

On Star Expressions and Coalgebraic Completeness Theorems

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An open problem posed by Milner asks for a proof that a certain axiomatisation, which Milner showed is sound with respect to bisimilarity for regular expressions, is also complete. One of the main difficulties of the problem is the lack of a full *Kleene theorem*, since there are automata that can not be specified, up to bisimilarity, by an expression. Grabmayer and Fokkink (2020) characterise those automata that can be expressed by regular expressions without the constant 1, and use this characterisation to give a positive answer to Milner’s question for this subset of expressions. In this paper, we analyse Grabmayer and Fokkink’s proof of completeness from the perspective of universal coalgebra, and thereby give an abstract account of their proof method. We then compare this proof method to another approach to completeness proofs from coalgebraic language theory. This culminates in two abstract proof methods for completeness, what we call the *local* and *global* approaches, and a description of when one method can be used in place of the other.

1 Introduction

In 1984, Robin Milner gave a non-standard operational interpretation of regular expressions [28], viewing them as nondeterministic processes rather than regular languages. Comparing them for bisimilarity rather than language equivalence affects the semantics in two key ways. First, there are finite nondeterministic processes that do not behave like any regular expression up to bisimilarity (the problem of characterising those that do was solved first in [3]). This draws a stark contrast with the language semantics of regular expressions, where Kleene’s theorem gives a one-to-one correspondence between finite automata and expressions. Second, there are axioms in Salomaa’s complete axiomatisation of the algebra of regular expressions [31] that are unsound in the process interpretation. Milner offers a modified version of Salomaa’s axioms and shows that they are sound with respect to bisimilarity. Completeness is left as an open problem in [28], a full solution to which has yet to appear in the literature.

Several partial solutions to Milner’s problem are known. For instance, by omitting the constants 0 and 1 from the language and replacing the Kleene star with its binary version,¹ interpreted as iteration on its first argument before proceeding with the second, one obtains the calculus introduced in [5]. The corresponding axiomatisation was shown to be complete with respect to bisimilarity in [11, 12]. Later, soundness and completeness were shown for the fragments including either (or both) of 0 and 1, but with a perpetual loop operator $(-)^{\omega}$ in place of the star [10]. These partial solutions led up to the recent paper of Grabmayer and Fokkink [14], which solves the completeness problem for the fragment of 1-free regular expressions, and so subsumes much of the previous work on the problem. However, what

*Silva’s work was partially supported by ERC grant Autoprobe (grant agreement 101002697) and a Royal society Wolfson fellowship.

¹In fact, Kleene’s original star operation was binary. However, the binary star operation seems to have fallen out of fashion by the time [31] was written.

specifically interests us in their work is that it perfectly illustrates a subtle difficulty in completeness proofs.

Grabmayer and Fokkink’s approach consists of four key parts, as is the case for many related completeness proofs that go through automata. The first is the production of models from expressions through the operational semantics. The second is a sort of inverse to the first, a notion of *solution to a model* in the class of expressions. The third is the identification of a distinguished class of models that includes the semantics of the expressions, every member of which admits a *unique* solution modulo the axioms. This gives a one-to-one correspondence between models in the distinguished class and expressions modulo the axioms. The fourth is the ability to combine or reduce and compare models (without leaving the distinguished class), as well as their solutions.

The last part is subtler than the first three. In a classical proof such as [31], but also in more recent coalgebraic formulations (e.g., [34, 19]), the distinguished class typically consists of all finite (or locally finite) automata, and comparing automata consists of finding a bisimulation between them. Bisimulations between finite automata are trivially finite, so the fourth step is rarely worth mentioning in this situation. Here, the highest hurdle to clear seems to be the issue of proving that solutions are unique. This is in stark contrast with the setting of Grabmayer and Fokkink’s paper [14], where the class of models they consider are the so-called *LLEE-charts*. The necessity of identifying such a non-trivial class is caused by the above-mentioned issue that, up to bisimilarity, not all finite processes are characterised by regular expressions. For LLEE-charts, uniqueness of solutions is not a triviality, but also does not warrant a proof in the main body of [14].² In comparison, a great amount of ingenuity is involved in establishing the fourth of the moving parts mentioned above: The ability to reduce equivalent LLEE-charts to a common LLEE-chart.

Grabmayer and Fokkink’s solution to this problem is highly innovative and technical, and makes use of new tools carefully crafted for proving the compositionality result mentioned above. The abstract view we present here is no replacement for the detailed combinatorial arguments found in [14]. Instead, the intent of the present paper is to unpack its contents by situating them in the context of universal coalgebra. Universal coalgebra is a well-established general framework for state-based systems, subsuming constructs like bisimilarity and behaviour [30, 20]. We give a coalgebraic spin on some of the results of [14], strengthening some while simplifying the proofs of others:

- We show that solutions to automata are in one-to-one correspondence with coalgebra homomorphisms into the expressions modulo the axioms.
- We elucidate the four moving parts of Grabmayer and Fokkink’s completeness proof mentioned above and prove that they are sufficient in a general coalgebraic setting.
- We generalise the *connect-through-to* operation from [14] to a purely coalgebraic construction. We coin the term *rerouting* for this construction, and show that a prevalence of reroutings can be used to establish the fourth moving part of completeness proofs.
- Finally, we give a general account of a related approach to completeness proofs found in [19, 34, 27, 6] and show how the method used by Grabmayer and Fokkink can be restructured to fit this mould.

Overall, we use the structure of the completeness proof in [14] as a case study in completeness proof methods from coalgebra that do not rely on a one-to-one correspondence between expressions and all finite automata. This culminates in two abstract proof methods for completeness, what we call the *local*

²It appears in the extended version [15].

and *global* approaches, and a description of those situations in which the latter method can be used in place of the former.

The paper is organized as follows: In section 2, we introduce the 1-free fragment of regular expressions in parallel with its coalgebraic aspects. In section 3, we discuss the four moving parts of Grabmayer and Fokkink's completeness proof and show that they are sufficient in a general coalgebraic setting. In section 4, we give an alternative description of LLEE-charts and show how Grabmayer and Fokkink's technique for reducing LLEE-charts can be strengthened. It is in this section that we generalize their *connect-through-to* operation. Lastly, in section 5, we give a general account of a related approach to completeness proofs, found in [19, 34, 27, 6], and show how the method used by Grabmayer and Fokkink can be restructured to fit this mould.

2 Coalgebras and 1-free Star Expressions

For a fixed finite set A of *atomic actions*, the set of *1-free star expressions*, or *star expressions* for short, is generated by the BNF grammar

$$\text{SExp} \ni e, f ::= a \in A \mid 0 \mid e + f \mid ef \mid e * f$$

The expression $e * f$ denotes the regular expression $e^* f$ from [28], but we write $*$ as an infix to emphasize that it is a binary operation in this formalism, as in Kleene's seminal paper [24].

Operationally, each star expression specifies a labelled transition system with outputs in 2^A , called its *chart*. Following [14], a *chart* consists of a set of *states* X , a *transition relation* $\xrightarrow{(-)} \subseteq X \times A \times X$, an *output relation* $\Rightarrow \subseteq X \times A$, and a *start state* $x \in X$ from which every other state is reachable via a path of finite length. We impose the additional assumption that charts are *finitely branching*, ie. for any $x \in X$ and $a \in A$, $x \xrightarrow{a} y$ for finitely many $y \in X$.

