Quantum Set Theory Extending the Standard Probabilistic Interpretation of Quantum Theory (Extended Abstract)

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The notion of equality between two observables will play many important roles in foundations of quantum theory. However, the standard probabilistic interpretation based on the conventional Born formula does not give the probability of equality relation for a pair of arbitrary observables, since the Born formula gives the probability distribution only for a commuting family of observables. In this paper, quantum set theory developed by Takeuti and the present author is used to systematically extend the probabilistic interpretation of quantum theory to define the probability of equality relation for a pair of arbitrary observables. Applications of this new interpretation to measurement theory are discussed briefly.

1 Introduction

Set theory provides foundations of mathematics. All the mathematical notions like numbers, functions, relations, and structures are defined in the axiomatic set theory, ZFC (Zermelo-Fraenkel set theory with the axiom of choice), and all the mathematical theorems are required to be provable in ZFC [15]. Quantum set theory instituted by Takeuti [14] and developed by the present author [11] naturally extends the logical basis of set theory from classical logic to quantum logic [1]. Accordingly, quantum set theory extends quantum logical approach to quantum foundations from propositional logic to predicate logic and set theory. Hence, we can expect that quantum set theory will provide much more systematic interpretation of quantum theory than the conventional quantum logic approach [3].

The notion of equality between quantum observables will play many important roles in foundations of quantum theory, in particular, in the theory of measurement and disturbance [9, 10]. However, the standard probabilistic interpretation based on the conventional Born formula does not give the probability of equality relation for a pair of arbitrary observables, since the Born formula gives the probability distribution only for a commuting family of observables [7].

In this paper, quantum set theory is used to systematically extend the probabilistic interpretation of quantum theory to define the probability of equality relation for a pair of arbitrary observables, based on the fact that real numbers constructed in quantum set theory exactly corresponds to quantum observables [14, 11]. It is shown that all the observational propositions on a quantum system correspond to statements in quantum set theory with the same projection-valued truth value assignments and the same probability assignments in any state. In particular, the equality relation for real numbers in quantum set theory naturally provides the equality relation for quantum mechanical observables. It has been broadly accepted that we cannot speak of the values of quantum observables without assuming a hidden variable theory, which severely constrained by Kochen-Specker type no-go theorems [5, 13]. However, quantum

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set theory enables us to do so without assuming hidden variables but alternatively with the consistent use of quantum logic. Applications of this new interpretation to measurement theory are discussed briefly.

Section 2 provides preliminaries on commutators in complete orthomodular lattices, which play a fundamental role in quantum set theory. Section 3 introduces quantum logic on Hilbert spaces and section 4 introduces quantum set theory and the transfer principle from theorems in ZFC to valid statements in quantum set theory established in Ref. [11]. Section 5 introduces the Takeuti correspondence between reals in quantum set theory to observables in quantum theory found by Takeuti [14]. Section 6 formulates the standard probabilistic interpretation of quantum theory and also shows that observational propositions for a quantum system can be embedded in statements in quantum set theory with the same projection-valued truth value assignment. Section 7 extends the standard interpretation by introducing state-dependent joint determinateness relation. Section 8 extends the standard interpretation by introducing state-dependent equality for arbitrary two observables. Section 9 and 10 provide applications to quantum measurement theory.

2 Complete orthomodular lattices and commutators

A *complete orthomodular lattice* is a complete lattice $\mathscr{Q}$ with an *orthocomplementation*, a unary operation $\perp$ on $\mathscr{Q}$ satisfying (C1) if $P \le Q$ then $Q^{\perp} \le P^{\perp}$, (C2) $P^{\perp\perp} = P$, (C3) $P \vee P^{\perp} = 1$ and $P \wedge P^{\perp} = 0$, where $0 = \bigwedge \mathscr{Q}$ and $1 = \bigvee \mathscr{Q}$, that satisfies the *orthomodular law* (OM) if $P \le Q$ then $P \vee (P^{\perp} \wedge Q) = Q$. In this paper, any complete orthomodular lattice is called a *logic*. A non-empty subset of a logic $\mathscr{Q}$ is called a *subalgebra* iff it is closed under $\wedge$, $\vee$, and $\perp$. A subalgebra $\mathscr{A}$ of $\mathscr{Q}$ is said to be *complete* iff it has the supremum and the infimum in $\mathscr{Q}$ of an arbitrary subset of $\mathscr{A}$. For any subset $\mathscr{A}$ of $\mathscr{Q}$, the subalgebra generated by $\mathscr{A}$ is denoted by $\Gamma_0\mathscr{A}$. We refer the reader to Kalmbach [4] for a standard text on orthomodular lattices.

