Ludics without Designs I: Triads

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In this paper, we introduce the concept of triad. Using this notion, we study, revisit, discover and rediscover some basic properties of ludics from a very general point of view.

1 Introduction

An orthodox introduction of a paper on ludics should begin as follows. First, the authors say what ludics is commonly intended to be: typically, they would say that it is a kind of game semantics which is close to the more popular categorical game models for linear logic and *PCF* introduced in the last twenty years. Having set up the context, then they could informally describe ludics as an untyped and monistic framework which provides a semantics for proofs of a linear (i.e., without exponentials) polarized fragment of linear logic. The authors should also stress that ludics is a semantics which is based on interaction. Finally, — trying not to frighten the casual reader — the authors should give an intuitive account of some of the basic constituents of ludics: the notions of design, orthogonality, behaviour, etc., putting more emphasis on the concepts which are more related to the contribution of the paper.

Of course, there is *nothing* wrong (or bad) in starting an article on ludics in the "orthodox way" described above. However, for this paper we find more instructive to take another approach. Namely, we give from the very beginning the most important definition of our work.

Definition 1.1 (Triad)**.** A **triad** is an ordered triple $A = (\mathscr{P}_A, \mathscr{N}_A, \perp_A)$ where:

- $\mathscr{P}_A = \{ p, q, r, \dots \}$ is a set. Its elements are said to be **positive terms**.
- $\mathscr{N}_A = \{ n, m, \ell, \dots \}$ is a set. Its elements are said to be **negative terms**.
- The sets $\mathscr{P}_A$ and $\mathscr{N}_A$ are *disjoint*. We call the set $\mathscr{P}_A \cup \mathscr{N}_A$ the **domain of** A and we denote it as $\mathrm{dom}(A)$. Elements of $\mathrm{dom}(A)$ are also said to be **terms**.
- $\perp_A$ is a relation $\perp_A \subseteq \mathscr{P}_A \times \mathscr{N}_A$ called **orthogonality**. For $p \in \mathscr{P}_A$ and $n \in \mathscr{N}_A$, we write $p \perp_A n$ and $p \not\perp_A n$ for $(p,n) \in \perp_A$ and $(p,n) \notin \perp_A$, respectively. △

We now give a simple example of triad.

Example 1.2. Let $I \overset{\text{DEF}}{=} (\mathscr{P}_I, \mathscr{N}_I, \perp_I)$ be the ordered triple given as follows.

- Let P and N be two distinct symbols.
- Let $\mathscr{P}_I \overset{\text{DEF}}{=} \{ 0,1,2 \} \times \{ P \}$ and $\mathscr{N}_I \overset{\text{DEF}}{=} \{ 0,1,2 \} \times \{ N \}$. Clearly, $\mathscr{P}_I$ and $\mathscr{N}_I$ are disjoint sets. The domain of I is, of course, the set $\{ (0,P),(1,P),(2,P),(0,N),(1,N),(2,N) \}$.
- $\perp_I$ is given as follows:

$$(r,P) \perp_I (s,N) \overset{\text{DEF}}{\iff} r = s \ , \qquad \text{for } (r,P) \in \mathscr{P}_I \text{ and } (s,N) \in \mathscr{N}_I \ .$$

The triple I is a triad in our sense. △

*Supported by the ANR project ANR-2010-BLAN-021301 LOGOI.

S. Alves and I. Cervesato (Eds): Third International Workshop on Linearity 2014 (LINEARITY'14)
EPTCS 176, 2015, pp. 49–63, doi:10.4204/EPTCS.176.5

The notion of triad is not at all a new mathematical concept. Except for minor details, similar structures have already been defined and investigated in the literature of several fields of research. To be short, we only mention the notion of *context* in *formal concept analysis* (see e.g., [7, 15]) and the notion of *classification domain* (or *classification*) in *information theory* (see e.g., [1, 2]).

In the field of research which concerns this paper — i.e., the *proof theory* related to *linear logic* — we remark that our notion of triad is very similar to the concept of *Boolean–valued game* [9] or *Boolean–valued Chu space* (see e.g., [11, 15]), as we discuss in more detail in Section 4.

How is the notion of triad related to ludics?

As the title of this work should suggest ("ludics without designs" is indeed quite provocative; it sounds like "proof theory without proofs") here we do not consider a *design* — the very central concept of ludics — as the *well–specified* and concrete proof–like object defined in [8] (see also [4, 13] for other more or less equivalent definitions of design). Rather, a design is seen as special case of what we are calling *term*: just an *unspecified* and primitive element of the domain of a given triad A.

Similarly, in ludics there is a *well–specified* orthogonality relation [8]: the one which relate two elements p and n if and only if the procedure of normalization between the designs p and n successfully terminates. By contrast, here we consider a more general situation: we are interested in all possible orthogonality relations. Given two disjoint sets $\mathscr{P}_A$ and $\mathscr{N}_A$, *any* subset of $\mathscr{P}_A \times \mathscr{N}_A$ is an orthogonality in our sense. In particular, we do not need to recall or introduce any kind of procedure of normalization.

What can we do with triads?

- In Subsection 2.1 we use the orthogonality relation $\perp_A$ to define *closed sets* (in the sense of [5]) in $\mathscr{P}_A$ and $\mathscr{N}_A$, and we study some basic properties. Closed sets are called *behaviours* in ludics [8], and they are the semantical counter–part of the syntactical notion of formula in logic.
- In Subsection 2.2 we introduce the *specialization relation* on $\mathscr{P}_A$ and $\mathscr{N}_A$. Our relation of specialization is exactly the *precedence relation* between designs defined in [8]. Furthermore, we generalize specialization to a new relation that we call *semantical consequence* and study some of its properties.
- In Subsection 2.3 we introduce the notion of *entailment system*. This notion can be seen as the natural adaptation of the concept *information system* [12] (see also [1, 15]) to our setting. We show that triads equipped by relations of semantical consequence are entailment systems. This result is useful to us because it allows us to understand properties of the relation of semantical consequence for triads in terms of "structural rules" of entailment systems.

Next, in Section 3 we introduce and study the concept of *functional* for a triad. In ludics, functionals are introduced in [3], and they are the designs which constitute — categorically speaking — the *morphisms* in ludics. In view of their importance, one of the aims of this paper is to study some fundamental properties of functionals at a more general and abstract level.

- In Subsection 3.1 we generalize the notion of functional introduced in [3] to our setting.
- In Subsection 3.2 we define the notion of *continuous* functional and the notion of functional which *preserves the relation of semantical consequence*. We show that these two notions are equivalent.
- In Subsection 3.3 we define the notion of *regular* functional. This notion is crucial in our work because in ludics every functional (in the sense of [3]) is regular. Regularity has also very pleasant consequences: *regular functionals are continuous and preserve the relation of semantical consequence*.

In Section 4 we give a couple of examples of triads and functionals. In particular, we show that designs and functionals as given in [3] meet our conditions. In Section 5 we conclude.

Our methodology is the following: except for Section 4, we always work with *arbitrary triads*. This (obviously) means that the results we are going to show hold in any triad, and — more importantly

— that these results hold in ludics without explicitly introducing the notion of design nor the specific orthogonality relation of ludics. The reader should be able to understand our abstract results of Section 2 and Section 3 without any previous knowledge of ludics: all we need is in Definition 1.1.

To conclude this section, we would like point out that some of the results we show in this paper are perhaps not new, as structures similar to our triads have been studied extensively in the literature. On the other hand, we also remark that — to the best of our knowledge — many constructions we are considering in this paper and, consequently, many results stated in Section 2 and Section 3 seem to be new, *when concretely applied to ludics*. More specifically, we mainly refer to:

- the construction of the relation of semantical consequence — which allows us to understand closed sets (i.e., behaviours) in terms of sets of consequences of entailment systems;
- the results on functionals concerning the notions of continuity and regularity — which allow us to better understand the nature of the functionals of ludics (in the sense of [3]).

We collect the most significant results in Theorem 5.1.

2 Triads: Basic Theory

In this section, we study some basic property of triads. In Subsection 2.1 we introduce the notion of closed set in our setting. In Subsection 2.2 we introduce the relations of specialization and semantical consequence, respectively, and study some properties. Finally, in Subsection 2.3 we introduce the notion of entailment system and relate this concept with the relation of semantical consequence.

We now fix some notation and terminology. Let $A = (\mathscr{P}_A, \mathscr{N}_A, \perp_A)$ be a triad.

(1) We use $a, b, c, \ldots$ to range over terms (i.e., over elements of the domain of A).

(2) We use the letter $\mathscr{O}_A$ as a variable ranging over $\{\ \mathscr{P}_A, \mathscr{N}_A\ \}$. Furthermore, if $\mathscr{O}_A = \mathscr{P}_A$, then we write $\overline{\mathscr{O}_A}$ for $\mathscr{N}_A$. Similarly, if $\mathscr{O}_A = \mathscr{N}_A$, then we write $\overline{\mathscr{O}_A}$ for $\mathscr{P}_A$. Note that $\overline{\overline{\mathscr{O}_A}} = \mathscr{O}_A$.

(3) By an abuse of notation, for $a \in \mathscr{O}_A$ and $b \in \overline{\mathscr{O}_A}$ we write $a \perp_A b$, or equivalently $b \perp_A a$, for

$$\text{the positive term in } \{\ a, b\ \} \quad \perp_A \quad \text{the negative term in } \{\ a, b\ \}\ .$$

This notation makes sense precisely because the sets $\mathscr{O}_A$ and $\overline{\mathscr{O}_A}$ are supposed to be disjoint.

(4) We write $\emptyset_{\mathscr{O}_A}$ to mean that the empty–set $\emptyset$ has to be intended as a subset of $\mathscr{O}_A$.

(5) Given a set E we write $\mathrm{pow}(E)$ for the power–set of E (i.e., the set of its subsets).

(6) We use the expression “iff” as an abbreviation for “if and only if.”