Where $\mathcal{P}_\omega(X) = \{U \subseteq X \mid |U| < \omega\}$, a transition relation is equivalent to a function $\partial : X \rightarrow \mathcal{P}_\omega(X)^A$, an output relation is equivalent to a function $o : X \rightarrow 2^A$, and they can be given together by a function

$$\langle o, \partial \rangle : X \rightarrow 2^A \times \mathcal{P}_\omega(X)^A.$$

Since $\langle o, \partial \rangle$ says nothing about a start state, we call a pair $(X, \langle o, \partial \rangle)$ a *prechart*.

Precharts fit nicely into the framework of universal coalgebra. For an endofunctor G on the category **Sets** of sets and functions, a *G-coalgebra* is a pair (X, δ_X) consisting of a set X of *states* and a *structure map* $\delta_X : X \rightarrow GX$. Thus, if $P(X) = 2^A \times \mathcal{P}_\omega(X)^A$ and $P(f : X \rightarrow Y)(o, h)(a) = (o(a), f(h(a)))$, precharts are precisely P -coalgebras. Given a prechart $(X, \langle o, \partial \rangle)$, its transition and output relations can be recovered by writing $x \xrightarrow{a} y$ to denote $y \in \partial(x)(a)$ and $x \Rightarrow a$ to denote $o(x)(a) = 1$.

To obtain a chart from each star expression, Grabmayer and Fokkink begin by giving the set of star expressions SExp the structure of a prechart. The transitions of SExp are built inductively from the interpretations of expressions as processes: The constant 0 is deadlock, $a \in A$ is the process that performs the action a and then terminates, $e + f$ and ef are alternative and sequential composition respectively, and $e * f$ iterates e before executing f . Formally, the transitions and outputs of SExp are those derivable from the rules in Figure 1.

Given an expression $e \in \text{SExp}$, the *chart interpretation of e* is the smallest subset of SExp containing e and closed under the transition and output relations. The resulting prechart is denoted $\langle e \rangle$, and coincides with the smallest *subcoalgebra* of SExp containing e , ie. if $U \subseteq \text{SExp}$ contains e and $(U, \langle o_U, \partial_U \rangle)$ is a

$$\begin{array}{ccccc}
\frac{a \in A}{a \Rightarrow a} & \frac{e_i \Rightarrow a}{e_1 + e_2 \Rightarrow a} & \frac{e_i \xrightarrow{a} f}{e_1 + e_2 \xrightarrow{a} f} & \frac{e_1 \Rightarrow a}{e_1 e_2 \xrightarrow{a} e_2} & \frac{e_1 \xrightarrow{a} f}{e_1 e_2 \xrightarrow{a} f e_2} \\
\\
\frac{e_2 \Rightarrow a}{e_1 * e_2 \Rightarrow a} & \frac{e_2 \xrightarrow{a} f}{e_1 * e_2 \xrightarrow{a} f} & \frac{e_1 \xrightarrow{a} f}{e_1 * e_2 \xrightarrow{a} f(e_1 * e_2)} & \frac{e_1 \Rightarrow a}{e_1 * e_2 \xrightarrow{a} e_1 * e_2} &
\end{array}$$

Figure 1: The prechart $(\text{SExp}, \langle \partial_{\text{SExp}}, \partial_{\text{SExp}} \rangle)$.

P -coalgebra such that

$$\begin{array}{ccc}
U & \xleftarrow{\text{in}_U} & \text{SExp} \\
\langle \partial_U, \partial_U \rangle \downarrow & & \downarrow \langle \partial_{\text{SExp}}, \partial_{\text{SExp}} \rangle \\
P(U) & \xleftarrow{P(\text{in}_U)} & P(\text{SExp})
\end{array} \tag{1}$$

commutes, then $\langle e \rangle \subseteq U$. In general, a *chart* is a prechart of the form $\langle x \rangle$ for some $(X, \langle \partial, \partial \rangle)$ with $x \in X$.

In coalgebraic terminology, (1) states that the set U carries a P -coalgebra structure such that the inclusion of U into SExp is a P -coalgebra homomorphism. For a general endofunctor G on **Sets**, a G -coalgebra homomorphism from (X, δ_X) to (Y, δ_Y) is a map $h : X \rightarrow Y$ such that $\delta_Y \circ h = G(h) \circ \delta_X$. We write $X \cong Y$ if there is a bijective coalgebra homomorphism $X \rightarrow Y$, and say that X and Y are *isomorphic*. Homomorphisms coincide with the standard notion of *functional bisimulation*.

Lemma 2.1. *A function $h : X \rightarrow Y$ between precharts is a coalgebra homomorphism if and only if for any $x \in X$, $y' \in Y$, and $a \in A$, (i) $x \Rightarrow a$ if and only if $h(x) \Rightarrow a$, and (ii) $h(x) \xrightarrow{a} y'$ if and only if there is an $x' \in X$ such that $h(x') = y'$ and $x \xrightarrow{a} x'$.*

Bisimulation can also be captured coalgebraically. For a general endofunctor G , a *bisimulation* between two coalgebras (X, δ_X) and (Y, δ_Y) is a relation $R \subseteq X \times Y$ carrying a coalgebra structure (R, δ_R) such that the projection maps $\pi_1 : R \rightarrow X$ and $\pi_2 : R \rightarrow Y$ are G -coalgebra homomorphisms. It follows from Lemma 2.1 that a relation $R \subseteq X \times Y$ between precharts is a bisimulation if and only if for any $(x, y) \in R$ and $a \in A$, (i) $x \Rightarrow a$ if and only if $y \Rightarrow a$; (ii) if $x \xrightarrow{a} x'$, then there is a $y' \in Y$ such that $(x', y') \in R$ and $y \xrightarrow{a} y'$; and (iii) if $y \xrightarrow{a} y'$, then there is an $x' \in X$ such that $x \xrightarrow{a} x'$ and $(x', y') \in R$. Conversely, a map h is a coalgebra homomorphism if and only if its graph $\text{Gr}(h) = \{(x, h(x)) \mid x \in X\}$ is a bisimulation. If there is a bisimulation R relating $x \in X$ and $y \in Y$, we say x and y are *bisimilar* and write $x \underline{\Leftrightarrow} y$. Restricted to a single coalgebra, $\underline{\Leftrightarrow} \subseteq X \times X$ is a *bisimulation equivalence*, a bisimulation that is also an equivalence relation.

Within SExp , bisimilarity satisfies a number of intuitive equivalences, keeping in mind the interpretation of star expressions as processes. For instance, $0e \underline{\Leftrightarrow} 0$ and $e + f \underline{\Leftrightarrow} f + e$ for any $e, f \in \text{SExp}$. These are captured by two of the axioms suggested by Milner in [28], appearing as (B7) and (B1) in Grabmayer and Fokink's adaptation of Milner's axioms to star expressions summarised in Figure 2. We define \equiv to be the smallest congruence relation on SExp containing the pairs $e \equiv f$ found in Figure 2.

In general, an equivalence relation \equiv on the state space of a G -coalgebra E is *sound* with respect to bisimilarity if \equiv is a bisimulation equivalence, and *complete* with respect to bisimilarity if $e \equiv f$ whenever $e \underline{\Leftrightarrow} f$. The following theorem says that the axioms in Figure 2 build a sound equivalence relation on SExp .