We say that P and Q in a logic $\mathscr{Q}$ *commute*, in symbols $P \,{}^{\mid}_{\circ}\, Q$, iff $P = (P \wedge Q) \vee (P \wedge Q^{\perp})$. A logic $\mathscr{Q}$ is a Boolean algebra if and only if $P \,{}^{\mid}_{\circ}\, Q$ for all $P, Q \in \mathscr{Q}$ [4, pp. 24–25]. For any subset $\mathscr{A} \subseteq \mathscr{Q}$, we denote by $\mathscr{A}^{!}$ the *commutant* of $\mathscr{A}$ in $\mathscr{Q}$ [4, p. 23], i.e.,

$$\mathscr{A}^{!} = \{P \in \mathscr{Q} \mid P \,{}^{\mid}_{\circ}\, Q \text{ for all } Q \in \mathscr{A}\}.$$

Then, $\mathscr{A}^{!}$ is a complete subalgebra of $\mathscr{Q}$. A *sublogic* of $\mathscr{Q}$ is a subset $\mathscr{A}$ of $\mathscr{Q}$ satisfying $\mathscr{A} = \mathscr{A}^{!!}$. For any subset $\mathscr{A} \subseteq \mathscr{Q}$, the smallest logic including $\mathscr{A}$ is $\mathscr{A}^{!!}$ called the *sublogic generated by* $\mathscr{A}$. Then, it is easy to see that a subset $\mathscr{A}$ is a Boolean sublogic, or equivalently a distributive sublogic, if and only if $\mathscr{A} = \mathscr{A}^{!!} \subseteq \mathscr{A}^{!}$.

Let $\mathscr{Q}$ be a logic. Marsden [6] introduced the *commutator* $\mathrm{com}(P,Q)$ of two elements P and Q of $\mathscr{Q}$ by

$$\mathrm{com}(P,Q) = (P \wedge Q) \vee (P \wedge Q^{\perp}) \vee (P^{\perp} \wedge Q) \vee (P^{\perp} \wedge Q^{\perp}).$$

Generalizing this notion to an arbitrary subset $\mathscr{A}$ of $\mathscr{Q}$, Takeuti [14] defined the commutator $\mathrm{com}(\mathscr{A})$ of $\mathscr{A}$ by

$$\mathrm{com}(\mathscr{A}) = \bigvee \{E \in \mathscr{A}^{!} \mid P_1 \wedge E \,{}^{\mid}_{\circ}\, P_2 \wedge E \text{ for all } P_1, P_2 \in \mathscr{A}\}.$$

Subsequently, Chevalier [2] proved the relation

$$\mathrm{com}(\mathscr{A}) = \bigwedge \{\mathrm{com}(P,Q) \mid P, Q \in \Gamma_0(\mathscr{A})\},$$

which concludes $\mathrm{com}(\mathscr{A}) \in \mathscr{A}^{!!} \cap \mathscr{A}^{!}$. For any $P,Q \in \mathscr{Q}$, the *interval* $[P,Q]$ is the set of all $X \in \mathscr{Q}$ such that $P \le X \le Q$. For any $\mathscr{A} \subseteq \mathscr{Q}$ and $P,Q \in \mathscr{A}$, we write $[P,Q]_{\mathscr{A}} = [P,Q] \cap \mathscr{A}$. The following theorem clarifies the significance of commutators.

Theorem 1. *Let $\mathscr{A}$ be a subset of a logic $\mathscr{Q}$. Then, $\mathscr{A}^{!!}$ is isomorphic to the direct product of the complete Boolean algebra $[0,\mathrm{com}(\mathscr{A})]_{\mathscr{A}^{!!}}$ and the complete orthomodular lattice $[0,\mathrm{com}(\mathscr{A})^{\perp}]_{\mathscr{A}^{!!}}$ without non-trivial Boolean factor.*

3 Quantum logic on Hilbert spaces

Let $\mathscr{H}$ be a Hilbert space. For any subset $S \subseteq \mathscr{H}$, we denote by $S^{\perp}$ the orthogonal complement of S. Then, $S^{\perp\perp}$ is the closed linear span of S. Let $\mathscr{C}(\mathscr{H})$ be the set of all closed linear subspaces in $\mathscr{H}$. With the set inclusion ordering, the set $\mathscr{C}(\mathscr{H})$ is a complete lattice. The operation $M \mapsto M^{\perp}$ is an orthocomplementation on the lattice $\mathscr{C}(\mathscr{H})$, with which $\mathscr{C}(\mathscr{H})$ is a complete orthomodular lattice. Denote by $\mathscr{B}(\mathscr{H})$ the algebra of bounded linear operators on $\mathscr{H}$ and $\mathscr{Q}(\mathscr{H})$ the set of projections on $\mathscr{H}$. For any $M \in \mathscr{C}(\mathscr{H})$, denote by $\mathscr{P}(M) \in \mathscr{Q}(\mathscr{H})$ the projection operator of $\mathscr{H}$ onto M. Then, $M \le N$ if and only if $\mathscr{P}(M) \subseteq \mathscr{P}(N)$ for any $M,N \in \mathscr{C}(\mathscr{H})$, and $\mathscr{Q}(\mathscr{H})$ with the operator ordering is a complete orhtomodular lattice isomorphic to $\mathscr{C}(\mathscr{H})$.

Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$. We denote by $\mathscr{A}'$ the *commutant of* $\mathscr{A}$ *in* $\mathscr{B}(\mathscr{H})$. A self-adjoint subalgebra $\mathscr{M}$ of $\mathscr{B}(\mathscr{H})$ is called a *von Neumann algebra* on $\mathscr{H}$ iff $\mathscr{M}'' = \mathscr{M}$. We denote by $\mathscr{P}(\mathscr{M})$ the set of projections in a von Neumann algebra $\mathscr{M}$. For any $P,Q \in \mathscr{Q}(\mathscr{H})$, we have $P \,\rfloor\, Q$ iff $[P,Q] = 0$, where $[P,Q] = PQ - QP$. For any subset $\mathscr{A} \subseteq \mathscr{Q}(\mathscr{H})$, we denote by $\mathscr{A}^{!}$ the *commutant* of $\mathscr{A}$ in $\mathscr{Q}(\mathscr{H})$. A *logic* on $\mathscr{H}$ is a sublogic of $\mathscr{Q}(\mathscr{H})$. For any subset $\mathscr{A} \subseteq \mathscr{Q}(\mathscr{H})$, the smallest logic including $\mathscr{A}$ is the logic $\mathscr{A}^{!!}$ called the *logic generated by* $\mathscr{A}$. Then, a subset $\mathscr{Q} \subseteq \mathscr{Q}(\mathscr{H})$ is a logic on $\mathscr{H}$ if and only if $\mathscr{Q} = \mathscr{P}(\mathscr{M})$ for some von Neumann algebra $\mathscr{M}$ on $\mathscr{H}$ [11].

We define the *implication* and the *logical equivalence* on $\mathscr{Q}$ by $P \to Q = P^{\perp} \vee (P \wedge Q)$ and $P \leftrightarrow Q = (P \to Q) \wedge (Q \to P)$. We have the following characterization of commutators in logics on Hilbert spaces [11].

Theorem 2. *Let $\mathscr{Q}$ be a logic on $\mathscr{H}$. For any subset $\mathscr{A} \subseteq \mathscr{Q}$, we have*

$$\mathrm{com}(\mathscr{A}) = \mathscr{P}\{\psi \in \mathscr{H} \mid [A,B]\psi = 0 \text{ for all } A,B \in \mathscr{A}''\}.$$

4 Quantum set theory

We denote by V the universe of the Zermelo-Fraenkel set theory with the axiom of choice (ZFC). Let $\mathscr{L}(\in)$ be the language for first-order theory with equality having a binary relation symbol $\in$, bounded quantifier symbols $\forall x \in y$, $\exists x \in y$, and no constant symbols. For any class U, the language $\mathscr{L}(\in,U)$ is the one obtained by adding a name for each element of U.

Let $\mathscr{Q}$ be a logic on $\mathscr{H}$. For each ordinal α, let

$$V_{\alpha}^{(\mathscr{Q})} = \{u \mid u : \mathscr{D}(u) \to \mathscr{Q} \text{ and } (\exists \beta < \alpha)\mathscr{D}(u) \subseteq V_{\beta}^{(\mathscr{Q})}\}.$$

The *$\mathscr{Q}$-valued universe* $V^{(\mathscr{Q})}$ is defined by

$$V^{(\mathscr{Q})} = \bigcup_{\alpha \in \mathrm{On}} V_{\alpha}^{(\mathscr{Q})},$$

where On is the class of all ordinals. For every $u \in V^{(\mathscr{Q})}$, the rank of u, denoted by $\mathrm{rank}(u)$, is defined as the least α such that $u \in V^{(\mathscr{Q})}_{\alpha+1}$. It is easy to see that if $u \in \mathscr{D}(v)$ then $\mathrm{rank}(u) < \mathrm{rank}(v)$.

For any $u, v \in V^{(\mathscr{Q})}$, the $\mathscr{Q}$-valued truth values of atomic formulas $u = v$ and $u \in v$ are assigned by by the following rules recursive in rank.

(i) $[\![u = v]\!]_{\mathscr{Q}} = \bigwedge_{u' \in \mathscr{D}(u)} (u(u') \to [\![u' \in v]\!]_{\mathscr{Q}}) \wedge \bigwedge_{v' \in \mathscr{D}(v)} (v(v') \to [\![v' \in u]\!]_{\mathscr{Q}})$.

(ii) $[\![u \in v]\!]_{\mathscr{Q}} = \bigvee_{v' \in \mathscr{D}(v)} (v(v') \wedge [\![u = v']\!]_{\mathscr{Q}})$.

To each statement ϕ of $\mathscr{L}(\in, V^{(\mathscr{Q})})$ we assign the $\mathscr{Q}$-valued truth value $[\![\phi]\!]_{\mathscr{Q}}$ by the following rules.

