure \mathcal{K} over AP whose set of traces consists of the traces whose first position has label $\{\#_1, \#_2, 0\}$. Let us consider the P(AB) formula φ_n with just one parameter given by $\varphi_n \stackrel{\text{def}}{=} \psi_n \rightarrow \langle A \rangle_{\leq u} \langle A \rangle (\text{len}_1 \wedge \$)$. Evidently, by Lemma 3, $V(\mathcal{K}, \varphi_n)$ is not empty and the minimal parameter valuation in $V(\mathcal{K}, \varphi_n)$ is doubly exponential in n . Hence, we obtain the following result.

Proposition 9. *There is a Kripke structure \mathcal{K} over a set AP of atomic propositions and a family $\{\varphi_n\}_{n \geq 1}$ of P(AB) formulas over AP with just one parameter such that for each $n \geq 1$, φ_n has size polynomial in n , $V(\mathcal{K}, \varphi_n) \neq \emptyset$, and the smallest parameter valuation in $V(\mathcal{K}, \varphi_n)$ is doubly exponential in n .*

5 Conclusion

We have introduced parametric HS (PHS), a parametric extension of the interval temporal logic HS under the trace-based semantics. The novel logic allows to express parametric timing constraints on the duration of the intervals. We have shown that the satisfiability and model checking problems for the whole logic are decidable, and for the fragment P($AB\overline{B}$) of PHS, the problems are EXPSPACE-complete. Moreover, for the fragment P($AB\overline{B}$), we gave tight bounds on optimal parameter values for the considered problems. An intriguing open question is the expressiveness of P($AB\overline{B}$) (or more in general PHS) versus parametric LTL (PLTL). We have shown that P($AB\overline{B}$) subsumes PLTL. In particular, given a PLTL formula φ , it is possible to construct in linear time a P($AB\overline{B}$) on the same set of parameters which is equivalent to φ for each parameter valuation. Is P($AB\overline{B}$) more expressive than PLTL? Another problem left open is whether PromptHS is strictly less expressive than full PHS.

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