Theorem 2.1. *The relation $\equiv \subseteq \text{SExp} \times \text{SExp}$ is a bisimulation equivalence on SExp .*

$$\begin{array}{ll}
\text{(B1)} & e_1 + e_2 \equiv e_2 + e_1 \\
\text{(B2)} & e_1 + (e_2 + e_3) \equiv (e_1 + e_2) + e_3 \\
\text{(B3)} & e_1 + e_1 \equiv e_1 \\
\text{(B4)} & (e_1 + e_2)e_3 \equiv e_1e_3 + e_2e_3 \\
\text{(B5)} & e_1(e_2e_3) \equiv (e_1e_2)e_3 \\
\text{(B6)} & e_1 + 0 \equiv e_1 \\
\text{(B7)} & 0e_1 \equiv 0 \\
\text{(BKS1)} & e_1 * e_2 \equiv e_1(e_1 * e_2) + e_2 \\
\text{(BKS2)} & (e_1 * e_2)e_3 \equiv e_1 * (e_2e_3) \\
\text{(RSP)} & \frac{e_3 \equiv e_1e_3 + e_2}{e_3 \equiv e_1 * e_2}
\end{array}$$

Figure 2: A sound and complete axiomatisation [14]. Here, $e_1, e_2, e_3 \in \text{SExp}$.

The role that bisimulation equivalences play in coalgebra is analogous to the role that congruences play in algebra. The kernel $\ker(h) = \{(x, x') \in X \times X \mid h(x) = h(x')\}$ of any coalgebra homomorphism h is a bisimulation equivalence,³ and every bisimulation equivalence R is the kernel of some coalgebra homomorphism $X \rightarrow X/R$ [30]. By Theorem 2.1, the set SExp/\equiv of star expressions modulo provable equivalence is itself a P -coalgebra, and the quotient map $[-]_{\equiv} : \text{SExp} \rightarrow \text{SExp}/\equiv$ is a coalgebra homomorphism.

2.1 Linear Systems and Solutions

Starting with an expression $e \in \text{SExp}$, obtaining a prechart X with a state $x \in X$ such that $e \underline{\leftrightarrow} x$ is only a matter of computing $\langle e \rangle$. However, going from a prechart X and a state $x \in X$ to an expression $e \in \text{SExp}$ such that $e \underline{\leftrightarrow} x$ is more difficult (and in fact, is not always possible). The following theorem hints at a method for doing so.

Theorem 2.2. *Let $e \in \text{SExp}$. Then $e \equiv \sum_{x \Rightarrow a} a + \sum_{x \xrightarrow{a} f} af$, where $\sum_{i=1}^n e_i = e_1 + (\sum_{i=2}^n e_i)$.*⁴

Given a finite prechart $(X, \langle o, \partial \rangle)$, its corresponding *linear system* is the set of equations

$$x = \sum_{x \Rightarrow a} a + \sum_{x \xrightarrow{a} x'} ax' \quad (2)$$

indexed by X , where we are thinking of each $x \in X$ as an indeterminate. A *solution* to the linear system associated with X is a map $s : X \rightarrow \text{SExp}$ such that

$$s(x) \equiv \sum_{x \Rightarrow a} a + \sum_{x \xrightarrow{a} x'} a s(x') \quad (3)$$

for all $x \in X$. Composing a solution s with the homomorphic image homomorphism $[-]_{\equiv} : \text{SExp} \rightarrow \text{SExp}/\equiv$, (3) becomes the equation

$$[s(x)]_{\equiv} = \sum_{x \Rightarrow a} a + \sum_{x \xrightarrow{a} x'} a [s(x')]_{\equiv} \quad (4)$$

It follows from (4) that $\{(x, [s(x)]_{\equiv}) \mid x \in X\}$ is a bisimulation between X and SExp/\equiv . Since this is the graph of the map $[-]_{\equiv} \circ s$, if $s : X \rightarrow \text{SExp}$ is a solution, then $[-]_{\equiv} \circ s : X \rightarrow \text{SExp}/\equiv$ is a coalgebra

³Actually, this is only true if G preserves weak pullbacks. This is a common assumption, however, and holds for each of the concrete functors we consider here.

⁴Here, the generalised sum on the left is well-defined up to the commutativity and associativity of $+$ assumed in Figure 2.

homomorphism. Conversely, if $[-]_{\equiv} \circ s$ is a homomorphism, then (4) holds. As (3) and (4) are equivalent, we obtain:

Lemma 2.2. *A map $s : X \rightarrow \text{SExp}$ is a solution iff $[-]_{\equiv} \circ s : X \rightarrow \text{SExp}/\equiv$ is a coalgebra homomorphism.*

We often identify solutions with their corresponding homomorphisms into SExp/\equiv .

3 A Local Approach

In the previous section, we observed that the axioms in Figure 2 are sound with respect to bisimilarity, and that *solutions* from [14] coincide with coalgebra homomorphisms into SExp/\equiv . In *loc. cit.*, Grabmayer and Fokkink show that the axiomatisation in Figure 2 is *complete* with respect to bisimilarity: that $e \equiv f$ whenever $e \xleftrightarrow{\quad} f$, for any $e, f \in \text{SExp}$. Next, we give an abstract description of Grabmayer and Fokkink's approach to proving soundness and completeness, which we call the *local approach*, and compare it to an approach found in classical automata theory. Grabmayer and Fokkink's approach can essentially be organized into four steps.

Step 1 is to show that the provable equivalence relation \equiv is a bisimulation equivalence. This is the content of Theorem 2.1 from section 2, and establishes soundness.

Step 2 is to identify a class \mathcal{C} of precharts such that for any $e \in \text{SExp}$, $\langle e \rangle \in \mathcal{C}$.

Step 3 is to show that for any $X \in \mathcal{C}$, there is a unique homomorphism $X \rightarrow \text{SExp}/\equiv$. By Lemma 2.2, homomorphisms into SExp/\equiv are identifiable with solutions, so this is the same as saying that precharts in \mathcal{C} admit unique solutions.

Step 4 is to show that \mathcal{C} is closed under binary coproducts and bisimulation collapses. That is, for any $X, Y \in \mathcal{C}$, we find $X \sqcup Y \in \mathcal{C}$ and $X/\xleftrightarrow{\quad} \in \mathcal{C}$ as well.

It should be noted that Grabmayer and Fokkink never explicitly show that their class \mathcal{C} is closed under binary coproducts, due to their focus being on charts, which do not have this property. Thus, the four steps above are a coalgebraic rephrasing of their approach that requires the introduction of coproducts. However, the coalgebraic analogue of Grabmayer and Fokkink's distinguished class of models is easily seen to be closed under binary coproducts, as we will see in section 4.

The four steps above are sufficient for showing soundness and completeness of an axiomatisation of bisimilarity in general. In fact, we can even replace step 4 with a weaker version:

Step 4 is to show that \mathcal{C} is *collapsible*, ie. for any $X, Y \in \mathcal{C}$ and any $x \in X$ and $y \in Y$ such that $x \xleftrightarrow{\quad} y$, there is a $Z \in \mathcal{C}$ and a pair of homomorphisms $p : X \rightarrow Z$ and $q : Y \rightarrow Z$ such that $p(x) = q(y)$.

Steps 1-4 constitute the *local approach*, leading to soundness and completeness via the following theorem.