(iii) $[\![\neg\phi]\!]_{\mathscr{Q}} = [\![\phi]\!]_{\mathscr{Q}}^{\perp}$.

(iv) $[\![\phi_1 \wedge \phi_2]\!]_{\mathscr{Q}} = [\![\phi_1]\!]_{\mathscr{Q}} \wedge [\![\phi_2]\!]_{\mathscr{Q}}$.

(v) $[\![\phi_1 \vee \phi_2]\!]_{\mathscr{Q}} = [\![\phi_1]\!]_{\mathscr{Q}} \vee [\![\phi_2]\!]_{\mathscr{Q}}$.

(vi) $[\![\phi_1 \to \phi_2]\!]_{\mathscr{Q}} = [\![\phi_1]\!]_{\mathscr{Q}} \to [\![\phi_2]\!]_{\mathscr{Q}}$.

(vii) $[\![\phi_1 \leftrightarrow \phi_2]\!]_{\mathscr{Q}} = [\![\phi_1]\!]_{\mathscr{Q}} \leftrightarrow [\![\phi_2]\!]_{\mathscr{Q}}$.

(viii) $[\![(\forall x \in u)\, \phi(x)]\!]_{\mathscr{Q}} = \bigwedge_{u' \in \mathscr{D}(u)} (u(u') \to [\![\phi(u')]\!]_{\mathscr{Q}})$.

(ix) $[\![(\exists x \in u)\, \phi(x)]\!]_{\mathscr{Q}} = \bigvee_{u' \in \mathscr{D}(u)} (u(u') \wedge [\![\phi(u')]\!]_{\mathscr{Q}})$.

(x) $[\![(\forall x)\, \phi(x)]\!]_{\mathscr{Q}} = \bigwedge_{u \in V^{(\mathscr{Q})}} [\![\phi(u)]\!]_{\mathscr{Q}}$.

(xi) $[\![(\exists x)\, \phi(x)]\!]_{\mathscr{Q}} = \bigvee_{u \in V^{(\mathscr{Q})}} [\![\phi(u)]\!]_{\mathscr{Q}}$.

We say that a statement ϕ of $\mathscr{L}(\in, V^{(\mathscr{Q})})$ *holds* in $V^{(\mathscr{Q})}$ if $[\![\phi]\!]_{\mathscr{Q}} = 1$. A formula in $\mathscr{L}(\in)$ is called a Δ_0-*formula* if it has no unbounded quantifiers $\forall x$ or $\exists x$. The following theorem holds [11].

Theorem 3 (Δ_0-Absoluteness Principle)**.** *For any Δ_0-formula $\phi(x_1, \ldots, x_n)$ of $\mathscr{L}(\in)$ and $u_1, \ldots, u_n \in V^{(\mathscr{Q})}$, we have*

$$[\![\phi(u_1, \ldots, u_n)]\!]_{\mathscr{Q}} = [\![\phi(u_1, \ldots, u_n)]\!]_{\mathscr{Q}(\mathscr{H})}.$$

Henceforth, for any Δ_0-formula $\phi(x_1, \ldots, x_n)$ and $u_1, \ldots, u_n \in V^{(\mathscr{Q})}$, we abbreviate $[\![\phi(u_1, \ldots, u_n)]\!] = [\![\phi(u_1, \ldots, u_n)]\!]_{\mathscr{Q}}$, which is the common $\mathscr{Q}$-valued truth value in all $V^{(\mathscr{Q})}$ such that $u_1, \ldots, u_n \in V^{(\mathscr{Q})}$.

The universe V can be embedded in $V^{(\mathscr{Q})}$ by the following operation $\vee : v \mapsto \check{v}$ defined by the $\in$-recursion: for each $v \in V$, $\check{v} = \{\check{u} \mid u \in v\} \times \{1\}$. Then we have the following [11].

Theorem 4 (Δ_0-Elementary Equivalence Principle)**.** *For any Δ_0-formula $\phi(x_1, \ldots, x_n)$ of $\mathscr{L}(\in)$ and $u_1, \ldots, u_n \in V$, we have $\langle V, \in \rangle \models \phi(u_1, \ldots, u_n)$ if and only if $[\![\phi(\check{u}_1, \ldots, \check{u}_n)]\!] = 1$.*

For $u \in V^{(\mathscr{Q})}$, we define the *support* of u, denoted by $L(u)$, by transfinite recursion on the rank of u by the relation

$$L(u) = \bigcup_{x \in \mathscr{D}(u)} L(x) \cup \{u(x) \mid x \in \mathscr{D}(u)\}.$$

For $\mathscr{A} \subseteq V^{(\mathscr{Q})}$ we write $L(\mathscr{A}) = \bigcup_{u \in \mathscr{A}} L(u)$ and for $u_1, \ldots, u_n \in V^{(\mathscr{Q})}$ we write $L(u_1, \ldots, u_n) = L(\{u_1, \ldots, u_n\})$. Let $\mathscr{A} \subseteq V^{(\mathscr{Q})}$. The *commutator of $\mathscr{A}$*, denoted by $\mathrm{com}(\mathscr{A})$, is defined by

$$\mathrm{com}(\mathscr{A}) = \mathrm{com}(L(\mathscr{A})).$$

For any $u_1, \ldots, u_n \in V^{(\mathscr{Q})}$, we write $\mathrm{com}(u_1, \ldots, u_n) = \mathrm{com}(\{u_1, \ldots, u_n\})$. For bounded theorems, the following transfer principle holds [11].