From now on, up to the end the paper, we fix an arbitrary triad $A = (\mathscr{P}_A, \mathscr{N}_A, \perp_A)$ and an arbitrary subset of terms $\mathscr{O}_A \in \{\ \mathscr{P}_A, \mathscr{N}_A\ \}$. To ease notation, in the rest of the paper we write $\mathscr{P}$, $\mathscr{N}$, $\perp$ and $\mathscr{O}$ for $\mathscr{P}_A$, $\mathscr{N}_A$, $\perp_A$ and $\mathscr{O}_A$, respectively.

2.1 Closed Sets

In this subsection, we use the orthogonality relation $\perp$ to equip the sets $\mathscr{P}$ and $\mathscr{N}$ with some topological structure. Namely, we introduce the concept of *closed set* in our setting. Here, closed sets are not to be intended as “closed sets in a *topological space*” but as “closed sets in a *closure space* [5]”, a slightly more general topological notion. Closed sets are important to us because they correspond to *behaviours* [8, 4, 13], the “semantical” notion of formula in ludics. Closed sets induced by orthogonality are not uncommon in the literature of theoretical computer science, see e.g., [14, 10].

Definition 2.1 (Orthogonal sets, closed sets)**.** Let $X \subseteq \mathscr{O}$. We define the **orthogonal set of** X as the subset $X^{\perp}$ of $\overline{\mathscr{O}}$ given by:

$$b \in X^{\perp} \overset{\text{DEF}}{\iff} a \perp b \text{ for every } a \in X \ , \qquad \text{for } b \in \overline{\mathscr{O}} \ .$$

We call **closed set in** $\mathscr{O}$ any set $X \subseteq \mathscr{O}$ such that $X = X^{\perp\perp}$. △

The following theorem justifies the terminology given in Definition 2.1.

Theorem 2.2 (Closure)**.** *Let X and Y be subsets of $\mathscr{O}$, and let $a \in \mathscr{O}$ and $b \in \overline{\mathscr{O}}$. Then, we have:*

(1) $X \subseteq X^{\perp\perp}$; (2) $X \subseteq Y$ *implies* $Y^{\perp} \subseteq X^{\perp}$;
(3) $X \subseteq Y$ *implies* $X^{\perp\perp} \subseteq Y^{\perp\perp}$; (4) $X^{\perp} = X^{\perp\perp\perp}$;
(5) $X^{\perp\perp} = X^{\perp\perp\perp\perp}$; (6) $X^{\perp}$ *is a closed set in* $\overline{\mathscr{O}}$;
(7) $X^{\perp\perp}$ *is a closed set in* $\mathscr{O}$; (8) X *is a closed set in* $\mathscr{O}$ *just in case* $X = Z^{\perp\perp}$ *for some* $Z \in \mathscr{O}$;
(9) $a \perp b$ *just in case* $a \in \{ b \}^{\perp}$; (10) $a \perp b$ *just in case* $b \in \{ a \}^{\perp}$.

Proof. (1) : Let $c \in X$. We have $c \perp d$ for every $d \in X^{\perp}$ by definition of $X^{\perp}$. Hence, $c \in X^{\perp\perp}$.

(2) : Let $c \in Y^{\perp}$. Then, $c \perp d$ for every $d \in Y$. As $X \subseteq Y$, we have $c \perp d$ for every $d \in X$. So, $c \in X^{\perp}$.

(3) : By applying (2) above two times, we get the result.

(4) : By (1) above, we have $X^{\perp} \subseteq (X^{\perp})^{\perp\perp} = X^{\perp\perp\perp}$. By (1) again, we obtain $X \subseteq X^{\perp\perp}$. By (2) above, we get $X^{\perp\perp\perp} = (X^{\perp\perp})^{\perp} \subseteq X^{\perp}$. Hence, we have $X^{\perp} = X^{\perp\perp\perp}$.

(5) : By (4) above, we have $X^{\perp} = X^{\perp\perp\perp}$. Hence, $X^{\perp\perp} = (X^{\perp})^{\perp} = (X^{\perp\perp\perp})^{\perp} = X^{\perp\perp\perp\perp}$.

(6) and (7) : By (4) and (5) above, we have $X^{\perp} = (X^{\perp})^{\perp\perp}$ and $X^{\perp\perp} = (X^{\perp\perp})^{\perp\perp}$, respectively.

(8) : Suppose that $X = Z^{\perp\perp}$ for some $Z \subseteq \mathscr{O}$. Then, we have $X^{\perp\perp} = Z^{\perp\perp\perp\perp} = Z^{\perp\perp} = X$ by using (5) above. As for the converse, suppose that X is a closed in $\mathscr{O}$, i.e., that $X = X^{\perp\perp}$ holds. Take $Z \overset{\text{DEF}}{=} X^{\perp\perp}$. Then, we have $Z^{\perp\perp} = X^{\perp\perp\perp\perp} = X^{\perp\perp} = X$ by using (5) above again.

(9) and (10) : They immediately follow from the definition of orthogonal set. □

We note that properties (1), (3) and (5) above say that $^{\perp\perp}$ is a closure operator on the sets $\mathscr{P}$ and $\mathscr{N}$, i.e., that $(\mathscr{P}, {}^{\perp\perp})$ and $(\mathscr{N}, {}^{\perp\perp})$ form two *closure spaces*, in the sense of [5]. However, we want to point out that our theory is richer than the theory of closure spaces: the latter can be "axiomatized" by using properties (1), (3) and (5) above and the double–orthogonality operator $^{\perp\perp}$ only; in particular, there is nothing there which corresponds to property (2) above, for instance.

In the sequel, we frequently use the properties listed in Theorem 2.2 without any explicit reference.

Example 2.3. Let I be the triad given in Example 1.2. We calculate the closed sets in $\mathscr{P}_I$ and $\mathscr{N}_I$.

- $(\emptyset_{\mathscr{P}_I})^{\perp_I\perp_I} = (\mathscr{N}_I)^{\perp_I} = \emptyset_{\mathscr{P}_I}$.
- $\{ (r,P) \}^{\perp_I\perp_I} = \{ (r,N) \}^{\perp_I} = \{ (r,P) \}$, for every $r \in \{ 0,1,2 \}$.
- $X^{\perp_I\perp_I} = (\emptyset_{\mathscr{N}_I})^{\perp_I} = \mathscr{P}_I$, for any other subset X of $\mathscr{P}_I$.
- $(\emptyset_{\mathscr{N}_I})^{\perp_I\perp_I} = (\mathscr{P}_I)^{\perp_I} = \emptyset_{\mathscr{N}_I}$.
- $\{ (s,N) \}^{\perp_I\perp_I} = \{ (s,P) \}^{\perp_I} = \{ (s,N) \}$, for every $s \in \{ 0,1,2 \}$.
- $X^{\perp_I\perp_I} = (\emptyset_{\mathscr{P}_I})^{\perp_I} = \mathscr{N}_I$, for any other subset X of $\mathscr{N}_I$. △

2.2 The Relation of Specialization and the Relation of Semantical Consequence

We now introduce the *specialization relation* in our setting. Specialization is a relation which has been extensively studied in topology, order theory and domain theory. In our setting, we define the specialization relation as follows.

Definition 2.4 (The specialization relation $\lhd_{\mathscr{O}}$)**.** We define the **specialization relation** as the binary relation $\lhd_{\mathscr{O}} \subseteq \mathscr{O} \times \mathscr{O}$ given by:

$$a \lhd_{\mathscr{O}} b \stackrel{\text{DEF}}{\Longleftrightarrow} \{ a \}^{\perp} \subseteq \{ b \}^{\perp} \ , \qquad \text{for } a \text{ and } b \text{ in } \mathscr{O} \ . \qquad \triangle$$

In the sequel, we read the expression $a \lhd_{\mathscr{O}} b$ as "a is more special than b."

Our definition of $\lhd_{\mathscr{O}}$ follows the analogous relation defined in [8]. There, the specialization relation is called *precedence* relation, and in [4, 13] it is called *observational* ordering (but note, however, that there the observational ordering is defined in a different manner). In ludics, we can read $a \lhd_{\mathscr{O}} b$ as "a is more defined than b" (see [8]).

We also point out that several authors define the specialization relation as the inverse of $\lhd_{\mathscr{O}}$. For instance, in [5] specialization for closure spaces is defined as in Proposition 2.6(iii) below, but with the role of a and b interchanged.

Example 2.5. Let I be the triad given in Example 1.2. We have:

- $(r,P) \lhd_{\mathscr{P}_I} (r',P)$ if and only if $r = r'$, for every r and r' in $\{ 0,1,2 \}$.
- Analogously, $(s,N) \lhd_{\mathscr{N}_I} (s',N)$ if and only if $s = s'$, for every s and s' in $\{ 0,1,2 \}$. $\triangle$

The following proposition gives us a useful characterization of the relation of specialization.

Proposition 2.6. *Let a and b in $\mathscr{O}$. Then, the following claims are equivalent:*

(i) $a \lhd_{\mathscr{O}} b$; (ii) $b \in \{ a \}^{\perp\perp}$; (iii) *For every $X \subseteq \mathscr{O}$, $a \in X^{\perp\perp}$ implies $b \in X^{\perp\perp}$* .

Proof. (i) implies (iii) : Let $X \subseteq \mathscr{O}$. Suppose that $a \in X^{\perp\perp}$, i.e., $\{ a \} \subseteq X^{\perp\perp}$. Then, we have $X^{\perp} = X^{\perp\perp\perp} \subseteq \{ a \}^{\perp}$. Suppose that $a \lhd_{\mathscr{O}} b$, i.e., $\{ a \}^{\perp} \subseteq \{ b \}^{\perp}$. Then, we have $X^{\perp} \subseteq \{ b \}^{\perp}$. Thus, $\{ b \}^{\perp\perp} \subseteq X^{\perp\perp}$. Therefore, $b \in \{ b \} \subseteq \{ b \}^{\perp\perp} \subseteq X^{\perp\perp}$.

(iii) implies (ii) : Let $X = \{ a \}$. We have $a \in \{ a \} \subseteq \{ a \}^{\perp\perp}$. So, $b \in \{ a \}^{\perp\perp}$ by (iii).

