

Between quantum logic and concurrency

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We start from two closure operators defined on the elements of a special kind of partially ordered sets, called causal nets. Causal nets are used to model histories of concurrent processes, recording occurrences of local states and of events. If every maximal chain (line) of such a partially ordered set meets every maximal antichain (cut), then the two closure operators coincide, and generate a complete orthomodular lattice. In this paper we recall that, for any closed set in this lattice, every line meets either it or its orthocomplement in the lattice, and show that to any line, a two-valued state on the lattice can be associated. Starting from this result, we delineate a logical language whose formulas are interpreted over closed sets of a causal net, where every line induces an assignment of truth values to formulas. The resulting logic is non-classical; we show that maximal antichains in a causal net are associated to Boolean (hence “classical”) substructures of the overall quantum logic.

1 Introduction

Partially ordered sets are a natural framework to model concurrent processes, namely processes where several components evolve in parallel, possibly interacting with each other. In Petri net theory (see, for example, [8], [3]) the behaviour of concurrent systems is modelled by causal nets, a class of Petri nets which records a partial order between occurrences of local states and events.

The partial order reflects relations of causal dependence, while concurrency (or independence) is given by lack of mutual order. Petri axiomatized such partial orders, with the aim of describing the flow of information in non-sequential processes, by drawing from the laws of physics, and especially from the theory of relativity ([9]). Among other properties, Petri introduced *K-density*, which requires that any maximal antichain (or *cut*) in the partial order and any maximal chain (or *line*) have a non-empty intersection. A line can be interpreted as the history of a sequential subprocess (a particle, a signal), while cuts correspond to time instants, where time is to be intended in a way analogous to the time coordinate in relativistic spacetime. K-density then requires that, at any instant, any sequential subprocess must be in a definite local state or involved in a change of state, modelled by an event (see [3]).

Two binary relations, corresponding to causal dependence and to concurrency, can be defined on the elements of an occurrence net. This suggests an analogy with relativistic spacetime, and in particular with Minkowski spacetime. Several authors have studied algebraic structures derived from Minkowski spacetime (in particular, see [5, 6]). The main result related to our work consists in defining a lattice whose elements are special sets of points in Minkowski, or even in more general, spacetimes, and in showing that such a lattice is complete and orthomodular.

Following similar ideas, but working on discrete partial orders, in [2] a closure operator is defined, whose closed sets are subsets of points of the partial order. It is furthermore shown that under a weak form of K-density, which holds for Petri causal nets, the closed sets form a complete orthomodular lattice.

In the same work it is shown that, in a discrete and locally finite framework, K -dense posets are exactly those in which, chosen an arbitrary closed set, any line intersects either it or its orthocomplement.

In this paper, we strengthen this last result by showing that each line identifies a two-valued state, in the sense of quantum logic. This suggests to look at the closed sets as propositions in a logical language (see Section 3.1), where orthocomplementation corresponds to negation, so that any line induces an interpretation. The resulting logical framework is obviously non-classical.

While lines (or maximal chains) correspond, in this sense, to two-valued states in a quantum logic, cuts (or maximal antichains) correspond to Boolean substructures of the logic, as stated in Section 3.

In the next section, we recall definitions and properties to be used later, related to quantum logics, closure operators, and Petri nets. Section 3 collects our main results.

2 Preliminaries

In this section, we recall some basic definitions and results, which will be used in the rest of the paper, related to quantum logics, Petri nets, and closure operators.

2.1 Quantum logic

The main reference for this section is [10]. A useful treatment of quantum logic, also in relation to an alternative formulation based on partial Boolean algebras, can be found in [7].

Definition 1 [10] *A quantum logic $(P, \leq, 0, 1, (\cdot)')$ is a partially ordered set (P, \leq) , equipped with a minimum element, denoted by 0 , and a maximum element, denoted by 1 , and with a map $(\cdot)': P \rightarrow P$ — called orthocomplementation — such that the following conditions are satisfied (where \vee and \wedge denote, respectively, the least upper bound and the greatest lower bound with respect to \leq , when they exist):*

$\forall x, y \in P$

1. $(x')' = x$
2. $x \leq y \Rightarrow y' \leq x'$
3. $x \wedge x' = 0$ and $x \vee x' = 1$

Two elements $x, y \in P$ are orthogonal, denoted $x \perp y$, if $x \leq y'$. P is orthocomplete when every countable subset of pairwise orthogonal elements of P has a least upper bound. Moreover, P is orthomodular when the following condition, called orthomodular law,

$$x \leq y \quad \Rightarrow \quad y = x \vee (y \wedge x')$$

is satisfied.