Theorem 5 (Δ_0-ZFC Transfer Principle)**.** *For any Δ_0-formula $\phi(x_1, \ldots, x_n)$ of $\mathscr{L}(\in)$ and $u_1, \ldots, u_n \in V^{(\mathscr{Q})}$, if $\phi(x_1, \ldots, x_n)$ is provable in ZFC, then we have*

$$\mathrm{com}(u_1, \ldots, u_n) \le [\![\phi(u_1, \ldots, u_n)]\!].$$

5 Real numbers in quantum set theory

Let $\mathbf{Q}$ be the set of rational numbers in V. We define the set of rational numbers in the model $V^{(\mathscr{Q})}$ to be $\check{\mathbf{Q}}$. We define a real number in the model by a Dedekind cut of the rational numbers. More precisely, we identify a real number with the upper segment of a Dedekind cut assuming that the lower segment has no end point. Therefore, the formal definition of the predicate $\mathbf{R}(x)$, "x is a real number," is expressed by

$$\mathbf{R}(x) := \forall y \in x(y \in \check{\mathbf{Q}}) \wedge \exists y \in \check{\mathbf{Q}}(y \in x) \wedge \exists y \in \check{\mathbf{Q}}(y \notin x) \wedge \forall y \in \check{\mathbf{Q}}(y \in x \leftrightarrow \forall z \in \check{\mathbf{Q}}(y < z \rightarrow z \in x)).$$

The symbol ":=" is used to define a new formula, here and hereafter. We define $\mathbf{R}^{(\mathscr{Q})}$ to be the interpretation of the set $\mathbf{R}$ of real numbers in $V^{(\mathscr{Q})}$ as follows.

$$\mathbf{R}^{(\mathscr{Q})} = \{u \in V^{(\mathscr{Q})} |\ \mathscr{D}(u) = \mathscr{D}(\check{\mathbf{Q}}) \text{ and } [\![\mathbf{R}(u)]\!] = 1\}.$$

The set $\mathbf{R}_{\mathscr{Q}}$ of real numbers in $V^{(\mathscr{Q})}$ is defined by

$$\mathbf{R}_{\mathscr{Q}} = \mathbf{R}^{(\mathscr{Q})} \times \{1\}.$$

For any $u, v \in \mathbf{R}^{(\mathscr{Q})}$, Then, the following relations hold in $V^{(\mathscr{Q})}$ [11].

(i) $[\![(\forall u \in \mathbf{R}_{\mathscr{Q}})u = u]\!] = 1$.

(ii) $[\![(\forall u, v \in \mathbf{R}_{\mathscr{Q}})u = v \rightarrow v = u]\!] = 1$.

(iii) $[\![(\forall u, v, w \in \mathbf{R}_{\mathscr{Q}})u = v \wedge v = w \rightarrow u = w]\!] = 1$.

(iv) $[\![(\forall v \in \mathbf{R}_{\mathscr{Q}})(\forall x, y \in v)x = y \wedge x \in v \rightarrow y \in v]\!]$.

(v) $[\![(\forall u, v \in \mathbf{R}_{\mathscr{Q}})(\forall x \in u)x \in u \wedge u = v \rightarrow x \in v]\!]$.

From the above, the equality is an equivalence relation between real numbers in $V^{(\mathscr{Q})}$. For any $u_1, \dots, u_n \in \mathbf{R}^{(\mathscr{Q})}$, we have

$$[\![u_1 = u_2 \wedge \cdots \wedge u_{n-1} = u_n]\!] \le \mathrm{com}(u_1, \dots, u_n),$$

and hence commutativity follows from equality in $\mathbf{R}^{(\mathscr{Q})}$ [11].