(ii) implies (i) : Assume $b \in \{ a \}^{\perp\perp}$, i.e., $\{ b \} \subseteq \{ a \}^{\perp\perp}$. Then, we have $\{ a \}^{\perp} = \{ a \}^{\perp\perp\perp} \subseteq \{ b \}^{\perp}$. Hence, $a \lhd_{\mathscr{O}} b$. □

Corollary 2.7. *For every a and b in $\mathscr{O}$, we have $b \in \{ a \}^{\perp\perp}$ if and only if $a \lhd_{\mathscr{O}} b$.*

Proof. It immediately follows from the equivalence of properties (i) and (ii) of Proposition 2.6. □

By Corollary 2.7, for any singleton subset $\{ a \}$ of $\mathscr{O}$, the closed set $\{ a \}^{\perp\perp}$ can be completely described by using specialization: the members of $\{ a \}^{\perp\perp}$ are exactly the terms $b \in \mathscr{O}$ such that $a \lhd_{\mathscr{O}} b$ holds. Our next step is to generalize this kind of property to arbitrary sets, i.e., not only singletons. To do this, we need to generalize the relation of specialization. This motivates the following definition.

Definition 2.8 (The semantical consequence relation $\blacktriangleleft_{\mathscr{O}}$)**.** We define the relation of **semantical consequence** as the binary relation $\blacktriangleleft_{\mathscr{O}} \subseteq \mathrm{pow}(\mathscr{O}) \times \mathscr{O}$ given by:

$$X \blacktriangleleft_{\mathscr{O}} b \stackrel{\text{DEF}}{\Longleftrightarrow} \bigcap_{a \in X} \{ a \}^{\perp} \subseteq \{ b \}^{\perp} \ , \qquad \text{for } X \subseteq \mathscr{O} \text{ and } b \in \mathscr{O} \ . \qquad \triangle$$

In the sequel, we read the expression $X \blacktriangleleft_{\mathscr{O}} b$ as "b is a semantical consequence of X."

Regarding our terminology, we call the relation $\blacktriangleleft_{\mathscr{O}}$ *semantical consequence* because so it is called in similar contexts (e.g., in [1]). Indeed, if we consider the elements of $\mathscr{O}$ as *sentences* (in a language for first–order logic), and for each $c \in \mathscr{O}$ the set $\{ c \}^{\perp}$ as the class of *structures* (i.e., models) in which c is true, then $X \blacktriangleleft_{\mathscr{O}} b$ states that the class of structures in which all a in X are true is a subclass of the class of structures in which b is true. In this sense, the definition of $X \blacktriangleleft_{\mathscr{O}} b$ is very similar in spirit to the standard definition of the relation of semantical consequence in logic.

We now observe that the relation $\blacktriangleleft_{\mathscr{O}}$ is indeed a generalization of $\lhd_{\mathscr{O}}$.

Proposition 2.9. *For every a and b in $\mathscr{O}$, we have $a \lhd_{\mathscr{O}} b$ if and only if $\{ a \} \blacktriangleleft_{\mathscr{O}} b$.*

Proof. Let a and b in $\mathscr{O}$. Since $\bigcap_{c \in \{a\}} \{ c \}^{\perp} = \{ a \}^{\perp}$, we have

$$a \lhd_{\mathscr{O}} b \quad \text{iff} \quad \{ a \}^{\perp} \subseteq \{ b \}^{\perp} \quad \text{iff} \quad \bigcap_{c \in \{a\}} \{ c \}^{\perp} \subseteq \{ b \}^{\perp} \quad \text{iff} \quad \{ a \} \blacktriangleleft_{\mathscr{O}} b \ .$$ □

Example 2.10. Let I be the triad given in Example 1.2.

- For every $r \in \{ 0,1,2 \}$ and every $X \subseteq \mathscr{P}_I$, we have $X \blacktriangleleft_{\mathscr{P}_I} (r,P)$ if and only if either $X = \{ (r,P) \}$ or X contains at least two elements.
- Similarly, for every $s \in \{ 0,1,2 \}$ and every $X \subseteq \mathscr{N}_I$, we have $X \blacktriangleleft_{\mathscr{N}_I} (s,N)$ if and only if either $X = \{ (s,N) \}$ or X contains at least two elements. △

Lemma 2.11. *For every $X \subseteq \mathscr{O}$, we have $X^{\perp} = \bigcap_{a \in X} \{ a \}^{\perp}$.*

Proof. Let $X \subseteq \mathscr{O}$, and let $b \in \overline{\mathscr{O}}$. We have:

$$b \in \bigcap_{a \in X} \{ a \}^{\perp} \quad \text{iff} \quad b \in \{ a \}^{\perp} \text{ for every } a \in X \quad \text{iff} \quad a \perp b \text{ for every } a \in X \quad \text{iff} \quad b \in X^{\perp} \ .$$ □

We now characterize the orthogonality relation $\perp$ in terms of the relations of semantical consequence.

Proposition 2.12. *Let $a \in \mathscr{O}$ and $b \in \overline{\mathscr{O}}$. Then, the following claims are equivalent:*

$$(1) \quad a \perp b\,; \qquad (2) \quad \{ b \}^{\perp} \blacktriangleleft_{\mathscr{O}} a\,; \qquad (3) \quad \{ a \}^{\perp} \blacktriangleleft_{\overline{\mathscr{O}}} b \ .$$

Proof. (1) implies (2) : Assume $a \perp b$. Then, we have $a \in \{ b \}^{\perp}$, i.e., $\{ a \} \subseteq \{ b \}^{\perp}$. So, $\{ b \}^{\perp\perp} \subseteq \{ a \}^{\perp}$. As $\{ b \}^{\perp\perp} = (\{ b \}^{\perp})^{\perp}$, we conclude $\{ b \}^{\perp} \blacktriangleleft_{\mathscr{O}} a$ by Lemma 2.11.

(2) implies (3) : Assume $\{ b \}^{\perp} \blacktriangleleft_{\mathscr{O}} a$. By Lemma 2.11, we obtain $\{ b \}^{\perp\perp} = (\{ b \}^{\perp})^{\perp} \subseteq \{ a \}^{\perp}$. Thus, $(\{ a \}^{\perp})^{\perp} = \{ a \}^{\perp\perp} \subseteq \{ b \}^{\perp\perp\perp} = \{ b \}^{\perp}$. By Lemma again, 2.11, we finally get $\{ a \}^{\perp} \blacktriangleleft_{\overline{\mathscr{O}}} b$.

(3) implies (1) : Assume $\{ a \}^{\perp} \blacktriangleleft_{\overline{\mathscr{O}}} b$. By Lemma 2.11, we get $\{ a \}^{\perp\perp} = (\{ a \}^{\perp})^{\perp} \subseteq \{ b \}^{\perp}$. Hence, $a \in \{ a \} \subseteq \{ a \}^{\perp\perp} \subseteq \{ b \}^{\perp}$. Thus, we conclude $a \perp b$. □

The following theorem is the generalization of Corollary 2.7 which we are looking for.

Theorem 2.13. *For every $X \subseteq \mathscr{O}$ and every $b \in \mathscr{O}$, we have $b \in X^{\perp\perp}$ if and only if $X \blacktriangleleft_{\mathscr{O}} b$.*

Proof. Let $X \subseteq \mathscr{O}$, and let $b \in \mathscr{O}$. Suppose that $b \in X^{\perp\perp}$. Then, we have $\{ b \} \subseteq X^{\perp\perp}$. Thus, $X^{\perp\perp\perp} = X^{\perp} \subseteq \{ b \}^{\perp}$. By Lemma 2.11, we have $X^{\perp} = \bigcap_{a \in X} \{ a \}^{\perp}$. Hence, we obtain $X \blacktriangleleft_{\mathscr{O}} b$. As for the converse, assume that $X \blacktriangleleft_{\mathscr{O}} b$, i.e., $\bigcap_{a \in X} \{ a \}^{\perp} \subseteq \{ b \}^{\perp}$. By Lemma 2.11, we have $X^{\perp} \subseteq \{ b \}^{\perp}$. So, we obtain $\{ b \}^{\perp\perp} \subseteq X^{\perp\perp}$. As $b \in \{ b \}^{\perp\perp}$, we conclude $b \in X^{\perp\perp}$. □

2.3 Entailment Systems

We now introduce the notion of entailment system. This notion can be seen as the natural adaptation of the concept of *information system* [12] (see also [1, 15]) to our setting. We do not claim at all that our concept of entailment system constitutes a novelty: structures of the same nature — often called *consequence relations* — has already been studied, for different purposes, in the literature of abstract algebraic logic (see e.g., [6]).

In this paper, we introduce this notion in order to show that the set $\mathscr{O}$ equipped by the relation of semantical consequence $\blacktriangleleft_{\mathscr{O}}$ forms an entailment system (Theorem 2.16). One of the consequences of this fact is that we can use the "structural rules" of Proposition 2.15 to derive properties of terms.

Definition 2.14 (Entailment system)**.** We call **entailment systems** any ordered pair $(T, \Vdash)$ where:

- T is a set. Its elements are said to be **tokens**, and we use $u, v, w, \ldots$ to range over them.

- $\Vdash \subseteq \mathrm{pow}(T) \times T$ is a binary relation such that for each $u \in T$, every $U \subseteq T$ and every $V \subseteq T$:

$$\begin{array}{rll} \text{Axiom} & : & u \in U \quad \text{implies} \quad U \Vdash u \ ; \\ \text{Cut} & : & U \Vdash v \ \text{ for every } v \in V \ \text{ and } \ V \Vdash u \quad \text{imply} \quad U \Vdash u \ . \end{array}$$

We call $\Vdash$ the **entailment relation** of the entailment system and read "$U \Vdash u$" as "U entails u." Given $U \subseteq T$, we call $\{ u \in T \mid U \Vdash u \}$ the **set of consequences of** U. △

Proposition 2.15 (Structural rules)**.** *Let $(T, \Vdash)$ be an entailment system. Let U and V be subsets of T, and let u and v be tokens. Then, we have*

$$\begin{array}{ll} \text{Axiom}_0 : \ \{ u \} \Vdash u \ ; & \text{Weakening} : \ V \Vdash u \textit{ and } V \subseteq U \ \textit{ imply } \ U \Vdash u \ ; \\ \text{Cut}_0 : \ U \Vdash v \textit{ and } U \cup \{ v \} \Vdash u \ \textit{ imply } \ U \Vdash u \ ; & \text{Transitivity} : \ U \Vdash v \textit{ and } \{ v \} \Vdash u \ \textit{ imply } \ U \Vdash u \ . \end{array}$$

Proof. Axiom$_0$: We always have $u \in \{ u \}$. Hence, $\{ u \} \Vdash u$ by Axiom.