Figure 1 shows one finite and one infinite quantum logic. In a quantum logic, \perp is a symmetric relation and the De Morgan laws hold: $(x \vee y)' = x' \wedge y'$, $(x \wedge y)' = x' \vee y'$ whenever one of the members of the equation exists. We will sometimes use *meet* and *join* to denote, respectively, \wedge and \vee with the obvious extension to families of elements, denoted by \bigwedge and \bigvee . In the following we will sometimes use *logic* as a shorthand for *quantum logic*.

Definition 2 *Two elements x, y of a logic P are said to be compatible — denoted by $x \leftrightarrow y$ — if there exist in P three mutually orthogonal elements x_1, z and y_1 such that $x = x_1 \vee z$ and $y = y_1 \vee z$.*

Definition 3 *A logic P is called regular if for any set $\{x, y, z\}$ of pairwise compatible elements in P , it holds that $x \leftrightarrow (y \vee z)$.*

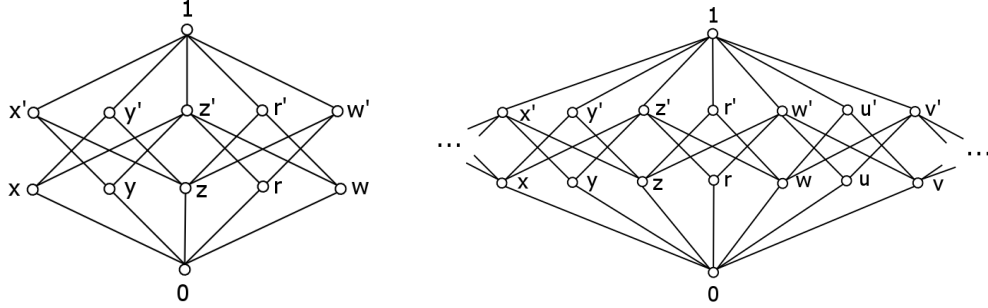


Figure 1: Examples of orthomodular quantum logics.

Every logic whose supporting poset is a lattice is regular ([10], proposition 1.3.27).

A regular quantum logic admits an alternative characterization, as a *partial Boolean algebra* (see [7]). We will not give the formal definitions related to this view, but only recall it informally. A partial Boolean algebra is a family of partially overlapping Boolean algebras, which share the minimum and the maximum element, and satisfy a set of axioms on the shared elements.

Definition 4 [10] A two-valued state on a logic P is a map $s : P \rightarrow \{0, 1\}$ such that, for any sequence $(a_i)_{i \in I}$ of mutually orthogonal elements in P :

1. $s(1) = 1$
2. $s(\bigvee_{i \in I} a_i) = \sum_{i \in I} s(a_i)$

The set of two-valued states on a logic P will be called $\mathcal{S}_2(P)$. If the elements of a quantum logic are interpreted as propositions of a logical language, two-valued states can be considered as consistent assignments of truth values to those propositions.

2.2 Petri nets and causal nets

A Petri net is a discrete model of a concurrent system, based on the notions of local state (or condition) and of local change of state (or event). Formally, they are bipartite graphs, where the arcs encode immediate causal relations.

Here, we will focus on a special case of Petri nets, where the underlying graph induces a partial order on the set of local states and events.

Definition 5 A net is a triple $N = (B, E, F)$, where B and E are countable sets, $F \subseteq (B \times E) \cup (E \times B)$, and

1. $B \cap E = \emptyset$;
2. $\text{dom}(F) \cup \text{ran}(F) = B \cup E$.

The elements of B are called *conditions*, the elements of E are called *events* and F is called *flow relation*.

The standard graphical notation for nets represents conditions as circles, events as squares and the flow relation as directed arcs.

The intuitive meaning associated to conditions and events is that conditions represent *local states* of a system while events represent *local changes of state*.

For each $x \in B \cup E$, define $\bullet x = \{y \in B \cup E \mid (y, x) \in F\}$, $x^\bullet = \{y \in B \cup E \mid (x, y) \in F\}$. For $e \in E$, an element $b \in B$ is a *precondition* of e if $b \in \bullet e$; it is a *postcondition* of e if $b \in e^\bullet$. In the basic model of Petri nets, which we refer to in this paper, a condition is either *true* or *false*; an event can *fire* (happen) if its preconditions are all true and its postconditions are false; the effect of an event firing consists in making its preconditions false and its postconditions true.