Let $\mathscr{M}$ be a von Neumann algebra on a Hilbert space $\mathscr{H}$ and let $\mathscr{Q} = \mathscr{P}(\mathscr{M})$. A closed operator A (densely defined) on $\mathscr{H}$ is said to be *affiliated* with $\mathscr{M}$, in symbols $A\,\eta\,\mathscr{M}$, iff $U^*AU = A$ for any unitary operator $U \in \mathscr{M}'$. Let A be a self-adjoint operator (densely defined) on $\mathscr{H}$ and let $A = \int_{\mathbf{R}} \lambda\, dE^A(\lambda)$ be its spectral decomposition, where $\{E^A(\lambda)\}_{\lambda \in \mathbf{R}}$ is the resolution of identity belonging to A [7, p. 119]. It is well-known that $A\,\eta\,\mathscr{M}$ if and only if $E^A(\lambda) \in \mathscr{Q}$ for every $\lambda \in \mathbf{R}$. Denote by $\overline{\mathscr{M}}_{SA}$ the set of self-adjoint operators affiliated with $\mathscr{M}$. Two self-adjoint operators A and B are said to *commute*, in symbols $A \,\rotatebox[origin=c]{180}{$!$}\, B$, iff $E^A(\lambda) \,\rotatebox[origin=c]{180}{$!$}\, E^B(\lambda')$ for every pair λ, λ' of reals.

Let $\mathscr{B}$ be a Boolean logic on $\mathscr{H}$. For any $u \in \mathbf{R}^{(\mathscr{B})}$ and $\lambda \in \mathbf{R}$, we define $E^u(\lambda)$ by

$$E^u(\lambda) = \bigwedge_{\lambda < r \in \mathbf{Q}} u(\check{r}).$$

Then, it can be shown that $\{E^u(\lambda)\}_{\lambda \in \mathbf{R}}$ is a resolution of identity in $\mathscr{B}$ and hence by the spectral theorem there is a self-ajoint operator $\hat{u}\,\eta\,\mathscr{B}''$ uniquely satisfying $\hat{u} = \int_{\mathbf{R}} \lambda\, dE^u(\lambda)$. On the other hand, let $A\,\eta\,\mathscr{B}''$ be a self-ajoint operator. We define $\tilde{A} \in V^{(\mathscr{B})}$ by

$$\mathscr{D}(\tilde{A}) = \mathscr{D}(\check{\mathbf{Q}}) \text{ and } \tilde{A}(\check{r}) = E^A(r) \text{ for all } r \in \mathbf{Q}.$$

Then, it is easy to see that $\tilde{A} \in \mathbf{R}^{(\mathscr{B})}$ and we have $(\hat{u})\tilde{} = u$ for all $u \in \mathbf{R}^{(\mathscr{B})}$ and $(\tilde{A})\hat{} = A$ for all $A \in \overline{(\mathscr{B}'')}_{SA}$. Therefore, the correspondence between $\mathbf{R}^{(\mathscr{B})}$ and $\overline{(\mathscr{B}'')}_{SA}$ is a one-to-one correspondence. We call the above correspondence the *Takeuti correspondence*. Now, we have the following [11].

Theorem 6. *Let $\mathcal{Q}$ be a logic on $\mathcal{H}$. The relations*

(i) $E^A(\lambda) = \bigwedge_{\lambda < r \in \mathbf{Q}} u(\check{r})$ *for all* $\lambda \in \mathbf{Q}$,

(ii) $u(\check{r}) = E^A(r)$ *for all* $r \in \mathbf{Q}$,

for all $u = \tilde{A} \in \mathbf{R}^{(\mathcal{Q})}$ *and* $A = \hat{u} \in \overline{(\mathcal{Q}'')}_{SA}$ *sets up a one-to-one correspondence between* $\mathbf{R}^{(\mathcal{Q})}$ *and* $\overline{(\mathcal{Q}'')}_{SA}$.

6 Standard probabilistic interpretation

Let $\mathcal{H}$ be a Hilbert space describing a quantum system **S**. For the system **S**, the *observables* are defined as self-adjoint operators on $\mathcal{H}$, the *states* are defined as density operators, and a *vector state* ψ is identified with the state $|\psi\rangle\langle\psi|$. We denote by $\mathcal{O}(\mathcal{H})$ the set of observables, by $\mathcal{S}(\mathcal{H})$ the space of density operators, and by $\mathcal{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$. Observables $X_1,\dots,X_n \in \mathcal{O}(\mathcal{H})$ are said to be *mutually commuting* if $X_j \,\natural\, X_k$ for all $j,k = 1,\dots,n$. If $X_1,\dots,X_n \in \mathcal{O}(\mathcal{H})$ are bounded, this condition is equivalent to $[X_j, X_k] = 0$ for all $j,k = 1,\dots,n$. The standard probabilistic interpretation of quantum theory defines the *joint probability distribution function* $F_\rho^{X_1,\dots,X_n}(x_1,\dots,x_n)$ for mutually commuting observables $X_1,\dots,X_n \in \mathcal{O}(\mathcal{H})$ in $\rho \in \mathcal{S}(\mathcal{H})$ by the *Born statistical formula*:

$$F_\rho^{X_1,\dots,X_n}(x_1,\dots,x_n) = \mathrm{Tr}[E^{X_1}(x_1)\cdots E^{X_n}(x_n)\rho].$$

To clarify the logical structure presupposed in the standard probabilistic interpretation, we define *observational propositions* for **S** by the following rules.