Theorem 3.1. *Let \equiv be a bisimulation equivalence on a fixed G -coalgebra E , and \mathcal{C} be a collapsible class of G -coalgebras containing $\langle e \rangle$ for each $e \in E$. If there is exactly one homomorphism $X \rightarrow E/\equiv$ for every G -coalgebra $X \in \mathcal{C}$, then $e \equiv f$ if and only if $e \xleftrightarrow{\quad} f$ for any $e, f \in E$.*

A class of G -coalgebras that is closed under binary coproducts and bisimulation collapses is collapsible: If X and Y are in the class, and $x \xleftrightarrow{\quad} y$ for some $x \in X$ and $y \in Y$, let $Z = (X \sqcup Y)/\xleftrightarrow{\quad}$ and take $p = [-]_{\xleftrightarrow{\quad}} \circ \text{in}_X$ and $q = [-]_{\xleftrightarrow{\quad}} \circ \text{in}_Y$. Here, $\text{in}_X : X \hookrightarrow X \sqcup Y$ is the inclusion of X into the coproduct $X \sqcup Y$, and similarly for in_Y , and $[-]_{\xleftrightarrow{\quad}} : X \sqcup Y \rightarrow Z$ is the bisimulation collapse homomorphism. Because $x \xleftrightarrow{\quad} y$ in $X \sqcup Y$, $p(x) = q(y)$, from which collapsibility follows.

Steps 1 through 3 of the local approach should be familiar to readers acquainted with equational axiomatisations in classical automata theory. Some aspects of the soundness and completeness theorems of regular algebra can be seen to trace each of the first three steps above. Since two states of a deterministic automaton are bisimilar if and only if they recognize the same language, one could reasonably expect that the local approach to proving soundness and completeness, of any one of the existing axiomatisations of language equivalence for regular expressions, should be successful.

Kleene proved in [24] that a language is regular if and only if it is recognized by a state in a deterministic finite automaton (or DFA). This motivates choosing DFAs as the distinguished class of coalgebras. This trivializes step 4, as finiteness is preserved under binary coproducts and bisimulation collapses. Thus, the central difficulties surpassed in the first completeness proofs of regular algebra lay in step 3 [31, 25].

Although all four steps had been taken, neither of the completeness proofs in [31, 25] conclude with an observation like Theorem 3.1. Instead, bisimulations between DFAs are treated as DFAs, and solutions are pulled back across projection homomorphisms. As Grabmayer and Fokkink point out in [14], this use of bisimulations does not translate to the case of 1-free regular expressions. This is due to the fact that the distinguished class \mathcal{C} , consisting of the precharts for which they could prove the existence and uniqueness of solutions, does not include every bisimulation between precharts in \mathcal{C} . This is where the need for collapsibility becomes apparent.

Comparing the difficulties in Salomaa's approach with the difficulties in Grabmayer and Fokkink's approach reveals a crucial aspect of discovering soundness and completeness theorems in general: When choosing a distinguished class of models \mathcal{C} , there is a balance to be kept between the difficulty of finding solutions to models in \mathcal{C} and proving their uniqueness on the one hand, and ensuring desirable structural qualities of \mathcal{C} on the other. Salomaa circumvented the difficulties of steps 2 and 4 by including every finite automaton in his distinguished class, but this made step 3 a difficult problem. Grabmayer and Fokkink were able to take step 3 and prove uniqueness of solutions for precharts in their distinguished class with relative ease, but step 4 took great ingenuity.

4 Layered Loop Existence and Elimination

Grabmayer and Fokkink prove that Milner's axioms are complete with respect to bisimilarity for the 1-free fragment by modelling star expressions with charts. They single out a specific class of charts, namely those satisfying their *layered loop existence and elimination property*, or *LLEE-property* for short. Roughly, a prechart is said to satisfy the LLEE-property if there is a labelling of its transitions by natural numbers such that an edge descending into a loop accompanies a descent in natural number labellings, and such that no successful termination can occur mid-loop. The existence of such a labelling ensures that loops are never mutually nested, and requires threads to finish every task in a loop before termination. Every chart interpretation of a star expression has the LLEE-property, and every prechart with the LLEE-property admits a unique solution.

In this section, we discuss a coalgebraic version of Grabmayer and Fokkink's distinguished class of models, the class of so-called *LLEE-precharts*, and review the proof of its collapsibility. As it so happens, a slight variation of Grabmayer and Fokkink's proof of collapsibility shows something much stronger: That the class of finite *LLEE-precharts* is closed under arbitrary homomorphic images. The main tool used in the proof of collapsibility is the *connect-through-to* operation, which preserves bisimilarity while it identifies bisimilar states. We generalize Grabmayer and Fokkink's *connect-through-to* operation, and show that it can be used to establish closure under homomorphic images in general.

4.1 Well-layeredness

We give an equivalent but different characterisation of LLEE-precharts that makes them easier to describe coalgebraically, and rename the property *well-layeredness*. While we recall all of the necessary details, much of what is covered here can be found in more detail in [14].

A simple but interesting observation about well-layeredness is that it makes no reference to the action labels of a prechart. In other words, well-layeredness is really a property of *transition systems with output* (*transition systems* for short), coalgebras for the endofunctor $2 \times \mathcal{P}_\omega(-)$.

A well-layered transition system is a transition system that carries a particular labelling, called an *entry/body labelling*, that satisfies a few extra conditions. Here, an *entry/body labelling* of a transition system $(X, \langle \underline{o}, \underline{\partial} \rangle)$ is a coalgebra $(X, \langle \underline{o}, \underline{\partial}^\bullet \rangle)$ for the endofunctor $2 \times \mathcal{P}_\omega(\{e, b\} \times (-))$ such that $\underline{\partial}(x) = \pi_2(\underline{\partial}^\bullet(x))$ for any $x \in X$. We typically denote an entry/body labelling of a transition system \underline{X} with X^\bullet .

To state the extra conditions on the labellings that define well-layeredness, we need some notation. Given an entry/body labelling $X^\bullet = (X, \langle \underline{o}, \underline{\partial}^\bullet \rangle)$, the following glyphs are used to denote its various transition types: For any $x, y \in X$, $x \Rightarrow$ means $\underline{o}(x) = 1$, $x \rightarrow_e y$ means $(e, y) \in \underline{\partial}^\bullet(x)$, and $x \rightarrow_b y$ means $(b, y) \in \underline{\partial}^\bullet(x)$. Furthermore, $x \curvearrowright y$ means

$$(\exists v_1, \dots, v_k) x \rightarrow_e v_1 \rightarrow_b \dots \rightarrow_b v_k \rightarrow_b y, x \notin \{v_1, \dots, v_l, y\}$$

and $y \subset x$ means

$$(\exists v_1, \dots, v_k) x \rightarrow_e v_1 \rightarrow_b \dots \rightarrow_b v_k \rightarrow_b x, y \in \{v_1, \dots, v_l\}, x \notin \{v_1, \dots, v_l\}.$$

Transitions of the form $x \rightarrow_e y$ and $x \rightarrow_b y$ are called *entry* and *body* transitions, respectively. We enclose a relation in $(-)^+$ or $(-)^*$ to denote its transitive or transitive-reflexive closure respectively.

Definition 4.1. A *layering witness* is an entry/body labelling X^\bullet that is

1. *locally finite*, meaning that $\langle x \rangle$ is finite for all $x \in X$;
2. *flat*, meaning that $x \rightarrow_e y$ implies $\neg(x \rightarrow_b y)$ for all $x, y \in X$;
3. *fully specified*, meaning that for all $x, y \in X$,
 - (a) $\neg(x \rightarrow_b^+ x)$ and
 - (b) if $x \rightarrow_e y$ for some $y \neq x$, then $y \rightarrow^+ x$.
4. *layered*, meaning that the directed graph (X, \curvearrowright) is acyclic; and
5. *goto-free*, meaning that $x \curvearrowright y$ implies $\neg(y \Rightarrow)$, for all $x, y \in X$.

A transition system is *well-layered* if it is the underlying transition system of a layering witness.