Weakening : Suppose that $V \Vdash u$ and $V \subseteq U$. Let $v \in V$. Since $V \subseteq U$, we have $v \in U$. Hence, $U \Vdash v$ by Axiom. Since this holds for every $v \in V$, we obtain $U \Vdash v$ for every $v \in V$. Hence, $U \Vdash u$ by Cut.

Cut$_0$: Suppose that $U \Vdash v$ and $U \cup \{ v \} \Vdash u$. Let $w \in U \cup \{ v \}$. If $w \in U$, then $U \Vdash w$ by Axiom. If $w = v$, then $U \Vdash v$ by assumption. Hence, we have $U \Vdash w$ for every $w \in U \cup \{ v \}$. By assumption, $U \cup \{ v \} \Vdash u$. Therefore, $U \Vdash u$ by Cut.

Transitivity : Suppose that $\{ v \} \Vdash u$. Then, we have $U \cup \{ v \} \Vdash u$ by Weakening. Since $U \Vdash v$ holds by assumption, we conclude $U \Vdash u$ by Cut$_0$. □

Theorem 2.16. *The pair $(\mathscr{O}, \blacktriangleleft_{\mathscr{O}})$ is an entailment system.*

Proof. We have to show that conditions Axiom and Cut of Definition 2.14 hold. As for Axiom, note that we have $\bigcap_{a \in U} \{ a \}^\perp \subseteq \{ a \}^\perp$ for every $a \in U$. As for Cut, assume that $\bigcap_{a \in U} \{ a \}^\perp \subseteq \{ b \}^\perp$ for every $b \in V$ and that $\bigcap_{b \in V} \{ b \}^\perp \subseteq \{ c \}^\perp$. Then, we have $\bigcap_{a \in U} \{ a \}^\perp \subseteq \bigcap_{b \in V} \{ b \}^\perp$. Since $\bigcap_{b \in V} \{ b \}^\perp \subseteq \{ c \}^\perp$, we conclude $\bigcap_{a \in U} \{ a \}^\perp \subseteq \{ c \}^\perp$. □

By Theorem 2.13, we have $\{ b \in \mathscr{O} \mid X \blacktriangleleft_{\mathscr{O}} b \} = X^{\perp\perp}$, for every $X \subseteq \mathscr{O}$. Since, by Theorem 2.16, the pair $(\mathscr{O}, \blacktriangleleft_{\mathscr{O}})$ is an entailment system, we conclude that every closed set $X^{\perp\perp}$ in $\mathscr{O}$ can be precisely described as *the set of consequences of X*. Furthermore, by Theorem 2.16 again, we can use the "structural rules" of Proposition 2.15 in the entailment system $(\mathscr{O}, \blacktriangleleft_{\mathscr{O}})$. We now use some of them to derive some simple properties of terms.

Proposition 2.17. *For every $X \subseteq \mathscr{O}$ and every $b \in (\emptyset_{\mathscr{O}})^{\perp\perp}$, we have $X \blacktriangleleft_{\mathscr{O}} b$.*

Proof. By Theorem 2.13, we have $b \in (\emptyset_{\mathscr{O}})^{\perp\perp}$ if and only if $\emptyset_{\mathscr{O}} \blacktriangleleft_{\mathscr{O}} b$. Thus, we can conclude $X \blacktriangleleft_{\mathscr{O}} b$ by a simple application of Weakening. □

We now show a property which connects the relations $\blacktriangleleft_{\mathscr{O}}$ and $\blacktriangleleft_{\overline{\mathscr{O}}}$ to the orthogonality relation $\perp$.

Proposition 2.18. *Let a and a' be elements of $\mathscr{O}$, and let b and b' be elements of $\overline{\mathscr{O}}$. Suppose that $a \perp b$, $\{ a \} \blacktriangleleft_{\mathscr{O}} a'$ and $\{ b \} \blacktriangleleft_{\overline{\mathscr{O}}} b'$ holds. Then, $a' \perp b'$ holds as well. Graphically,*

$$\begin{array}{ccc} \{ a \} & \blacktriangleleft_{\mathscr{O}} & a' \\ \perp & & \\ \{ b \} & \blacktriangleleft_{\overline{\mathscr{O}}} & b' \end{array} \qquad \textit{implies} \qquad \begin{array}{ccc} \{ a \} & \blacktriangleleft_{\mathscr{O}} & a' \\ \perp & & \perp \\ \{ b \} & \blacktriangleleft_{\overline{\mathscr{O}}} & b' \end{array} .$$

Proof. Suppose that $a \perp b$, $\{ a \} \blacktriangleleft_{\mathscr{O}} a'$ and $\{ b \} \blacktriangleleft_{\overline{\mathscr{O}}} b'$. Then, we have $\{ b \}^\perp \blacktriangleleft_{\mathscr{O}} a$ by Proposition 2.12. From this and $\{ a \} \blacktriangleleft_{\mathscr{O}} a'$ we obtain $\{ b \}^\perp \blacktriangleleft_{\mathscr{O}} a'$ by Transitivity. Since $\{ b \} \blacktriangleleft_{\overline{\mathscr{O}}} b'$ is equivalent to $\{ b \}^\perp \subseteq \{ b' \}^\perp$, we obtain $\{ b' \}^\perp \blacktriangleleft_{\mathscr{O}} a'$ by Weakening. By Proposition 2.12, this means $a' \perp b'$. □

3 Functionals

In this section, we introduce the notion of functional in our setting and study some properties of functionals. In Subsection 3.1 we give the formal definition of functionals, and in Subsection 3.2 we define and study the notions of continuous functional and functional which preserves the relation of semantical consequence. Finally, in Subsection 3.3 we introduce the notion of regular functional and show that regular functionals are continuous and preserve the relation of semantical consequence.

3.1 Functionals

We now define the concept of collection of functionals for a triad. Recall that $A = (\mathscr{P}, \mathscr{N}, \perp)$ is the arbitrary triad that we fixed at the beginning of Section 2. Also, remember that $\mathrm{dom}(A)$, the domain of A, is the set $\mathscr{P} \cup \mathscr{N}$ (see Definition 1.1).

Definition 3.1 (Collection of functionals for a triad)**.** A **collection of functionals for** A is an ordered pair $F = (\mathscr{F}_F, \widehat{\ }^F)$ where:

- $\mathscr{F}_F = \{ f, g, h, \dots \}$ is a set. Its members are called **functionals**.
- $\widehat{\ }^F$ is a function, that we call **interpretation**, which maps each functional $f \in \mathscr{F}_F$ to a function $\widehat{f}^F$ from $\mathrm{dom}(A)$ to $\mathrm{dom}(A)$. Furthermore, the function $\widehat{\ }^F$ has to satisfy the following condition, called **preservation of polarity**:

$$\widehat{f}^F(p) \in \mathscr{P} \quad \text{and} \quad \widehat{f}^F(n) \in \mathscr{N} \ , \qquad \text{for every } f \in \mathscr{F}_F \text{, every } p \in \mathscr{P} \text{ and every } n \in \mathscr{N} \ . \qquad \triangle$$

Example 3.2. Let I be the triad given in Example 1.2. Let $W \stackrel{\mathrm{DEF}}{=} (\mathscr{F}_W, \widehat{\ }^W)$ be the pair given by:

- $\mathscr{F}_W \stackrel{\mathrm{DEF}}{=} \{ \sharp, \flat, \natural \}$, where $\sharp$, $\flat$ and $\natural$ are just three pairwise distinct symbols.
- $\widehat{\ }^W$ is the function which maps $\sharp$, $\flat$ and $\natural$ to the functions $\widehat{\sharp}^W$, $\widehat{\flat}^W$ and $\widehat{\natural}^W$ from $\mathrm{dom}(I)$ to $\mathrm{dom}(I)$ respectively given as follows:

$$\widehat{\sharp}^W(a) \stackrel{\mathrm{DEF}}{=} \begin{cases} (1,P) & \text{if } a \in \mathscr{P}_I \\ (1,N) & \text{if } a \in \mathscr{N}_I \ , \end{cases} \qquad \widehat{\flat}^W(a) \stackrel{\mathrm{DEF}}{=} \begin{cases} (1,P) & \text{if } a \in \{ (0,P), (1,P) \} \\ (2,P) & \text{if } a = (2,P) \\ (1,N) & \text{if } a \in \mathscr{N}_I \ , \end{cases} \qquad \widehat{\natural}^W(a) \stackrel{\mathrm{DEF}}{=} a \ ,$$

  for $a \in \mathrm{dom}(I)$. Note that the function $\widehat{\ }^W$ satisfies the condition of preservation of polarity.

According to our definition, the pair W is a collection of functionals for I. $\triangle$

From now on, up to the end of the paper, we fix an arbitrary collection of functionals $F = (\mathscr{F}_F, \widehat{\ }^F)$ for A and an arbitrary functional $f \in \mathscr{F}_F$. To ease notation, in the sequel we write $\mathscr{F}$, $\widehat{\ }$ and $\widehat{f}$ for $\mathscr{F}_F$, $\widehat{\ }^F$ and $\widehat{f}^F$, respectively. Similarly, we write $\widehat{\sharp}$, $\widehat{\flat}$ and $\widehat{\natural}$ for $\widehat{\sharp}^W$, $\widehat{\flat}^W$ and $\widehat{\natural}^W$, respectively.

Let us now discuss Definition 3.1.

Intuitively, if we think of the triad A as a structure (i.e., model) for a first–order language, then

- $\mathscr{F}$ can be seen as the set of (unary) function symbols of a first–order language;
- $\widehat{\ }$ can be seen as an interpretation of the function symbols in the structure A i.e., as a function which maps each function symbol f in $\mathscr{F}$ to a (unary) function $\widehat{f}$ from the domain of A to itself.