(R1) For any $X \in \mathcal{O}(\mathcal{H})$ and $x \in \mathbf{R}$, the expression $X \le x$ is an observational proposition.

(R2) If ϕ_1 and ϕ_2 are observational propositions, $\neg\phi_1$ and $\phi_1 \wedge \phi_2$ are also observational propositions.

Thus, every observational proposition is built up from "atomic" observational propositions $X \le x$ by means of the connectives $\neg$ and $\wedge$. We introduce the connective $\vee$ by definition.

(D1) $\phi_1 \vee \phi_2 := \neg(\neg\phi_1 \wedge \neg\phi_2)$.

For each observational proposition ϕ, we assign its projection-valued truth value $[\![\phi]\!]_o \in \mathcal{Q}(\mathcal{H})$ by the following rules [1].

(T1) $[\![X \le x]\!]_o = E^X(x)$.

(T2) $[\![\neg\phi]\!]_o = [\![\phi]\!]_o^\perp$.

(T3) $[\![\phi_1 \wedge \phi_2]\!]_o = [\![\phi_1]\!]_o \wedge [\![\phi_2]\!]_o$.

From (D1), (T2) and (T3), we have

(D2) $[\![\phi_1 \vee \phi_2]\!]_o = [\![\phi_1]\!]_o \vee [\![\phi_2]\!]_o$.

We define the *probability* $\Pr\{\phi\|\rho\}$ of an observational proposition ϕ in a state ρ by

(P1) $\Pr\{\phi\|\rho\} = \mathrm{Tr}[[\![\phi]\!]_o\rho]$.

We say that *an observational proposition ϕ holds in a state ρ* if $\Pr\{\phi\|\rho\} = 1$.

The standard interpretation of quantum theory restricts observational propositions to be standard defined as follows.

(W1) An observational proposition including atomic formulas $X_1 \le x_1,\dots,X_n \le x_n$ is called *standard* if $X_1,\dots,X_n$ are mutually commuting.

All the standard observational propositions including only given mutually commuting observables $X_1,\dots,X_n$ comprise a complete Boolean algebra under the logical order $\le$ defined by $\phi \le \phi'$ iff $[\![\phi]\!]_o \le [\![\phi']\!]_o$ and obey inference rules in classical logic. Suppose that $X_1,\dots,X_n \in \mathscr{O}(\mathscr{H})$ are mutually commuting. Let $x_1,\dots,x_n \in \mathbf{R}$. Then, $X_1 \le x_1 \wedge \cdots \wedge X_n \le x_n$ is a standard observational proposition. We have

$$[\![X_1 \le x_1 \wedge \cdots \wedge X_n \le x_n]\!]_o = E^{X_1}(x_1) \wedge \cdots \wedge E^{X_n}(x_n) = E^{X_1}(x_1)\cdots E^{X_n}(x_n).$$

Hence, we reproduce the Born statistical formula as

$$\Pr\{X_1 \le x_1 \wedge \cdots \wedge X_n \le x_n \| \rho\} = \mathrm{Tr}[E^{X_1}(x_1)\cdots E^{X_n}(x_n)\rho].$$

From the above, our definition of the truth vales of observational propositions are consistent with the standard probabilistic interpretation of quantum theory.

In order to make the counter part of $r \in \mathbf{R}$ in $\mathbf{R}^{(\mathscr{Q})}$, for any $r \in \mathbf{R}$, we define $\tilde{r} \in \mathbf{R}^{(\mathscr{Q})}$ by

$$\mathscr{D}(\tilde{r}) = \mathscr{D}(\check{\mathbf{Q}}) \quad \text{and} \quad \tilde{r}(\check{t}) = [\![\check{r} \le \check{t}]\!]$$

for all $t \in \mathbf{Q}$. Then, $\tilde{r} \in \mathbf{R}^{(\mathscr{Q})}$ corresponds to $(\widetilde{r1})$, where 1 is the identity operator, under the Takeuti correspondence.

For every observational proposition ϕ the corresponding statement $\tilde{\phi}$ in $\mathscr{L}(\in,\mathbf{R}^{(\mathscr{Q})})$ is given by the following rules for any $X \in \mathscr{O}(\mathscr{H})$, $x \in \mathbf{R}$, and observational propositions ϕ,ϕ_1,ϕ_2.

(Q1) $\widetilde{X \le x} := \tilde{X} \le \tilde{x}$.

(Q2) $\widetilde{\neg\phi} := \neg\tilde{\phi}$.

(Q3) $\widetilde{\phi_1 \wedge \phi_2} := \tilde{\phi}_1 \wedge \tilde{\phi}_2$.

Then, it is easy to see that the relation

$$[\![\tilde{\phi}]\!] = [\![\phi]\!]_o$$

holds for any observational proposition ϕ. Thus, all the observational propositions are embedded in statements in $\mathscr{L}(\in,\mathbf{R}^{(\mathscr{Q})})$ with the same projection-valued truth value assignments.