Every prechart $(X, \langle o, \partial \rangle)$ also comes with an underlying transition system $\underline{X} = (X, \langle \underline{o}, \underline{\partial} \rangle)$, given by

$$\underline{o} = \bigvee_{a \in A} o(a) \quad \underline{\partial}(x) = \bigcup_{a \in A} \partial(x)(a)$$

for any $x \in X$. A layering witness *for a prechart* is a layering witness for its underlying transition system, and a prechart is said to be *well-layered* if it has a layering witness.

Remark 4.1. Every bisimulation R between precharts X and Y carries an underlying bisimulation \underline{R} between the transition systems \underline{X} and \underline{Y} . However, not every bisimulation between \underline{X} and \underline{Y} lifts to a bisimulation between X and Y : Such relations ignore action labels in general, while bisimulations between precharts do not.

Remark 4.2. It can be checked that the underlying transition system of a locally finite prechart X is well-layered if and only if X has an LLEE-witness [14]. To obtain an LLEE-witness from a layering witness, replace each $x \rightarrow_e y$ with a weighted transition $x \xrightarrow{[|x|_{en}]} y$, where

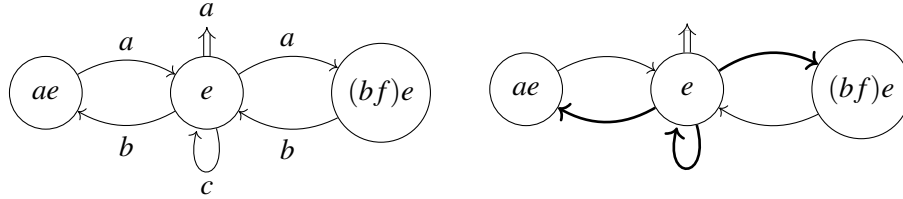
$$|x|_{en} = \max\{m \in \mathbb{N} \mid (\exists x_1, \dots, x_m) x \rightsquigarrow x_1 \rightsquigarrow \dots \rightsquigarrow x_m \text{ s.t. } x \neq x_i \neq x_j \text{ for } i \neq j\}$$

and each $x \rightarrow_b y$ with $x \xrightarrow{[0]} y$. This is a well-defined translation because we have assumed that $\langle x \rangle$ is finite and $(\langle x \rangle, \rightsquigarrow)$ is acyclic. To obtain a layering witness from an LLEE-witness, replace each $x \xrightarrow{a}_{[n]} y$ by $x \rightarrow_e y$ if $n > 0$ and $y \rightarrow^+ x$, or by $x \rightarrow_b y$ otherwise. This entry/body labelling is flat because every resulting entry transition appears in a minimal cycle, and every minimal cycle contains precisely one entry transition by (W1) and (W2)(b) from [14]. Each of the remaining conditions are by construction, or are a direct consequence of the LLEE-witness conditions. For example, full specification follows from (a) local finiteness and (W1) in *loc. cit.*, and (b) our assumption that $x \rightarrow_e y$ implies $y \rightarrow^+ x$ for all x, y (see [33, Proposition C.1]).

By restricting a layering witness X^\bullet to a subcoalgebra U of X , one obtains a layering witness U^\bullet for U . It follows from this observation and the lemma below that $\langle e \rangle$ is well-layered for any 1-free star expression e .

Lemma 4.1. *The prechart SExp is well-layered.*

This completes Step 2 from section 3: Where \mathcal{C} is the set of finite well-layered precharts, we find $\langle e \rangle \in \mathcal{C}$ for any $e \in \text{SExp}$. For a concrete example, let $f = (ab) * (ba)$ and $e = f * a$, where $a, b, c \in A$. The prechart $\langle e \rangle$ is depicted below along with a layering witness.



It is important to note that not every well-layered prechart has a unique layering witness. The prechart $\langle (aa) * 0 \rangle$, for example, has exactly two.

4.2 Existence and uniqueness of solutions

Steps 1 and 2 consisted of showing that \equiv is a bisimulation equivalence and $\langle e \rangle$ is a well-layered prechart for each $e \in \text{SExp}$. To complete step 3 of the local approach, Grabmayer and Fokink give an explicit description of a solution to a chart $X = \langle v \rangle$ with layering witness X^\bullet , and show that it is equivalent to any other solution to X . For any $x \in X$, let

$$s_X(x) \equiv \left(\sum_{x \xrightarrow{a}_e x} a + \sum_{\substack{x \xrightarrow{a}_e y \\ x \neq y}} a t_X(y, x) \right) * \left(\sum_{x \Rightarrow a} a + \sum_{x \xrightarrow{a}_b y} a s_X(y) \right) \quad (5)$$

where

$$t_X(x, z) \equiv \left(\sum_{x \xrightarrow{a} e^x} a + \sum_{\substack{x \xrightarrow{a} e^y \\ x \neq y}} a t_X(y, x) \right) * \left(\sum_{x \xrightarrow{a}} a + \sum_{x \xrightarrow{a} b^y} a t_X(y, z) \right)$$

Both functions are well-defined by induction on the pair $(|x|_{en}, |x|_b)$, where $|x|_{en}$ is given in Remark 4.2 and $|x|_b = \max\{m \mid (\exists x_1, \dots, x_m) x \rightarrow_b x_1 \rightarrow_b \dots \rightarrow_b x_m\}$, with respect to the lexicographical ordering on $\mathbb{N} \times \mathbb{N}$.

It is shown in [14] that for any solution $s : X \rightarrow \text{SExp}$, $s(x) \equiv s_X(x)$ for all $x \in X$. This proves that well-layered charts have unique solutions. The same result readily extends to the prechart case: If X is an arbitrary well-layered prechart and $x \in X$, then $s_{\langle x \rangle}(x)$ is a well-defined expression, as $\langle x \rangle$ is a subcoalgebra of X and is therefore also well-layered. By uniqueness of solutions for charts, the map $s_X : X \rightarrow \text{SExp}$ given by $s_X(x) = s_{\langle x \rangle}(x)$ is a well-defined solution to X . Furthermore, since every solution to X restricts to a solution to $\langle x \rangle$ for each $x \in X$, s_X is the unique solution to X .

Lemma 4.2. *If X is a well-layered prechart, then there is a unique solution $s_X : X \rightarrow \text{SExp} / \equiv$ to X .*

4.3 Reroutings and Closure under homomorphic images

The crucial step in Grabmayer and Fokkink’s proof is step 4 of the local approach, showing that the bisimulation collapse of a finite well-layered chart is also well-layered. This is done in a step-by-step procedure that exhaustively identifies bisimilar states. In each step, a specially chosen pair (w_1, w_2) of distinct bisimilar states is reduced to the singleton w_2 by rerouting all of w_1 ’s incoming transitions to w_2 and then deleting w_1 .