A net $N = (B, E, F)$ is *simple* if for each $x, y \in B \cup E$: $(\bullet x = \bullet y \text{ and } x^\bullet = y^\bullet) \Rightarrow x = y$.

By F^+ we denote the irreflexive, transitive closure of F , by F^* we denote $F^+ \cup \text{id}_X$.

Definition 6 A causal net is a net in which the following conditions hold:

1. $\forall b \in B : |\{e \in E \mid (e, b) \in F\}| \leq 1 \wedge |\{e \in E \mid (b, e) \in F\}| \leq 1$;
2. $\forall x, y \in B \cup E : (x, y) \in F^+ \Rightarrow (y, x) \notin F^+$.

A causal net is a net without cycles, and such that branches can occur only at events. In the standard terminology, no *choices* are allowed.

The structure (X, \sqsubseteq) derived from a causal net N by putting $X = B \cup E$ and $\sqsubseteq = F^*$ is a partially ordered set (shortly a *poset*). A subset $S \subseteq X$ is said to be *convex* if, for each pair of its elements, S contains the interval between them: $\forall x, y \in S : [x, y] \subseteq S$ where $[x, y] = \{z \in X \mid x \sqsubseteq z \sqsubseteq y\}$.

Definition 7 (X, \sqsubseteq) is interval-finite $\Leftrightarrow \forall x, y \in X : |[x, y]| < \infty$. (X, \sqsubseteq) is degree-finite $\Leftrightarrow \forall x \in X : |\bullet x| < \infty$ and $|x^\bullet| < \infty$. When (X, \sqsubseteq) is both interval and degree-finite, we will say that it is *locally finite*. We will apply these terms also to the causal net from which (X, \sqsubseteq) is obtained.

When using discrete partial orders to model processes of real systems, local finiteness is a natural assumption since synchronization of infinite events or causal dependence at infinite distance between the events cannot be realized.

On the poset (X, \sqsubseteq) , the relations $\mathbf{li} = \sqsubseteq \cup \sqsubseteq^{-1}$, and $\mathbf{co} = (X \times X) \setminus \mathbf{li}$ can be defined. Intuitively, $x \mathbf{li} y$ means that x and y are connected by a causal relation while $x \mathbf{co} y$ means that x and y are causally independent. The relations \mathbf{li} and \mathbf{co} are symmetric and not transitive. Moreover, \mathbf{li} is a reflexive relation, while \mathbf{co} is irreflexive.

Given an element $x \in X$ and a set $S \subseteq X$, we write $x \mathbf{co} S$ if $\forall y \in S : x \mathbf{co} y$. Given two sets $S_1 \subseteq X$ and $S_2 \subseteq X$, we write $S_1 \mathbf{co} S_2$ if $\forall x \in S_1, \forall y \in S_2 : x \mathbf{co} y$.

A *clique* of a binary relation is a set of pairwise related elements; a clique of $\mathbf{co} \cup \text{id}_X$ will be also called a *coset*; a coset made of conditions only will be called a *B-coset*. Maximal cliques of $\mathbf{co} \cup \text{id}_X$ and \mathbf{li} are called, respectively, *cuts* and *lines*. Given a poset (X, \sqsubseteq) , its cuts and lines will be denoted, respectively, as:

$$\mathcal{C}(X) = \{c \subseteq X \mid c \text{ is a maximal clique of } \mathbf{co} \cup \text{id}_X\}$$

and

$$\mathcal{L}(X) = \{l \subseteq X \mid l \text{ is a maximal clique of } \mathbf{li}\}.$$

We assume the axiom of choice, so any clique of $\mathbf{co} \cup \text{id}_X$ and of \mathbf{li} can be extended to a maximal clique. This will be used in particular in the proof of Theorem 2 below.

In the following, (X, \sqsubseteq) will always be the poset obtained from a causal net N , we will indicate the cuts and lines in X as well as $\mathcal{C}(N)$ and $\mathcal{L}(N)$. When a cut is composed exclusively of conditions, it will be referred to as a *B-cut*. If $A \subseteq X$, the definition is restricted naturally to the maximal cliques in A with the notation $\mathcal{C}(A)$ and $\mathcal{L}(A)$.