With this analogy in mind, it is clear that functionals are *not* required to be functions. For instance, in Example 3.2, the *symbols* $\sharp$ and $\flat$ and $\natural$ are certainly not functions, but they are *interpreted* in the triad I as the functions $\widehat{\sharp}$, $\widehat{\flat}$ and $\widehat{\natural}$ from domain of I to itself given above.

In this paper, we are not considering functionals because we want to form a category, say with $\mathrm{dom}(A)$ as the unique object and with $\mathscr{F}$ as the collection of morphisms (essentially, this is what is done in [3]). In fact, the set $\mathscr{F}$ need not contain any functional intended to be the identity morphism of $\mathrm{dom}(A)$. Also, functionals are not equipped with any operation of composition. In this article, we want to study functionals from a different point of view. Namely, we want to analyze their relationship with the notions of closed set, continuity, semantical consequence, and regularity.

The condition of preservation of polarity comes from ludics: in that setting, functionals (as defined in [3]) always satisfy this property (see also Example 4.4). Except for this condition, note that we do not impose any restriction on the nature of the interpretation function $\widehat{\ }$. In particular, it may happen that the interpretation function $\widehat{\ }$ maps two *distinct* functionals g and h in $\mathscr{F}$ to the *same* function.

3.2 Properties of Functionals

In this subsection, we relate functionals to closed sets, continuity, and the relation of semantical consequence. To begin with, it is convenient to introduce some auxiliary notions and notation.

Definition 3.3 (Image, pre–image)**.** Let $X \subseteq \mathscr{O}$. We call **image of** X **under** f and **pre–image of** X **under** f the subsets $f^{\rightarrow}(X)$ and $f^{\leftarrow}(X)$ of $\mathscr{O}$ given by:

$$f^{\rightarrow}(X) \overset{\text{DEF}}{=} \{\, \widehat{f}(a) \mid a \in X \,\} \qquad \text{and} \qquad f^{\leftarrow}(X) \overset{\text{DEF}}{=} \{\, a \mid \widehat{f}(a) \in X \,\} \quad ,$$

respectively. Equivalently, for $a \in \mathscr{O}$ we have

$$a \in f^{\rightarrow}(X) \overset{\text{DEF}}{\iff} a = \widehat{f}(b) \text{ for some } b \in X \qquad \text{and} \qquad a \in f^{\leftarrow}(X) \overset{\text{DEF}}{\iff} \widehat{f}(a) \in X \ . \qquad \triangle$$

Example 3.4. Let I be the triad given in Example 1.2. Let W be the collection of functionals for I given in Example 3.2. We calculate the pre–images of some closed sets in $\mathscr{P}_I$ and $\mathscr{N}_I$ (cf. Example 2.3).

- For $X \subseteq \mathscr{P}_I$, we have
$$\sharp^{\leftarrow}(X^{\perp_I \perp_I}) \;=\; \begin{cases} \emptyset_{\mathscr{P}_I} & \text{if } X \in \{\, \emptyset_{\mathscr{P}_I}, \{\, (0,P) \,\}, \{\, (2,P) \,\} \,\} \\ \mathscr{P}_I & \text{otherwise} \ . \end{cases}$$
Similarly, for $X \subseteq \mathscr{N}_I$, we have
$$\sharp^{\leftarrow}(X^{\perp_I \perp_I}) = \begin{cases} \emptyset_{\mathscr{N}_I} & \text{if } X \in \{\, \emptyset_{\mathscr{N}_I}, \{\, (0,N) \,\}, \{\, (2,N) \,\} \,\} \\ \mathscr{N}_I & \text{otherwise} \ . \end{cases}$$
- We have $\flat^{\leftarrow}(\{\, (1,P) \,\}^{\perp_I \perp_I}) = \{\, (0,P), (1,P) \,\}$, and $\flat^{\leftarrow}(X^{\perp_I \perp_I}) = \sharp^{\leftarrow}(X^{\perp_I \perp_I})$ for every $X \subseteq \mathscr{N}_I$.
- We have $\natural^{\leftarrow}(X^{\perp_I \perp_I}) = X^{\perp_I \perp_I}$ for every $X \subseteq \mathscr{P}_I$, and $\natural^{\leftarrow}(X^{\perp_I \perp_I}) = X^{\perp_I \perp_I}$ for every $X \subseteq \mathscr{N}_I$. $\triangle$

The following lemma establishes some simple but fundamental facts that we need in the sequel.

Lemma 3.5. *Let X and Y be subsets of $\mathscr{O}$. Then, we have:*

(1) $X \subseteq Y$ *implies* $f^{\rightarrow}(X) \subseteq f^{\rightarrow}(Y)$; (2) $X \subseteq Y$ *implies* $f^{\leftarrow}(X) \subseteq f^{\leftarrow}(Y)$;
(3) $f^{\rightarrow}\big(f^{\leftarrow}(X)\big) \subseteq X$; (4) $X \subseteq f^{\leftarrow}\big(f^{\rightarrow}(X)\big)$.

Proof. (1) : Suppose that $a \in f^{\rightarrow}(X)$, and assume $X \subseteq Y$. Then, $a = \widehat{f}(b)$ for some $b \in X$, by definition of image. Since $X \subseteq Y$, we have $b \in Y$. So, $a = \widehat{f}(b)$ for some $b \in Y$. Thus, $a \in f^{\rightarrow}(Y)$.

(2) : Suppose that $a \in f^{\leftarrow}(X)$, and assume $X \subseteq Y$. Then, $\widehat{f}(a) \in X$, by definition of pre–image. Since $X \subseteq Y$, we have $\widehat{f}(a) \in Y$. Therefore, $a \in f^{\leftarrow}(Y)$.

(3) : Suppose that $a \in f^{\rightarrow}\big(f^{\leftarrow}(X)\big)$. Then, $a = \widehat{f}(b)$ for some $b \in f^{\leftarrow}(X)$, by definition of image. Also, we have $\widehat{f}(b) \in X$ by definition of pre–image. Hence, $a \in X$.

(4) : Suppose that $a \in X$. Then, $\widehat{f}(a) \in f^{\rightarrow}(X)$ by definition of image. Hence, $a \in f^{\leftarrow}\big(f^{\rightarrow}(X)\big)$, by definition of pre–image. $\square$

We are now in position to define the notion of continuous functional. We recall from [5] that a *continuous function* from a closure space (X,Γ) to a closure space (Y,Δ) (here X,Y are sets and Γ,Δ are closure operators) is a function F from X to Y such that for every closed set W in (Y,Δ) the pre–image of W under F is a closed set in (X,Γ). In our setting, we define the concept of continuity for functionals in a similar fashion.

Definition 3.6 (Continuous functional)**.** We say that f is **continuous in** $\mathscr{O}$ if for every $X \subseteq \mathscr{O}$ the set $f^{\leftarrow}(X^{\perp\perp})$ is a closed set in $\mathscr{O}$. That is,

$$f^{\leftarrow}(X^{\perp\perp})^{\perp\perp} = f^{\leftarrow}(X^{\perp\perp}) \ . \qquad \triangle$$

Example 3.7. In the same notation of Example 3.4, the following facts hold.

- The functional $\sharp$ is continuous in $\mathscr{P}_I$ and $\mathscr{N}_I$.
- The functional $\flat$ is *not* continuous in $\mathscr{P}_I$ (because $\flat^{\leftarrow}(\{\ (1,P)\ \}^{\perp_I\perp_I}) = \{\ (0,P),(1,P)\ \}$ and $\{\ (0,P), (1,P)\ \}^{\perp_I\perp_I} = \mathscr{P}_I \neq \{\ (0,P),(1,P)\ \}$). On the other hand, the functional $\flat$ is continuous in $\mathscr{N}_I$.
- The functional $\natural$ is continuous in $\mathscr{P}_I$ and $\mathscr{N}_I$. △

We now define the notion of preservation of the relation of specialization for functionals.

Definition 3.8 (Preservation of the relation of specialization $\lhd_{\mathscr{O}}$)**.** We say that f **preserves the relation of specialization** $\lhd_{\mathscr{O}}$ if

$$a \lhd_{\mathscr{O}} b \ \text{ implies } \ \widehat{f}(a) \lhd_{\mathscr{O}} \widehat{f}(b) \ , \qquad \text{for every } a \text{ and } b \text{ in } \mathscr{O} \ . \qquad \triangle$$

Generalizing the previous notion, we naturally obtain the definition of preservation of the relation of semantical consequence.

Definition 3.9 (Preservation of the relation of semantical consequence $\blacktriangleleft_{\mathscr{O}}$)**.** We say that f **preserves the relation of semantical consequence** $\blacktriangleleft_{\mathscr{O}}$ if

$$X \blacktriangleleft_{\mathscr{O}} b \ \text{ implies } \ f^{\rightarrow}(X) \blacktriangleleft_{\mathscr{O}} \widehat{f}(b) \ , \qquad \text{for every } X \subseteq \mathscr{O} \text{ and every } b \in \mathscr{O} \ . \qquad \triangle$$

The following theorem gives us some equivalent characterizations of the notion of continuity for functionals. The most important one is the equivalence between (1) and (4), because it allows us to understand the concept of continuity in $\mathscr{O}$ as an "inference rule" of the entailment system $(\mathscr{O}, \blacktriangleleft_{\mathscr{O}})$.