Formally, given a prechart $(X, \langle o, \partial \rangle)$ and a pair (x_1, x_2) of distinct states of X , the *connect- x_1 -through-to- x_2 construction* returns the prechart $X[x_2/x_1] = (X - \{x_1\}, \partial[x_2/x_1])$, where

$$\partial[x_2/x_1](x)(a) = \begin{cases} \{x_2\} \cup (\partial(x)(a) - \{x_1\}) & \text{if } x_1 \in \partial(x)(a), \\ \partial(x)(a) & \text{otherwise} \end{cases}$$

The connect- x_1 -through-to- x_2 operation preserves bisimilarity, in the sense that if R is a bisimulation equivalence on X , then $R \cap (X \times X - \{x_1\})$ is a bisimulation between X and $X[x_2/x_1]$. This has the following consequence: If the only pairs of distinct states in R are (x_1, x_2) and (x_2, x_1) , then $R_1 = R \cap (X \times X - \{x_1\})$ is the graph of a homomorphism between X and $X[x_2/x_1]$, and consequently $X[x_2/x_1] \cong X/R$. Otherwise, $R|_{X - \{x_1\}} = R \cap (X - \{x_1\})^2$ is a bisimulation equivalence containing a pair of distinct states (x_3, x_4) . If (x_3, x_4) and (x_4, x_3) are the only such pairs, then $R_2 = R|_{X - \{x_1\}} \cap (X - \{x_1\} \times X - \{x_1, x_3\})$ is the graph of a homomorphism $X[x_2/x_1] \rightarrow X[x_2/x_1][x_4/x_3]$, and therefore $R_1 \circledast R_2 = R \cap (X \times X - \{x_1, x_3\})$ is the graph of a homomorphism $X \rightarrow X[x_2/x_1][x_4/x_3]$, where \circledast denotes relational composition, and $X[x_2/x_1][x_4/x_3] \cong X/R$. Generally, if X is finite, then iterating this construction yields the graph $R_1 \circledast \dots \circledast R_m$ (for some m) of the homomorphism $X \rightarrow X/R$ (up to \cong). Taking $R = \leftrightarrow$, the bisimulation collapse of a finite prechart X can be computed by iterating the connect-through-to operation until no distinct pairs of bisimilar states are left.

For an arbitrary well-layered prechart X and a pair of distinct bisimilar states (x_1, x_2) , $X[x_2/x_1]$ may not be well-layered. An example discussed in [14] is the connect-through-to construction depicted in Figure 3, which takes a well-layered chart to a chart that does not admit a layering witness.

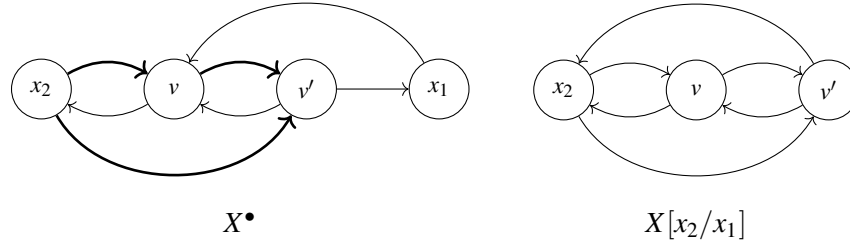


Figure 3: A bisimulation rerouting that does not preserve well-nestedness.

However, if (x_1, x_2) is chosen carefully, then the connect- x_1 -through-to- x_2 operation preserves well-layeredness. Where X^\bullet is a layering witness for X , it is shown in [14] that $X[w_2/w_1]$ is well-layered for any pair (w_1, w_2) of distinct bisimilar states satisfying one of the following three conditions in X^\bullet :

- (C1) $\neg(w_2 \rightarrow^* w_1)$; and if $(\exists x) x \sim w_1$, then $\neg(\exists y)(w_2 \rightarrow^* y \Rightarrow)$
- (C2) $w_2 \triangleright^+ w_1$
- (C3) $\neg(w_2 \rightarrow_b^* w_1)$; and $(\exists x) w_1 \triangleright x$ and $w_2 \triangleright^+ x$ and if $w_1 \triangleright y$, then $x \triangleright y$

As Grabmayer and Fokkink point out in *loc. cit.*, if X^\bullet is a layering witness for a finite prechart X such that $X \not\cong X/\underline{\leftrightarrow}$, then there is a pair (w_1, w_2) of distinct bisimilar states satisfying one of (C1)-(C3) in X^\bullet . A slight variation on their proof yields the following.

Lemma 4.3. *Let X^\bullet be a layering witness for X , and R be a bisimulation equivalence on X . If R is non-trivial, ie. $X \not\cong X/R$, then there is a pair $(w_1, w_2) \in R$ of distinct states satisfying one of (C1)-(C3).*

By iterating the connect-through-to construction on the pairs guaranteed to exist in Lemma 4.3, every homomorphic image of a finite well-layered prechart is seen to be well-layered.

Theorem 4.1. *Let X be a finite well-layered prechart, and R be a bisimulation equivalence on X . Then X/R is a well-layered prechart as well.*

This completes step 4 of Grabmayer and Fokkink's proof that Milner's axioms are complete with respect to bisimilarity for the 1-free fragment of regular expressions.

Theorem 4.2. *For any $e, f \in \text{SExp}$, if $e \underline{\leftrightarrow} f$, then $e \equiv f$.*

Proof. Let \mathcal{C} be the set of finite well-layered precharts. Lemma 4.1 tells us that $\langle e \rangle \in \mathcal{C}$ for any $e \in \text{SExp}$, and Theorem 4.2 tells us that precharts in \mathcal{C} admit unique solutions.

By Theorem 3.1, it suffices to show that \mathcal{C} is collapsible. We have already seen that a class of coalgebras closed under binary coproducts and homomorphic images is collapsible, so by Theorem 4.1 it suffices to show that \mathcal{C} is closed under binary coproducts. To this end, observe that if X^\bullet and Y^\bullet are layering witnesses for X and Y respectively, then $X^\bullet \sqcup Y^\bullet$ is a layering witness for $X \sqcup Y$. \square

4.4 Reroutings, in general

Interestingly, the connect-through-to construction can be performed on general G -coalgebras. For a fixed prechart X and a pair of states $x_1, x_2 \in X$, if $i : X - \{x_1\} \hookrightarrow X$ is the inclusion map and $j : X \twoheadrightarrow X - \{x_1\}$ is the map identifying x_2 with x_1 , then the prechart $X[x_2/x_1] = (X - \{x_1\}, \langle o, \partial[x_2/x_1] \rangle)$ obtained from the connect- x_1 -through-to- x_2 construction is given precisely by

$$\partial[x_2/x_1](x)(a) = j(\partial(x)(a)) = \mathcal{P}_\omega(j) \circ \partial \circ i(x)(a).$$

In other words, the following diagram commutes.

$$\begin{array}{ccc}
 X - \{x_1\} & \xleftarrow{i} & X \\
 \downarrow \langle o, \partial[x_2/x_1] \rangle & & \downarrow \langle o, \partial \rangle \\
 2 \times \mathcal{P}_\omega(X - \{x_1\}) & \xleftarrow{\text{id}_2 \times \mathcal{P}_\omega(j)^A} & 2 \times \mathcal{P}_\omega(X)^A
 \end{array}$$

Notice that (i, j) is a *splitting*, meaning $j \circ i = \text{id}_{X - \{x_1\}}$. In general, given any G -coalgebra X and any splitting (i, j) with $i : U \hookrightarrow X$, we define

$$X[i, j] = (U, G(j) \circ d \circ i),$$

and call $X[i, j]$ the *rerouting* by (i, j) of X . As is the case for the connect-through-to operation, reroutings that identify bisimilar states preserve bisimilarity.

Lemma 4.4. *Let R be a bisimulation equivalence on a prechart X , and (i, j) be a splitting with $i : U \hookrightarrow X$ and $\ker(j) \subseteq R$. Then $Q = R \cap (X \times U)$ is a bisimulation.*

A rerouting $X[i, j]$ is called an *R -rerouting* if R is a bisimulation and $\ker(j) \subseteq R$. In case $R = \leftrightarrow$, we will use the phrase *bisimulation rerouting* instead.