Definition 8 (X, \sqsubseteq) is K -dense $\Leftrightarrow \forall c \in \mathcal{C}(N), \forall l \in \mathcal{L}(N) : c \cap l \neq \emptyset$.

From their definition, it follows immediately that, if a line and a cut have a non-empty intersection, then the intersection will consist in exactly one point. When this is the case, we will say that the line and the cut *meet* at a point, or that the line *crosses* the cut, or viceversa.

2.3 Closure operators on causal nets

In this section, we recall the definition and some basic properties of two closure operators on the set of elements of a partially ordered set and, more specifically, on the set of elements of a causal net.

In general, by closure operator on a set Z , we mean a map $\gamma : \mathbf{P}(Z) \rightarrow \mathbf{P}(Z)$ (where $\mathbf{P}(Z)$ denotes the powerset of Z), satisfying the following, for all $A, B \subseteq Z$:

1. $A \subseteq \gamma(A)$ (increasing)
2. if $A \subseteq B$, then $\gamma(A) \subseteq \gamma(B)$ (monotone)
3. $\gamma(\gamma(A)) = \gamma(A)$ (idempotent)

A subset A of Z is called *closed* with respect to γ if $A = \gamma(A)$.

The first operator with which we will deal here is defined indirectly, starting from the definition of *causally closed sets*. These form a family of sets closed by intersection, and with a maximum, with respect to set inclusion. As usual, the closure of an arbitrary set $A \subseteq B \cup E$ is by definition the intersection of all the causally closed sets that contain A .

Let $N = (B, E, F)$ be a locally-finite causal net, and $X = B \cup E$. A subset of X is causally closed if it is convex with respect to the partial order induced by N , and if it is closed with respect to local causes (preconditions) and local effects (postconditions).

Definition 9 A set $C \subseteq X = B \cup E$ is a causally closed set if

- (i) $\forall e \in E, \bullet e \subseteq C \Rightarrow e \in C$,
- (ii) $\forall e \in E, e^\bullet \subseteq C \Rightarrow e \in C$,
- (iii) $\forall e \in E, e \in C \Rightarrow \bullet e \cup e^\bullet \subseteq C$,
- (iv) $\forall x, y \in C, x \text{ li } y \Rightarrow [x, y] \subseteq C$.

The family of causally closed sets of N will be called $\Gamma(N)$.

Define the *border* of a subset $A \subseteq X$ as those elements of A which are directly linked, by F -arcs, to elements outside of A : $\beta(A) = \{x \in A \mid \exists y \in X \setminus A : (x, y) \in F \cup F^{-1}\}$. As shown in [2], the border of a causally closed set is made of local states only:

$$\forall A \in \Gamma(N) \quad \beta(A) \subseteq B$$

Moreover, $\Gamma(N)$ is closed by intersection and $\emptyset \in \Gamma(N)$ and $B \cup E \in \Gamma(N)$. Hence, the family $\Gamma(N)$ forms a complete lattice — non orthocomplemented in the general case — where meet is given by set intersection and join is given by the causal closure of the set union of the operands. The associated closure operator, here denoted by ϕ , can now be defined as usual. Let $X = B \cup E$.

Definition 10 Define $\phi : \mathbf{P}(X) \rightarrow \mathbf{P}(X)$ as follows: $\forall A \subseteq X, \phi(A) = \bigcap \{C_i \mid C_i \in \Gamma(N) \text{ and } A \subseteq C_i\}$.

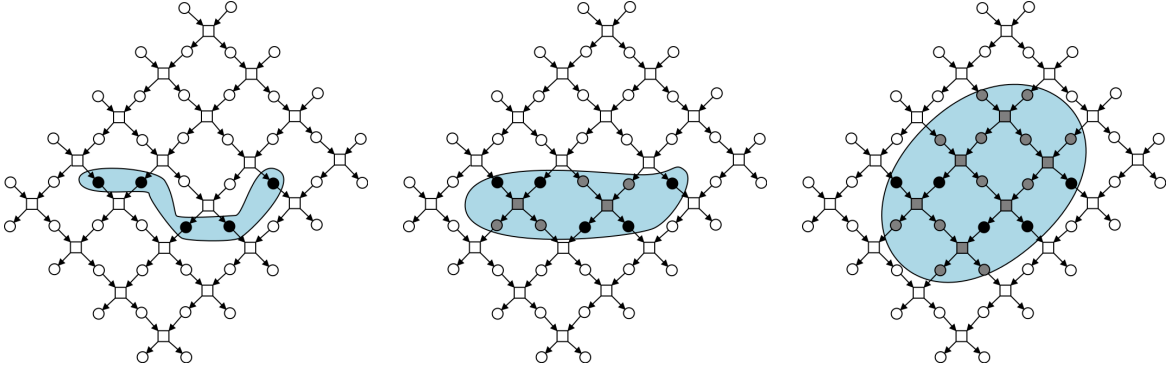


Figure 2: Iterative closure of a B-coset.