Theorem 3.10 (Equivalent characterizations of continuity)**.** *The following statements are equivalent.*

(1) *f is continuous in $\mathscr{O}$;*

(2) *$f^{\leftarrow}(X)^{\perp\perp} \subseteq f^{\leftarrow}(X^{\perp\perp})$, for every $X \subseteq \mathscr{O}$;*

(3) *$f^{\rightarrow}(X^{\perp\perp}) \subseteq f^{\rightarrow}(X)^{\perp\perp}$, for every $X \subseteq \mathscr{O}$;*

(4) *f preserves $\blacktriangleleft_{\mathscr{O}}$.*

Proof. (1) implies (2) : Let $X \subseteq \mathscr{O}$. As $X \subseteq X^{\perp\perp}$, we have $f^{\leftarrow}(X) \subseteq f^{\leftarrow}(X^{\perp\perp})$, by Lemma 3.5(2). Hence, $f^{\leftarrow}(X)^{\perp\perp} \subseteq f^{\leftarrow}(X^{\perp\perp})^{\perp\perp}$. Assume that f is continuous in $\mathscr{O}$. We have $f^{\leftarrow}(X^{\perp\perp})^{\perp\perp} = f^{\leftarrow}(X^{\perp\perp})$. Therefore, $f^{\leftarrow}(X)^{\perp\perp} \subseteq f^{\leftarrow}(X^{\perp\perp})$.

(2) implies (3) : Let $X \subseteq \mathscr{O}$. Let $Y \stackrel{\text{DEF}}{=} f^{\rightarrow}(X)$. By Lemma 3.5(4), we have $X \subseteq f^{\leftarrow}\big(f^{\rightarrow}(X)\big) = f^{\leftarrow}(Y)$. So, $X^{\perp\perp} \subseteq f^{\leftarrow}(Y)^{\perp\perp}$. Assume that (2) holds. We have $f^{\leftarrow}(Y)^{\perp\perp} \subseteq f^{\leftarrow}(Y^{\perp\perp})$. Thus, $X^{\perp\perp} \subseteq f^{\leftarrow}(Y^{\perp\perp})$. By Lemma 3.5(1), we obtain $f^{\rightarrow}(X^{\perp\perp}) \subseteq f^{\rightarrow}\big(f^{\leftarrow}(Y^{\perp\perp})\big)$. By Lemma 3.5(3), we have $f^{\rightarrow}\big(f^{\leftarrow}(Y^{\perp\perp})\big) \subseteq Y^{\perp\perp}$. Therefore, $f^{\rightarrow}(X^{\perp\perp}) \subseteq Y^{\perp\perp} = f^{\rightarrow}(X)^{\perp\perp}$.

(3) implies (4) : Let $X \subseteq \mathscr{O}$, and let $b \in \mathscr{O}$. Suppose that $X \blacktriangleleft_{\mathscr{O}} b$. By Theorem 2.13, this means $b \in X^{\perp\perp}$. So, we have $\widehat{f}(b) \in f^{\rightarrow}(X^{\perp\perp})$ by definition of image. Assume that (3) holds. Then, we have $f^{\rightarrow}(X^{\perp\perp}) \subseteq f^{\rightarrow}(X)^{\perp\perp}$. Thus, $\widehat{f}(b) \in f^{\rightarrow}(X)^{\perp\perp}$. By Theorem 2.13, this is equivalent to $f^{\rightarrow}(X) \blacktriangleleft_{\mathscr{O}} \widehat{f}(b)$.

(4) implies (1) : Let $X \subseteq \mathscr{O}$, and let $Y \stackrel{\text{DEF}}{=} f^{\leftarrow}(X^{\perp\perp})$. We have to show that $Y^{\perp\perp} = Y$. Clearly, $Y \subseteq Y^{\perp\perp}$. To show the converse, let $b \in Y^{\perp\perp}$. By Theorem 2.13, this means $Y \blacktriangleleft_{\mathscr{O}} b$. Assume that (4)

holds. Then, we have $f^{\rightarrow}(Y) \blacktriangleleft_{\mathscr{O}} \widehat{f}(b)$. By Theorem 2.13 again, this means $\widehat{f}(b) \in f^{\rightarrow}(Y)^{\perp\perp}$. Hence, we have $\widehat{f}(b) \in f^{\rightarrow}(Y)^{\perp\perp} = f^{\rightarrow}\big(f^{\leftarrow}(X^{\perp\perp})\big)^{\perp\perp} \subseteq (X^{\perp\perp})^{\perp\perp} = X^{\perp\perp}$, by using Lemma 3.5(3). Since $\widehat{f}(b) \in X^{\perp\perp}$, we have $b \in f^{\leftarrow}(X^{\perp\perp})$ by definition of pre–image. Since $f^{\leftarrow}(X^{\perp\perp}) = Y$, we conclude $Y^{\perp\perp} \subseteq Y$. This shows that f is continuous in $\mathscr{O}$. □

Analogously to what happens in the theory of closure spaces, in our setting we have that continuous functionals preserve the relation of specialization. Before showing this, we now prove a simple lemma.

Lemma 3.11. *Let $a \in \mathscr{O}$. Then, we have $\{\, \widehat{f}(a) \,\} = f^{\rightarrow}(\{\, a \,\})$.*

Proof. Let $b \in \mathscr{O}$. We have, by using the definition of image:

$$b \in \{\, \widehat{f}(a) \,\} \quad \text{iff} \quad b = \widehat{f}(a) \quad \text{iff} \quad b = \widehat{f}(c) \text{ for some } c \in \{\, a \,\} \quad \text{iff} \quad b \in f^{\rightarrow}(\{\, a \,\}) \ .$$ □

Corollary 3.12. *Suppose that f is continuous in $\mathscr{O}$. Then, f preserves the relation of specialization $\lhd_{\mathscr{O}}$.*

Proof. Suppose that f is continuous in $\mathscr{O}$. Then, by Theorem 3.10((1) implies (4)), the functional f preserves the relation of semantical consequence $\blacktriangleleft_{\mathscr{O}}$. Let a and b in $\mathscr{O}$, and suppose that $a \lhd_{\mathscr{O}} b$. By Proposition 2.9, $a \lhd_{\mathscr{O}} b$ is equivalent to $\{\, a \,\} \blacktriangleleft_{\mathscr{O}} b$. So, we obtain $f^{\rightarrow}(\{\, a \,\}) \blacktriangleleft_{\mathscr{O}} \widehat{f}(b)$ by preservation of $\blacktriangleleft_{\mathscr{O}}$. By Lemma 3.11, we have $f^{\rightarrow}(\{\, a \,\}) = \{\, \widehat{f}(a) \,\}$. Therefore, $\{\, \widehat{f}(a) \,\} \blacktriangleleft_{\mathscr{O}} \widehat{f}(b)$. The latter is equivalent to $\widehat{f}(a) \lhd_{\mathscr{O}} \widehat{f}(b)$, by using Proposition 2.9 again. □

3.3 Regularity

We now introduce the concept of regular functional. The reason for introducing this concept comes from ludics: in that setting every functional (in the sense of [3]) is regular (see Example 4.4).

Definition 3.13 (Regular functional)**.** We say that f is **regular** if the following condition holds:

$$\widehat{f}(p) \perp n \quad \text{if and only if} \quad p \perp \widehat{f}(n) \ , \qquad \text{for every } p \in \mathscr{P} \text{ and every } n \in \mathscr{N} \ .$$ △

We observe that our condition of regularity is analogous to the condition of *linearity* for maps in [9]. There, maps between *games* are said to be *linear* if they satisfies a similar condition (see Example 4.3). Unfortunately, the adjective "linear" is already present in the vocabulary of ludics [13, 3], and it denotes a property of designs which has nothing to do with the condition above. To avoid any sort of confusion, we decided to introduce a different terminology. We also remark that in the standard terminology for Chu spaces, the condition of linearity of [9] is commonly called *adjunction* condition.

We now show some equivalent characterization of the notion of regularity. Before doing this, it is convenient to introduce some now terminology.

Definition 3.14 (Various properties of functionals)**.** We say that:

- f is **semiregular in** $\mathscr{O}$ if $\widehat{f}(a) \perp b$ implies $a \perp \widehat{f}(b)$, for every $a \in \mathscr{O}$ and every $b \in \overline{\mathscr{O}}$;
- f is $\rightarrow\leftarrow$ **in** $\mathscr{O}$ if $f^{\rightarrow}(X)^{\perp} \subseteq f^{\leftarrow}(X^{\perp})$, for every $X \subseteq \mathscr{O}$;
- f is $\leftarrow\rightarrow$ **in** $\mathscr{O}$ if $f^{\leftarrow}(X^{\perp}) \subseteq f^{\rightarrow}(X)^{\perp}$, for every $X \subseteq \mathscr{O}$;
- f is **good in** $\mathscr{O}$ if $f^{\rightarrow}(X)^{\perp} = f^{\leftarrow}(X^{\perp})$, for every $X \subseteq \mathscr{O}$. △

Note that we have the following equivalences:

f is regular if and only if f is semiregular in $\mathscr{O}$ and semiregular in $\overline{\mathscr{O}}$;
f is good in $\mathscr{O}$ if and only if f is $\rightarrow\leftarrow$ in $\mathscr{O}$ and $\leftarrow\rightarrow$ in $\mathscr{O}$.

Proposition 3.15. *The following claims are equivalent.*

(1) f *is semiregular in* $\mathscr{O}$; (2) f *is* $\rightarrow\leftarrow$ *in* $\mathscr{O}$; (3) f *is* $\leftarrow\rightarrow$ *in* $\overline{\mathscr{O}}$.

Proof. (1) implies (2) : Let $X \subseteq \mathscr{O}$, and let $b \in \overline{\mathscr{O}}$. Assume that $b \in f^{\rightarrow}(X)^{\perp}$. Then, $c \perp b$ for every $c \in f^{\rightarrow}(X)$, by definition of orthogonal set. So, we have $\widehat{f}(a) \perp b$ for every $a \in X$ by definition of image. Assume (1). We obtain $a \perp \widehat{f}(b)$ for every $a \in X$. Hence, $\widehat{f}(b) \in X^{\perp}$ by definition of orthogonal set. To conclude, we get $b \in f^{\leftarrow}(X^{\perp})$ by definition of pre–image.