A common assumption in universal coalgebra is that the endofunctor G under consideration preserves weak pullbacks [30, 16]. This property is sufficient for ensuring that the relational composition of two bisimulations is again a bisimulation. In general, if R is an equivalence on X , and $Z \subseteq Y \subseteq X$, then $R \cap (X \times Z) = (R \cap (X \times Y)) \circledast (R \cap (Y \times Z))$ and $R \cap (Y \times Y)$ is an equivalence relation. Thus, by iterating Lemma 4.4, we obtain the following generalisation of Theorem 4.1.

Theorem 4.3. *Let G be an endofunctor that weakly preserves pullbacks, and \mathcal{C} be a class of finite G -coalgebras closed under isomorphism. Then the following two statements hold.*

1. *If for any $X \in \mathcal{C}$ and any nontrivial bisimulation equivalence $R \subseteq X \times X$ there is a nontrivial R -rerouting U of X such that $U \in \mathcal{C}$, then \mathcal{C} is closed under homomorphic images.*
2. *If for any $X \in \mathcal{C}$ such that $X \not\cong X / \leftrightarrow$ there is a nontrivial bisimulation rerouting U of X such that $U \in \mathcal{C}$, then \mathcal{C} is closed under bisimulation collapses.*

As closure under bisimulation collapses is often enough to establish collapsibility, Theorem 4.3 tells us that establishing an abundance of reroutings in the distinguished class can be a crucial step towards completeness.

5 A Global Approach

We now discuss a different approach to proving soundness and completeness theorems in process algebra, which we call the *global approach*, and show how the soundness and completeness theorems of [14] fit in this setting. Fitting Grabmayer and Fokkink’s proof into the mould of the global approach involves expanding the class of finite well-layered precharts to a much larger class that is closed under homomorphic images. We further show how the same remoulding technique can remould many local approach proofs into global ones.

The global approach originates in coalgebraic automata theory [19, 34, 35, 27, 6]. Its main goal is to show that the expression language modulo provable equivalence is isomorphic to a subcoalgebra of a *final* coalgebra. For example, in [19], Jacobs proves that the Kleene algebra axioms (see [25, 9])

are sound and complete with respect to language equivalence by exhibiting a coalgebra isomorphism between the initial Kleene algebra and the algebra of regular languages. The coalgebras that appear in Jacobs' paper are a standard choice for deterministic automata, the $2 \times (-)^A$ -coalgebras. This establishes the soundness and completeness of the Kleene Algebra axioms because bisimilarity and language equivalence coincide. Silva successfully applies the same method in [34] to a variety of expression languages and axiomatisations parametrized by the functor G , with Jacobs' proof given by the special case $G = 2 \times (-)^A$. Following the same pattern, Milius gives an expression language and axiomatisation of language equivalence for stream circuits in [27], and generalizes some of the results in [34] to endofunctors on categories other than **Sets**. Following a similar approach, all three of the above are unified in [6].

In order to explain precisely how the global approach works, fix a G -coalgebra E , thought of as an abstract expression language, and let \equiv be an equivalence relation on E . Similar to the local approach, the global approach involves a sequence of four steps:

Step 1 is showing that \equiv is a bisimulation equivalence. This establishes soundness.

Step 2 consists of identifying a class \mathcal{C} of G -coalgebras in which E/\equiv is *weakly final* in \mathcal{C} , ie. that $E/\equiv \in \mathcal{C}$ and every $X \in \mathcal{C}$ admits a homomorphism $X \rightarrow E/\equiv$. Again, homomorphisms into E/\equiv play the role of solutions, so it can be said that coalgebras in \mathcal{C} admit solutions.

Step 3 is a proof that E/\equiv is *final* in \mathcal{C} , ie. every $X \in \mathcal{C}$ admits exactly one solution.

Step 4 consists of showing that \mathcal{C} is closed under homomorphic images.

These four steps are sufficient for showing the soundness and completeness of the axiomatisations in each of the cases considered in [19, 34, 35, 27, 6] because the functors that are present there satisfy two key properties. The first key property is that there is a G -coalgebra Z that is *final*, ie. every G -coalgebra X admits a unique homomorphism $!_X : X \rightarrow Z$. Following the steps above, this implies that E/\equiv is a subcoalgebra of Z .

Lemma 5.1. *Assume that a final G -coalgebra exists, and call it Z . If \mathcal{C} is closed under homomorphic images and has a final object Y , then $!_Y : Y \rightarrow Z$ is injective.*

Proof. Where $!_Y : Y \rightarrow Z$ is the unique coalgebra homomorphism from Y into Z , let $J = !_Z(Y)$. The image of a coalgebra homomorphism is always a subcoalgebra of the codomain [30], so $J \in \mathcal{C}$ by closure under homomorphic images. Since Y is final in \mathcal{C} , J admits a unique coalgebra homomorphism $h : J \rightarrow Y$. Composing, $h \circ !_Y : Y \rightarrow Y$ is a homomorphism, so finality of Y in \mathcal{C} tells us that $h \circ !_Y = \text{id}_Y$. As $!_Y$ has a left inverse, it is injective. \square

This means that if every $X \in \mathcal{C}$ admits a unique solution and \mathcal{C} is closed under homomorphic images, then $[e]_{\equiv} = !_E(e)$ for any $e \in E$.⁵ The second key property is preservation of weak pullbacks.

Lemma 5.2 (Rutten [30]). *Let X and Y be G -coalgebras, $x \in X$, and $y \in Y$. Assume that a final G -coalgebra exists. If G preserves weak pullbacks, then $x \stackrel{\text{w}}{\leftrightarrow} y$ if and only if $!_X(x) = !_Y(y)$.*

Following steps 1 through 4 above, and assuming that G has a final coalgebra and preserves weak pullbacks, Lemmas 5.1 and 5.2 tell us that $[e]_{\equiv} = !_E(e) = !_E(f) = [f]_{\equiv}$ if and only if $e \stackrel{\text{w}}{\leftrightarrow} f$, for any $e, f \in \text{SEXP}$.

Theorem 5.1. *Assume G preserves weak pullbacks, and let \equiv be a bisimulation equivalence on a G -coalgebra E . Let \mathcal{C} be a class of G -coalgebras that is closed under homomorphic images. If E/\equiv is a final object in \mathcal{C} , then $e \equiv f$ if and only if $e \stackrel{\text{w}}{\leftrightarrow} f$ for any $e, f \in E$.*

⁵Here, we have identified E/\equiv with its isomorphic copy in Z .

It follows from standard observations about the prechart functor P that there is a final P -coalgebra [30]⁶ and that P preserves weak pullbacks [17]. This suggests the possibility that the global approach can be taken to proving Theorem 2.1 and Theorem 4.2. This is indeed the case, although the class of finite well-layered precharts needs to be extended so as to include SExp/\equiv .

5.1 A global approach to the 1-free fragment

Returning to the 1-free fragment of regular expressions, we have already seen that the class of well-layered precharts has SExp as a member. It is likely that SExp/\equiv is also well-layered, but proving this turns out to be unnecessary for our purposes.

In order to have the *global* approach go through for the 1-free fragment, we make a slight change in the distinguished class of precharts from section 4. Let \mathcal{C}_{loc} be the class of *locally well-layered* precharts, ie. $X \in \mathcal{C}_{loc}$ if and only if X is locally finite and every finite subcoalgebra of X is well-layered. Using the fact that the finite well-layered precharts are closed under homomorphic images, we obtain the following key lemma.