So, given two elements A and B in $\Gamma(N)$, $A \vee B$ is defined to be $\phi(A \cup B)$. As shown in [2], the causal closure of a B-coset can be defined by means of an iterative procedure, which adds new elements according to the axioms in Definition 9. An example is shown in Figure 2. The picture on the left shows a B-coset; the middle picture shows an intermediate step in the procedure, while the picture on the right shows the final step (other intermediate steps are omitted).

The second closure operator is defined following a well-known construction, to be found in [4], and recalled below.

Given a set Z , a symmetric relation $\alpha \subseteq Z \times Z$ and a subset A of Z , define

$$A' = \{x \in Z \mid \forall y \in A : (x, y) \in \alpha\}.$$

By applying twice this operator, we get a new operator $(\cdot)''$, which can be shown to be a closure operator. A subset A of Z is *closed* if $A = A''$. The family $L(Z)$ of all closed sets of Z , ordered by set inclusion, is then a complete lattice. When α is also irreflexive, the operator $(\cdot)'$, applied to elements of $L(Z)$, is an orthocomplementation; the structure $\mathbf{L} = (L(Z), \subseteq, \emptyset, Z, (\cdot)')$ then forms an orthocomplemented complete lattice [4]. Let us now consider the poset (X, \sqsubseteq) derived from a causal net N together with the irreflexive relation \mathbf{co} as in Section 2.2. By applying the construction above, with \mathbf{co} as irreflexive and symmetric relation, to the subsets of X , we obtain the orthocomplemented complete lattice $(L(N), \subseteq, \emptyset, X, (\cdot)')$, where $L(N)$ denotes the family of closed subsets of X .

Figure 3 shows a set A of elements of a causal net, together with its orthocomplement, A' , and its closure, A'' .

In [2] it is shown that this lattice is orthomodular if the construction above is applied to a locally finite poset satisfying a weak form of K-density, called N-density. All causal nets are N-dense ([3]), hence the lattice of closed sets derived from the concurrency relation in a locally finite causal net is complete orthomodular.

As already noted, in general, the lattice of causally closed sets of a locally finite causal net is not even orthocomplemented; however, if the causal net is K-dense, then the two closure operators coincide, and the lattice of causally closed sets is complete orthomodular.

Theorem 1 [2] *Let $N = (B, E, F)$ be a locally finite, K-dense causal net. Let $A \subseteq B \cup E$. Then $A \in \Gamma(N) \iff A \in L(N)$.*

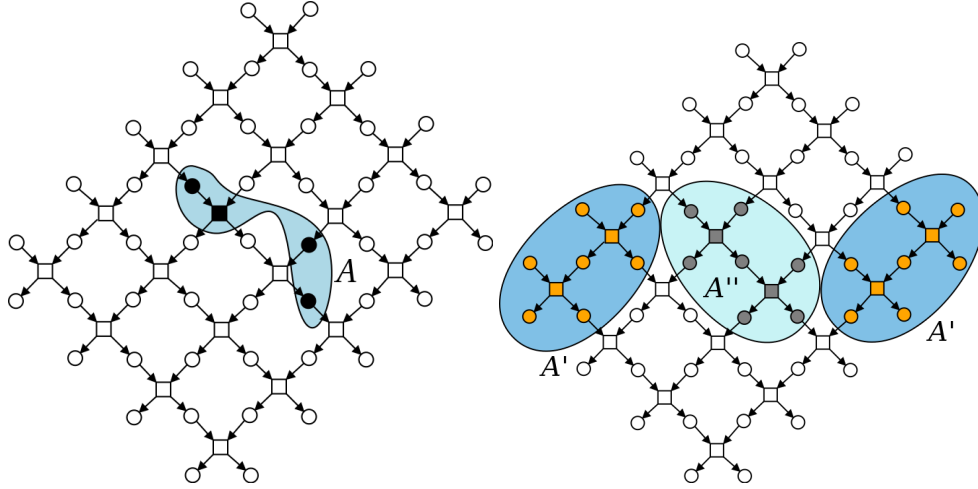


Figure 3: A set A , its orthocomplement A' , and its closure A'' .