(2) implies (3) : Let $Y \in \overline{\mathscr{O}}$, and let $a \in \mathscr{O}$. Assume $a \in f^{\leftarrow}(Y^{\perp})$. We have $\widehat{f}(a) \in Y^{\perp}$ by definition of pre–image. Thus, $\{ \widehat{f}(a) \} \subseteq Y^{\perp}$. Hence, $f^{\rightarrow}(\{ a \}) \subseteq Y^{\perp}$ by Lemma 3.11. So, $Y^{\perp\perp} \subseteq f^{\rightarrow}(\{ a \})^{\perp}$. Since $Y \subseteq Y^{\perp\perp}$, we obtain $Y \subseteq f^{\rightarrow}(\{ a \})^{\perp}$. Assume (2). Since $\{ a \} \subseteq \mathscr{O}$, we have $f^{\rightarrow}(\{ a \})^{\perp} \subseteq f^{\leftarrow}(\{ a \}^{\perp})$. So, $Y \subseteq f^{\leftarrow}(\{ a \}^{\perp})$. By Lemma 3.5(1) and (3), we have $f^{\rightarrow}(Y) \subseteq f^{\rightarrow}\big(f^{\leftarrow}(\{ a \}^{\perp})\big) \subseteq \{ a \}^{\perp}$. Thus, $f^{\rightarrow}(Y) \subseteq \{ a \}^{\perp}$. Hence, $\{ a \}^{\perp\perp} \subseteq f^{\rightarrow}(Y)^{\perp}$. As $a \in \{ a \}^{\perp\perp}$, we conclude $a \in f^{\rightarrow}(Y)^{\perp}$.

(3) implies (1) : Let $a \in \mathscr{O}$, and let $b \in \overline{\mathscr{O}}$. Assume that $\widehat{f}(a) \perp b$ holds, i.e., $\widehat{f}(a) \in \{ b \}^{\perp}$. From this, we obtain $\{ \widehat{f}(a) \} \subseteq \{ b \}^{\perp}$. By Lemma 3.11, we have $f^{\rightarrow}(\{ a \}) \subseteq \{ b \}^{\perp}$. By Lemma 3.5(2) and (4), we have $\{ a \} \subseteq f^{\leftarrow}\big(f^{\rightarrow}(\{ a \})\big) \subseteq f^{\leftarrow}\big(\{ b \}^{\perp}\big)$. So, $\{ a \} \subseteq f^{\leftarrow}\big(\{ b \}^{\perp}\big)$. Assume (3). As $\{ b \} \subseteq \overline{\mathscr{O}}$, we obtain $f^{\leftarrow}\big(\{ b \}^{\perp}\big) \subseteq f^{\rightarrow}(\{ b \})^{\perp}$. Thus, $\{ a \} \subseteq f^{\rightarrow}(\{ b \})^{\perp}$. By Lemma 3.11, we have $f^{\rightarrow}(\{ b \}) = \{ \widehat{f}(b) \}$. Hence, $\{ a \} \subseteq \{ \widehat{f}(b) \}^{\perp}$. Thus, we get $a \in \{ \widehat{f}(b) \}^{\perp}$, i.e., $a \perp \widehat{f}(b)$. □

Theorem 3.16 (Equivalent characterizations of regularity)**.** *The following statements are equivalent.*

(i) f *is regular ;* (ii) f *is good in* $\mathscr{O}$; (iii) f *is good in* $\overline{\mathscr{O}}$.

Proof. (i) implies (ii) : Assume that f is regular. Then, as f is semiregular in $\mathscr{O}$, it follows that f is $\rightarrow\leftarrow$ in $\mathscr{O}$ by Proposition 3.15((1) implies (2)). As f is also semiregular in $\overline{\mathscr{O}}$, we have that f is $\leftarrow\rightarrow$ in $\mathscr{O}$ by Proposition 3.15((1) implies (3)). As a consequence of this, f is good in $\mathscr{O}$.

(ii) implies (iii) : Suppose that f is good in $\mathscr{O}$. Then, since f is $\rightarrow\leftarrow$ in $\mathscr{O}$, we have that f is $\leftarrow\rightarrow$ in $\overline{\mathscr{O}}$ by Proposition 3.15((2) implies (3)). Similarly, as f is $\leftarrow\rightarrow$ in $\mathscr{O}$, we have that f is $\rightarrow\leftarrow$ in $\overline{\mathscr{O}}$ by Proposition 3.15((3) implies (2)). Therefore, f is good in $\overline{\mathscr{O}}$.

(iii) implies (i) : Finally, assume that f is good in $\overline{\mathscr{O}}$. Then, as f is $\rightarrow\leftarrow$ in $\overline{\mathscr{O}}$, we have that f is semiregular in $\overline{\mathscr{O}}$, by Proposition 3.15((2) implies (1)). Analogously, since f is $\leftarrow\rightarrow$ in $\overline{\mathscr{O}}$, we have that f is semiregular in $\mathscr{O}$, by Proposition 3.15((3) implies (1)). This shows that f is regular. □

We now show the main result of this section: regular functionals are continuous and preserve the relation of semantical consequence.

Theorem 3.17 (Regularity)**.** *Suppose that f is a regular functional. Then,*

f is continuous in $\mathscr{P}$ and $\mathscr{N}$ and preserves the relations of semantical consequence $\blacktriangleleft_{\mathscr{P}}$ and $\blacktriangleleft_{\mathscr{N}}$.

Proof. Suppose that f is regular. Let $X \subseteq \mathscr{O}$, and let $Y \overset{\text{DEF}}{=} X^{\perp}$. By Theorem 3.16((i) implies (iii)), we have that f is good in $\overline{\mathscr{O}}$. So, $f^{\rightarrow}(Y)^{\perp} = f^{\leftarrow}(Y^{\perp})$. Thus, $f^{\leftarrow}(X^{\perp\perp})^{\perp\perp} = f^{\leftarrow}(Y^{\perp})^{\perp\perp} = (f^{\rightarrow}(Y)^{\perp})^{\perp\perp} = f^{\rightarrow}(Y)^{\perp} = f^{\leftarrow}(Y^{\perp}) = f^{\leftarrow}(X^{\perp\perp})$. This show that f is continuous in $\mathscr{O}$. Now, by Theorem 3.10((1) implies (4)) we obtain that f preserves the relation of semantical consequence $\blacktriangleleft_{\mathscr{O}}$. □

Finally, we observe that regularity is a concept which is stronger than continuity. Namely, we show that continuity in $\mathscr{P}$ *and* $\mathscr{N}$ does *not* implies regularity in general. See Example 3.18(a) below.

Example 3.18. In the same notation of Example 3.4 and Example 3.7, we have:

(a) The functional $\sharp$ is continuous in $\mathscr{P}_I$ *and* $\mathscr{N}_I$. However, as $\widehat{\sharp}((0,P)) = (1,P)$ and $\widehat{\sharp}((1,N)) = (1,N)$, we have $\widehat{\sharp}((0,P)) \perp_I (1,N)$ and $(0,P) \not\perp_I \widehat{\sharp}((1,N))$. Therefore, the functional $\sharp$ is *not* regular.

(b) The functional $\flat$ is not continuous in $\mathscr{P}_I$ and hence, by Theorem 3.17, it cannot be regular.

(c) The functional $\natural$ is continuous in $\mathscr{P}_I$ and $\mathscr{N}_I$. It is also regular, as we have

$$\widehat{\natural}((r,P)) \perp_I (s,N) \quad \text{iff} \quad (r,P) \perp_I (s,N) \quad \text{iff} \quad (r,P) \perp_I \widehat{\natural}((s,N)) \ ,$$

for every $(r,P) \in \mathscr{P}_I$ and every $(s,N) \in \mathscr{N}_I$. △

4 Examples: Boolean–Valued Games and Ludics

We now give two "abstract" examples of triad.

Example 4.1 (Boolean–valued games)**.** A *Boolean–valued game* [9] (or *Boolean–valued Chu space* [11]) can be presented as an ordered triple $\mathsf{Z} = (\mathsf{P}, \mathsf{O}, \mathsf{R})$, where P ("strategies", "points") and O ("co–strategies", "open sets") are sets, and R is a subset of $\mathsf{P} \times \mathsf{O}$, i.e., a relation from P to O. Given $x \in \mathsf{P}$ and $y \in \mathsf{O}$, we also write $x \,\mathsf{R}\, y$ for $(x,y) \in \mathsf{R}$ in the sequel.

Every Boolean–valued game with P and O disjoint is a triad in our sense. △

Example 4.2 (Ludics)**.** We now show that ludics fits into our general setting. For convenience, we consider ludics as formulated in [13]. In Subsection 2.1 of [3] the reader can find all the notions required to understand this example and Example 4.4. Let $\mathscr{A}$ be a signature in the sense of [13, 3]. Consider the triple $\mathbb{A} \overset{\text{DEF}}{=} (\mathscr{P}_{\mathbb{A}}, \mathscr{N}_{\mathbb{A}}, \perp_{\mathbb{A}})$ where:

- $\mathscr{P}_{\mathbb{A}}$ is the set of all linear, cut–free and positive designs with at most x_0 as free variable (i.e., the set of the positive atomic designs of [3] *augmented by* Ω) over the signature $\mathscr{A}$.
- $\mathscr{N}_{\mathbb{A}}$ is the set of all linear, cut–free and negative designs without free variables (i.e., the set of the negative atomic designs of [3]) over the signature $\mathscr{A}$.
- For $p \in \mathscr{P}_{\mathbb{A}}$ and $n \in \mathscr{N}_{\mathbb{A}}$, we set
$$p \perp_{\mathbb{A}} n \overset{\text{DEF}}{\iff} [\![\ p[n/x_0]\ ]\!] = \maltese \ ,$$
where by "$[\![\ \ ]\!]$" we denote the *normal form function* (see [13, 3]). This orthogonality relation corresponds to the original orthogonality relation of ludics.