Lemma 5.3. *Let X be locally well-layered and $q : X \rightarrow Y$ a surjective coalgebra homomorphism. Then Y is locally well-layered as well.*

Every well-layered prechart is locally well-layered, so SExp is locally well-layered by Lemma 4.1. Since $\text{SExp} \in \mathcal{C}_{loc}$ and SExp/\equiv is the image of SExp under the homomorphism $[-]_{\equiv} : \text{SExp} \rightarrow \text{SExp}/\equiv$, Lemma 5.3 tells us that $\text{SExp}/\equiv \in \mathcal{C}_{loc}$ as well.

So far, we have taken step 4 and the first half of step 2 from the global approach. Interestingly, step 3 and the latter half of step 2 are possible because of Theorem 4.2, the uniqueness-of-solutions theorem for finite precharts. To see how this works, let $X \in \mathcal{C}_{loc}$. By Theorem 4.2, every finite subcoalgebra U of X admits a unique solution $s_U : U \rightarrow \text{SExp}/\equiv$. Since homomorphisms restrict to subcoalgebras, this clearly implies that X admits at most one solution. To see that SExp/\equiv is final in \mathcal{C}_{loc} , it suffices to construct a solution to X .

The unique solution to X is the map $s_X : X \rightarrow \text{SExp}/\equiv$ given by $s_X(x) = s_U(x)$ for any finite subcoalgebra U of X containing x . To see that this is well-defined, recall that X is locally finite, meaning that every state of X is contained in a finite subcoalgebra of X . If U and V are finite subcoalgebras of X with $x \in U$ and $x \in V$, then $U \cap V$ is a finite subcoalgebra of X containing x . We have assumed $U \cap V$ is well-layered, so by Theorem 4.2, $U \cap V$ admits a unique solution. Restricting s_U and s_V to $U \cap V$ also obtains a solution, so it must be that $s_U(x) = s_{U \cap V}(x) = s_V(x)$. To see that s is indeed a solution, observe that a map $h : X \rightarrow Y$ between locally finite coalgebras is a coalgebra homomorphism if $h|_U : U \rightarrow Y$ is a coalgebra homomorphism for any finite subcoalgebra U of X . Since the latter statement is true of s by definition, s is a solution to X . This establishes the lemma below.

Lemma 5.4. *Let \mathcal{C}_{loc} be the class of locally well-layered precharts. Then SExp/\equiv is a final object in the class \mathcal{C}_{loc} .*

Together, Theorems 2.1, 4.2, and Lemmas 5.3, 5.4 constitute steps 1 through 4 of the global approach to proving soundness and completeness of Milner's axioms for the 1-free fragment of regular expressions modulo bisimulation, thus providing an alternative proof of 4.2.

⁶Namely, that it is *bounded*.

5.2 From local to global

As Lemma 5.4 illustrates, there are instances in which a completeness proof taking the global approach can be obtained from the four steps in the local approach. This is particularly the case when the distinguished class of coalgebras is closed under binary coproducts and homomorphic images, like the well-layered precharts. Where E is a locally finite G -coalgebra and \equiv is a bisimulation equivalence on E , assume that in the four steps of the local approach we have obtained a class \mathcal{C} of finite G -coalgebras such that

- (a) each $X \in \mathcal{C}$ admits a unique homomorphism into E/\equiv ,
- (b) $\langle e \rangle \in \mathcal{C}$ for any $e \in E$, and
- (c) \mathcal{C} is closed under binary coproducts and homomorphic images.

Then the class \mathcal{C}_{loc} of *locally \mathcal{C} coalgebras*, locally finite coalgebras X such that every finite subcoalgebra of X is in \mathcal{C} , satisfies the necessary conditions for steps 2 through 4 of the global approach.

Going through the same motions as in the prechart case, for any $X \in \mathcal{C}_{loc}$ the unique solution $s_X : X \rightarrow E/\equiv$ is defined locally. If $x \in X$ and U is a finite subcoalgebra of X containing x , then $s_X(x) = s_U(x)$, where s_U is the unique solution to U . Furthermore, if $h : X \rightarrow Y$ is a surjective coalgebra homomorphism and $X \in \mathcal{C}_{loc}$, then for any finite subcoalgebra U of Y , $U = h(V)$ for some finite subcoalgebra V of X . By closure under homomorphic images, $U \in \mathcal{C}_{loc}$, and by extension $Y \in \mathcal{C}_{loc}$ as well. Lastly, $E \in \mathcal{C}_{loc}$ by definition, so $E/\equiv \in \mathcal{C}_{loc}$ by closure under homomorphic images. The following theorem obtains a global approach-style proof of completeness from the four steps of the local approach when \mathcal{C} is closed under coproducts and homomorphic images.

Theorem 5.2. *Let \mathcal{C} be a class of finite G -coalgebras satisfying (a)-(c) above. Then \mathcal{C}_{loc} is closed under homomorphic images, and E/\equiv is a final object of \mathcal{C}_{loc} .*

On the other hand, not every global approach-style completeness proof gives rise to a local one with such immediacy. For example, few of the distinguished classes of coalgebras found in the global approach-style proofs in [34] include the DFA interpretation of every expression in the language (each such DFA fails to be locally finite).

6 Discussion and Future Work

In this paper, we explore a coalgebraic take on Grabmayer and Fokkink's approach, what we call the local approach, to proving the completeness of Milner's axiomatisation of the 1-free star expressions modulo bisimilarity [14]. We use the insights gained from our exploration to give a general version of their method in section 3 that can be applied in other contexts. We do the same for a different proof method in section 5, what we call the global approach, originating in [19, 34, 6], and show how Grabmayer and Fokkink's proof can be remoulded to fit the global approach. At the end of the latter section, we give general conditions under which such a remoulding of a completeness proof that takes a local approach to a global one is possible.

A method is presented at the end of section 5 for turning a distinguished class \mathcal{C} from the local approach into a class \mathcal{C}_{loc} suitable for a global approach. Interestingly, the class \mathcal{C}_{loc} of locally \mathcal{C} coalgebras is closed under arbitrary coproducts, subcoalgebras, and homomorphic images when \mathcal{C} is closed under subcoalgebras and homomorphic images. In the case of the prechart functor P , and with \mathcal{C} the class of finite well-layered precharts, these structural qualities imply that \mathcal{C}_{loc} is a *covariety*, meaning that it can be presented by a predicate on a *cofree* P -coalgebra in some number κ of colours (that is, a *coequation* in

κ colours) [30]. The final P -coalgebra is a cofree coalgebra in one colour, but a covariety presented by a coequation in one colour is closed under bisimilarity [18], which we know from Figure 3 is not the case for \mathcal{C}_{loc} . We suspect that the number of colours needed to present the covariety of locally well-layered precharts is infinite, due to the infinitary nature of the layeredness condition, but more work needs to be done to be sure.

The use value of covarieties in the pursuit of completeness theorems is generally not well-understood. From Theorem 5.2, we expect there to be a deeper connection, but this is something that can only be uncovered by considering more examples. For instance, a covariety appears as the distinguished class of automata in the completeness proof in [32], the presenting coequation being the image of the expression language under the final coalgebra homomorphism. The situation in *loc. cit.* was similar to Grabmayer and Fokkink's, in that it was a completeness proof which lacked the use of a full Kleene theorem, and so could be an example of the phenomenon we are alluding to. We think other examples could be found by giving different operational interpretations of star expression languages considered in the literature, including as [26, 23, 21, 36, 32], as well as their fixed-point versions. Furthermore, we generally suspect that when going from semantics to expressions via solutions that depend only on generated subcoalgebras, a coequation should specify the distinguished class of models, thus enabling either of the two approaches to completeness discussed in this paper to go through.

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