3 Towards a logical view of closed sets

In this section, we collect the main results of our contribution. The first states that every line in a K -dense causal net N identifies a two-valued state in the lattice (or quantum logic) of closed sets, $L(N)$. This suggests to look at the closed sets as propositions in a logical language (see Section 3.1). The second result concerns the relation between B -cuts and Boolean subalgebras of a quantum logic.

Throughout this section, $N = (B, E, F)$ will denote an arbitrary locally finite, K -dense causal net.

The following lemma states that any closed set can be obtained as the closure of any of its local B -cuts.

Lemma 1 *Let $A \in L(N)$, and τ a B -cut of A . Then $A = \tau''$.*

Proof. The closure operator is idempotent and monotone; hence, from $\tau \subseteq A$, we get $\tau'' \subseteq A'' = A$. To show the inclusion in the other direction, take $x \in A$. If $x \in \tau$, then $x \in \tau''$. Suppose $x \notin \tau$. Since τ is a B -cut, $x \mathbf{li} z$ for some $z \in \tau$. By way of contradiction, suppose $x \notin \tau''$; then, there must be $w \in \tau'$, with $x \mathbf{li} w$.

Put $x < z$ (the symmetric case is treated analogously); then $x < w$, because $z \mathbf{co} w$. Choose a path from x to w ; this path must cross the border of A at an F -arc from a condition b to an event e . Then b must be concurrent to all the elements of τ , because A is convex, but this contradicts the hypothesis that τ is a cut (maximal antichain) of A .

The previous lemma implies that, for each $x \in B \cup E$, if $x \mathbf{co} \tau$, then $x \mathbf{co} A$. This will be used in later proofs.

Now we can prove a crucial relation between lines and closed sets. This is actually a corollary of Theorem 3.2 in [2], but we think that a direct proof could be useful. It says that, given a closed set A , a line crosses either A or A' (but not both). The statement is illustrated in Figure 4, where a line is shown with a thicker stroke.

Proposition 1 *Let $A \in L(N)$, and $\lambda \in \mathcal{L}(N)$. Then*

$$\lambda \cap A \neq \emptyset \Leftrightarrow \lambda \cap A' = \emptyset$$

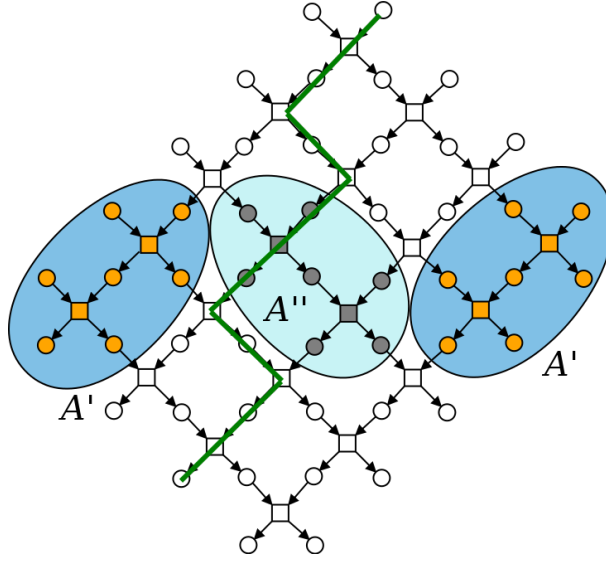


Figure 4: A line crosses either a closed set or its orthocomplement.

Proof. Suppose $\lambda \cap A \neq \emptyset$. Any element in A' is concurrent with any element in A . Since λ is a clique of the **li** relation, no element in λ can be in A' .

Suppose now that $\lambda \cap A = \emptyset$. Take a B-cut of A , say τ_1 . This is a B-coset of N , and can be extended to a B-cut of N , say τ . Since N is K-dense, τ crosses λ at a point $b \in B$. Then $b \mathbf{co} \tau_1$, and, by Lemma 1, $b \mathbf{co} A$, which means $b \in A'$.

Building on the previous proposition, we now define a map associated to a line in N . The map can be seen as the characteristic map of the family of closed sets that cross the given line.