Since in ludics the sets $\mathscr{P}_{\mathbb{A}}$ and $\mathscr{N}_{\mathbb{A}}$ are disjoint, the triple $\mathbb{A}$ is a triad in our sense. (In this example, Ω is a member of $\mathscr{P}_{\mathbb{A}}$. In ludics, this situation is usually not allowed. The only differences — w.r.t. more traditional presentations of ludics — are the following: (i) the set $\emptyset_{\mathscr{N}_{\mathbb{A}}}$ is a closed set in $\mathscr{N}_{\mathbb{A}}$, because $\emptyset_{\mathscr{N}_{\mathbb{A}}} = \{\ \Omega\ \}^{\perp_{\mathbb{A}}}$; (ii) the set $\mathscr{P}_{\mathbb{A}}$ is the unique closed set in $\mathscr{P}_{\mathbb{A}}$ which contains Ω, because $\{\ \Omega\ \}^{\perp_{\mathbb{A}}\perp_{\mathbb{A}}} = (\emptyset_{\mathscr{N}_{\mathbb{A}}})^{\perp_{\mathbb{A}}} = \mathscr{P}_{\mathbb{A}}$ and hence, $\{\ \Omega\ \} \subseteq X^{\perp_{\mathbb{A}}\perp_{\mathbb{A}}}$ implies $\mathscr{P}_{\mathbb{A}} = X^{\perp_{\mathbb{A}}\perp_{\mathbb{A}}}$.) △

We now give examples of collections of functionals for the triads given in the previous two examples.

Example 4.3 (Linear maps)**.** Let $\mathsf{Z} = (\mathsf{P}, \mathsf{O}, \mathsf{R})$ be a Boolean–valued game. A *linear map from* Z *to itself* is an ordered pair (p, o) of functions $\mathsf{p} : \mathsf{P} \longrightarrow \mathsf{P}$ and $\mathsf{o} : \mathsf{O} \longrightarrow \mathsf{O}$ such that

$$\mathsf{p}(x)\ \mathsf{R}\ y \qquad \text{if and only if} \qquad x\ \mathsf{R}\ \mathsf{o}(y) \ ,$$

for every $x \in \mathsf{P}$ and every $y \in \mathsf{O}$. (Here we are following the terminology introduced in [9]. We also mention that in the standard terminology for Chu spaces, linear maps are also called *Chu transforms*.)

Let $\mathsf{Z} = (\mathsf{P}, \mathsf{O}, \mathsf{R})$ be a Boolean–valued game with P and O disjoint. Recall that $\mathrm{dom}(\mathsf{Z}) = \mathsf{P} \cup \mathsf{O}$. Let $L \overset{\text{DEF}}{=} (\mathscr{F}_L, \widehat{\ }^L)$, where:

- $\mathscr{F}_L$ is *any subset* of the set of all linear maps from Z to itself.
- $\widehat{\ }^L$ maps each functional $(\mathsf{p}, \mathsf{o}) \in \mathscr{F}_L$ to the function $\widehat{(\mathsf{p}, \mathsf{o})}^L$ from $\mathrm{dom}(\mathsf{Z})$ to $\mathrm{dom}(\mathsf{Z})$ given by:

$$\widehat{(\mathsf{p},\mathsf{o})}^L(z) \stackrel{\text{DEF}}{=} \begin{cases} \mathsf{p}(z) & \text{if } z \in \mathsf{P} \\ \mathsf{o}(z) & \text{if } z \in \mathsf{O} \end{cases} , \quad \text{for } z \in \text{dom}(\mathsf{Z}) \ .$$

We claim that L *is a collection of functionals for* Z and that *all functionals in* $\mathscr{F}_L$ *are regular*. To prove these claims, let $(\mathsf{p},\mathsf{o}) \in \mathscr{F}_L$, $x \in \mathsf{P}$ and $y \in \mathsf{O}$.

By definition, we have $\widehat{(\mathsf{p},\mathsf{o})}^L(x) = \mathsf{p}(x) \in \mathsf{P}$ and $\widehat{(\mathsf{p},\mathsf{o})}^L(y) = \mathsf{o}(y) \in \mathsf{O}$, and this shows that preservation of polarity holds. As for regularity, we have that

$$\widehat{(\mathsf{p},\mathsf{o})}^L(x) \ \mathsf{R} \ y \quad \text{iff} \quad \mathsf{p}(x) \ \mathsf{R} \ y \quad \text{iff} \quad x \ \mathsf{R} \ \mathsf{o}(y) \quad \text{iff} \quad x \ \mathsf{R} \ \widehat{(\mathsf{p},\mathsf{o})}^L(y) \ . \qquad \triangle$$

Example 4.4 (Functionals in ludics)**.** Let $\mathbb{A}$ be the triad defined in Example 4.2. Recall that $\text{dom}(\mathbb{A}) = \mathscr{P}_{\mathbb{A}} \cup \mathscr{N}_{\mathbb{A}}$. Consider the pair $H = (\mathscr{F}_H, \widehat{\ }^H)$, where:

- $\mathscr{F}_H$ is *any subset* of the set of all linear, cut–free and negative designs with at most x_0 as free variable (i.e., the set of *functionals*, in the sense of Subsection 2.2 of [3]) over the signature $\mathscr{A}$.
- $\widehat{\ }^H$ maps each functional $g \in \mathscr{F}_H$ to the function $\widehat{g}^H$ from $\text{dom}(\mathbb{A})$ to $\text{dom}(\mathbb{A})$ given by:

$$\widehat{g}^H(a) \stackrel{\text{DEF}}{=} \begin{cases} [\![\, a[g/x_0] \,]\!] & \text{if } a \in \mathscr{P}_{\mathbb{A}} \\ [\![\, g[a/x_0] \,]\!] & \text{if } a \in \mathscr{N}_{\mathbb{A}} \end{cases} , \quad \text{for } a \in \text{dom}(\mathbb{A}) \ .$$

We now claim that H *is a collection of functionals for* $\mathbb{A}$ and that *all functionals in* $\mathscr{F}_H$ *are regular*. To show these claims, let $g \in \mathscr{F}_H$, $p \in \mathscr{P}_{\mathbb{A}}$ and $n \in \mathscr{P}_{\mathbb{A}}$.

Since $\mathscr{P}_{\mathbb{A}}$ and $\mathscr{N}_{\mathbb{A}}$ are disjoint, it follows from the definition of the normal form function that $\widehat{g}^H$ is a well–defined function from $\text{dom}(\mathbb{A})$ to itself. (It would be only a *partial* function if $\Omega \notin \mathscr{P}_{\mathbb{A}}$, as there are many $q \in \mathscr{P}_{\mathbb{A}}$ — different from Ω — and $h \in \mathscr{F}_H$ — in case $\mathscr{F}_H$ is the set of all functionals of [3] — such that $[\![\, q[h/x_0] \,]\!] = \Omega$. Exactly for this reason, we included Ω in $\mathscr{P}_{\mathbb{A}}$.) Furthermore, $\widehat{g}^H(p) \in \mathscr{P}_{\mathbb{A}}$ and $\widehat{g}^H(n) \in \mathscr{N}_{\mathbb{A}}$ again follow from the definition of normal form function. This shows that preservation of polarity holds. As for regularity, this property is a consequence of the *associativity* of normalization (see e.g., [3]). (This fact has also been observed, without proof, in [3]: see Lemma 2.4 and Equation (1) in Section 5 of [3].) Indeed, we have:

$$\begin{array}{rlr}
\widehat{g}^H(p) \perp_{\mathbb{A}} n & \text{iff} \quad [\![\, [\![\, p[g/x_0] \,]\!] \, [n/x_0] \,]\!] = \maltese & \text{(by definition of } \perp_{\mathbb{A}} \text{ and } \widehat{\ }^H) \\
& \text{iff} \quad [\![\, [\![\, p[g/x_0] \,]\!] [\, [\![\, n \,]\!]/x_0] \,]\!] = \maltese & \text{(because } n \text{ is cut–free)} \\
& \text{iff} \quad [\![\, p[g/x_0][n/x_0] \,]\!] = \maltese & \text{(by associativity)} \\
& \text{iff} \quad [\![\, p[g[n/x_0]/x_0] \,]\!] = \maltese & \text{(by substitution)} \\
& \text{iff} \quad [\![\, [\![\, p \,]\!] [\, [\![\, g[n/x_0] \,]\!]/x_0] \,]\!] = \maltese & \text{(by associativity)} \\
& \text{iff} \quad [\![\, p[\, [\![\, g[n/x_0] \,]\!]/x_0] \,]\!] = \maltese & \text{(because } p \text{ is cut–free)} \\
& \text{iff} \quad p \perp_{\mathbb{A}} \widehat{g}^H(n) \ . & \text{(by definition of } \perp_{\mathbb{A}} \text{ and } \widehat{\ }^H)
\end{array} \qquad \triangle$$

5 Conclusion

In this paper, we introduced the notion of triad in order to study, analyze, discover and rediscover some properties which hold in ludics from a more abstract and general perspective.

In particular, by applying of our abstract results to the concrete setting of ludics we arrive at the following conclusion.

Theorem 5.1 (Abstract results on triads and functionals applied to ludics)**.** *In the notation and terminology of Example* 4.2 *and Example* 4.4*, we have:*

- *The pairs* $(\mathscr{P}_{\mathbb{A}}, \blacktriangleleft_{\mathscr{P}_{\mathbb{A}}})$ *and* $(\mathscr{N}_{\mathbb{A}}, \blacktriangleleft_{\mathscr{N}_{\mathbb{A}}})$ *are entailment systems;*
- *Functionals in ludics are regular and therefore:*

- *Functionals in ludics are continuous in* $\mathscr{P}_{\mathbb{A}}$ *and* $\mathscr{N}_{\mathbb{A}}$*;*
- *Functionals in ludics preserve the relations of semantical consequence* $\blacktriangleleft_{\mathscr{P}_{\mathbb{A}}}$ *and* $\blacktriangleleft_{\mathscr{N}_{\mathbb{A}}}$ *and thus:*
  - *Functionals in ludics preserve the relations of specialization* $\lhd_{\mathscr{P}_{\mathbb{A}}}$ *and* $\lhd_{\mathscr{N}_{\mathbb{A}}}$*.*

Proof. By Example 4.2, Example 4.4, Theorem 2.16 , Theorem 3.17 and Corollary 3.12. □

To conclude the paper, we observe that from the point of view of ludics our paradigmatic vision of *designs as terms* is somehow limited because here terms only correspond to *atomic* designs. Even though terms capture the most important class of designs (in the opinion of the present author), in ludics there are plenty of non–atomic designs which do not fit into our framework. For future work, we plan to extend our setting in order to cope with them.

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