Definition 11 Let λ be a line of N . Define $\Delta(\lambda) = \{A \in L(N) \mid A \cap \lambda \neq \emptyset\}$, and $\delta_\lambda : L(N) \rightarrow \{0, 1\}$ in this way: for each $A \in L(N)$, $\delta_\lambda(A) = 1$ if $A \in \Delta(\lambda)$, 0 otherwise.

Theorem 2 The map δ_λ is a two-valued state of $L(N)$.

Proof. Let $(A_i)_{i \in I}$ be a family of pairwise orthogonal closed sets. Then, for each $i, j, i \neq j$, $A_i \subseteq A'_j$. Hence, if λ crosses one of the A_i s, it cannot cross any of the others because of Proposition 1. From this, it follows that, if $\delta_\lambda(A_i) = 1$, then $\delta_\lambda(A_j) = 0$ for any $j \neq i$, and $\delta_\lambda(\bigvee A_i) = 1$.

Suppose now that $\delta_\lambda(A_i) = 0$ for each $i \in I$. We will show that λ does not cross $\bigvee A_i$. For each $i \in I$, take a B-cut of A_i . The elements of all the B-cuts of the A_i 's are pairwise concurrent; hence their union is a B-coset, and can be extended to a B-cut of N . Since N is K-dense, this cut crosses λ at a point, say s , and s is concurrent with all the A_i s. By way of contradiction, suppose now that $\bigvee A_i$ crosses λ at a point b . Since this point is in the closure of $\bigcup A_i$, it must satisfy $b \mathbf{co} (\bigvee A_i)'$. But this implies $b \mathbf{co} s$, contradicting the hypothesis that b and s are on the same line.

The previous results relate lines and two-valued states in quantum logics. We now briefly point out a dual relation between cuts in a causal net and Boolean subalgebras in the lattice of closed sets.

We have already noted that a regular quantum logic can be seen as a family of partially overlapping Boolean algebras. In our context, these component Boolean algebras are atomic.

Consider a B-cut of N , say $\tau = \{b_1, \dots, b_i, \dots\}$. For each b_i in τ , compute $\{b_i\}'' = \beta_i$. The closed sets obtained in this way are pairwise orthogonal in the lattice of closed sets, which means that they are pairwise concurrent in N . Their join gives $B \cup E$.

Then, the set $\{\beta_1, \dots, \beta_i, \dots\}$ is the set of atoms of a (maximal) Boolean subalgebra of $L(N)$. More generally, given a family $(A_i)_{i \in I}$ of pairwise orthogonal (concurrent) closed sets, such that $\bigvee_{i \in I} A_i = B \cup E$, there is a Boolean subalgebra of $L(N)$ such that the A_i s are its atoms.

3.1 Logic

The statement of Theorem 2 suggests to look at the closed sets in $L(N)$ as propositions of a logical language, which admits a different interpretation for each line in N .

In fact, each line selects exactly one closed set for each pair (A, A') , and this selection is consistent with the structure of the lattice of closed sets: let us say that the proposition associated to A is true, with respect to a line λ , if λ crosses A , and false otherwise. Then, we can take $(\cdot)'$ as a negation, while the lattice operations correspond to the logical connectives, disjunction (join), and conjunction (meet). Theorem 2 guarantees that, under this interpretation, if two propositions are true, then also their conjunction is true, while the conjunction of two “compatible” propositions is true only if at least one of them is true, where two propositions are compatible if their corresponding closed sets are compatible in the quantum logic $L(N)$ (or, equivalently, if there is a Boolean subalgebra of $L(N)$ which contains both).

The resulting logic is obviously non-classical, since the lattice of closed sets is not, in general, distributive. It is, so to speak, locally classical, in the sense that the set of closed sets associated to true propositions, projected on a Boolean subalgebra of $L(N)$ gives an ultrafilter.

Formally, we define the propositional language \mathcal{F}_Π and its interpretation over the orthomodular lattice $L(N)$. Let $\Pi = (\pi_i)_{i \in I}$ be a set of *propositions*. Define the set \mathcal{F}_Π of *formulas* over Π , inductively, as follows:

- (i) every π_i is a formula;
- (ii) if f_1, f_2 are formulas, then $f_1 \vee f_2, f_1 \wedge f_2, \neg f_1, f_1 \rightarrow f_2$ are formulas;
- (iii) nothing else is a formula.

An interpretation of the language of formulas \mathcal{F}_Π is a pair $J = \langle h : \Pi \rightarrow L(N), \lambda \in \mathcal{L}(N) \rangle$.

To each formula f , we can associate an element of $L(N)$, by defining a map $i : \mathcal{F}_\Pi \rightarrow L(N)$, as follows. If $f = \pi_i$, then $i(f) = h(\pi_i)$; if $f = \neg(f_1)$, then $i(f) = (i(f_1))'$; if $f = f_1 \vee f_2$, then $i(f) = i(f_1) \vee i(f_2)$, where \vee is the join operation in the lattice $L(N)$; if $f = f_1 \wedge f_2$, then $i(f) = i(f_1) \wedge i(f_2)$, where \wedge is the meet operation in the lattice $L(N)$; if $f = f_1 \rightarrow f_2$, then $i(f) = (i(f_1))' \vee i(f_2)$.

With this definition, the implication connective and the partial order on $L(N)$ are related by the following statement: if $i(f_1) \subseteq i(f_2)$, then $i(f_1 \rightarrow f_2) = B \cup E$.

The choice of the line λ in an interpretation determines the assignment of truth values to formulas: a formula is true if λ crosses the closed set associated to the formula by the map i . Formally, we define a satisfiability relation between interpretations and formulas:

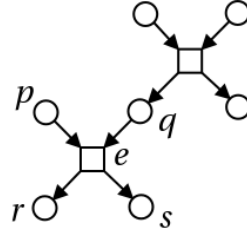
$$J \models f \quad \Leftrightarrow \quad i(f) \cap \lambda \neq \emptyset.$$

This definition is consistent, in the sense that, by direct verification, one can prove the following:

1. $J \models f \wedge g$ if, and only if, $J \models f$ and $J \models g$
2. $J \models \neg f$ if, and only if, $J \not\models f$

3. $J \models f \vee g$ if, and only if, $i(f) \leftrightarrow i(g)$, and either $J \models f$ or $J \models g$;
4. $J \models f \rightarrow g$ if, and only if, $i(f) \subseteq i(g)$.

The case of disjunctive formulas can be illustrated by means of a simple example. Consider the net below (which might be a fragment of a larger net).



Let f and g be two propositions interpreted, respectively, over $\{p\}$ and $\{r\}$, which are closed sets. Then $i(f \vee g) = \{p, q, e, r, s\}$. With respect to a line λ containing $\{q, e, s\}$, the formula $f \vee g$ is true, while f and g are false. In this case $i(f)$ and $i(g)$ are not compatible in $L(N)$.

Suppose now to interpret f and g over $\{p\}$ and $\{q\}$, which are compatible in $L(N)$. The interpretation of $f \vee g$ is the same as before, namely $\{p, q, e, r, s\}$, but any line crossing this closed set is bound to cross either $\{p\}$ or $\{q\}$, so that either f or g is true.

4 Conclusion and prospects

A first connection between orthomodular structures and concurrency theory emerged by studying an abstract notion of local state in automata and in Petri nets (see [1]).

Later, a different connection was found, which bears a more direct relation with special relativity theory. Several authors had previously shown that a closure operator, and a corresponding orthomodular lattice, can be derived from the “spacelike” relation between points in Minkowski spacetime ([5, 6]).

A similar construction was then applied to discrete partially ordered sets modelling the history of concurrent, or distributed, system. In the discrete case, the orthomodularity of the resulting lattice of closed sets depends on a feature of the partially ordered set which was called N -density by C.A. Petri ([2]).

In this context, the specific character of the causal nets introduced by Petri is the distinction between synchronization events and local properties changed by the occurrence of the events. With this distinction, *lines* can be interpreted as *signals* whose status is changed by the interaction event with other signals and preserving their local status until another interaction occurs. This in analogy with flows of particles in space whose mutual interactions are *collisions*.

For causal nets and in partial orders derived from causal nets, density properties were studied mainly with the aim of providing a sound set of axioms for the definition of the causal structures representing concurrent processes of systems. In this respect, [3] gives a comprehensive survey of the properties of partially ordered sets related to causal nets. In particular, the relations between N -density and K -density are presented in the specific context of models of concurrent systems.

Starting from what we have presented here, we will undertake several further steps: characterize those orthomodular lattices that can be obtained as lattices of closed sets induced by concurrency; building a causal net whose lattice of closed sets is isomorphic to a given lattice; study lattices of closed sets induced by concurrency in other classes of Petri nets; give a meaning to the logical language here defined.

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