Let $E^X(\lambda)$ be the resolution of identity belonging to $X \in \mathscr{O}(\mathscr{H})$. Let $a < b \in \mathbf{R}$. For the interval $I = (a,b]$, we define $E^X(I) = E^X(b) - E^X(a)$, and we define the corresponding interval $\tilde{I}$ of real numbers in $V^{(\mathscr{Q})}$ by $\mathscr{D}(\tilde{I}) = \mathbf{R}^{(\mathscr{Q})}$ and $\tilde{I}(u) = [\![u \le \tilde{a}]\!]^{\perp} \wedge [\![u \le \tilde{b}]\!]$ for all $u \in \mathbf{R}^{(\mathscr{Q})}$. Then, we have $[\![\tilde{X} \in \tilde{I}]\!] = E^X(I)$. The observational proposition $X \in I$, which will be also denoted by $a < X \le b$, is defined as

$$X \in I := \neg(a \le X) \wedge (X \le b).$$

Then, we have $[\![X \in I]\!]_o = [\![\tilde{X} \in \tilde{I}]\!]$. For mutually commuting observables $X_1,\dots,X_n \in \mathscr{O}(\mathscr{H})$ and intervals $I_1 = (a_1,b_1],\dots,I_n = (a_n.b_n]$ we have

$$\Pr\{X_1 \in I_1 \wedge \cdots \wedge X_n \in I_n \| \rho\} = \mathrm{Tr}[E^{X_1}(I_1)\cdots E^{X_n}(I_n)\rho].$$

7 Joint determinateness

Let $\mathscr{O}_\omega(\mathscr{H})$ be the set of observables on $\mathscr{H}$ with finite spectra. An observable $X \in \mathscr{O}(\mathscr{H})$ is said to be *finite* if $X \in \mathscr{O}_\omega(\mathscr{H})$, and *infinite* otherwise. Let $X \in \mathscr{O}_\omega(\mathscr{H})$. Let $\delta(X) = \min_{x,y \in \mathrm{Sp}(X), x \ne y}\{|x-y|/2, 1\}$. For any $x \in \mathbf{R}$, we define the formula $X = x$ by

$$X = x := x - \delta(X) < X \le x + \delta(X).$$

Then, it is easy to see that we have

$$[\![X = x]\!]_o = \mathscr{P}\{\psi \in \mathscr{H} \mid X\psi = x\psi\}$$

for all $x \in \mathbf{R}$.

For observational propositions $\phi_1, \dots, \phi_n$, we define the observational proposition $\bigvee_j \phi_j$ by $\bigvee_j \phi_j = \phi_1 \vee \cdots \vee \phi_n$. We denote by $\mathrm{Sp}(X)$ the spectrum of an observable $X \in \mathscr{O}(\mathscr{H})$. For any finite observables $X_1, \dots, X_n \in \mathscr{O}_\omega(\mathscr{H})$ we define the observational proposition $\underline{\vee}(X_1, \dots, X_n)$ by

$$\underline{\vee}(X_1, \dots, X_n) := \bigvee_{x_1 \in \mathrm{Sp}(X_1), \dots, x_n \in \mathrm{Sp}(X_n)} X_1 = x_1 \wedge \cdots \wedge X_n = x_n.$$

We say that observables $X_1, \dots, X_n$ are *jointly determinate* in a state ρ if the observational proposition $\underline{\vee}(X_1, \dots, X_n)$ holds in ρ. In general, we say that observables $X_1, \dots, X_n$ are *jointly determinate* in a state ρ with probability $\Pr\{\underline{\vee}(X_1, \dots, X_n) \| \rho\}$. Then, we have the following [12].

Theorem 7. *Finite observables $X_1, \dots, X_n \in \mathscr{O}_\omega(\mathscr{H})$ are jointly determinate in a vector state ψ if and only if the state ψ is a superposition of common eigenvectors of $X_1, \dots, X_n$.*

The joint determinateness is characterized by the commutator in quantum set theory as follows.

Theorem 8. *For any finite observables $X_1, \dots, X_n \in \mathscr{O}_\omega(\mathscr{H})$, we have*

$$[\![\underline{\vee}(X_1, \dots, X_n)]\!]_o = \mathrm{com}(\tilde{X}_1, \dots, \tilde{X}_n). \tag{1}$$

For self-adjoint operators $A_1, \dots, A_n$ on $\mathscr{H}$, the *von Neumann algebra generated by* $A_1, \dots, A_n$, denoted by $\{A_1, \dots, A_n\}''$, is the von Neumann algebra generated by projections $E^{A_j}(x)$ for all $j = 1, \dots, n$ and $x \in \mathbf{R}$. Under the Takeuti correspondence, the commutator of quantum reals are characterized as follows.

Theorem 9. *Let $\mathscr{Q}$ be a logic on $\mathscr{H}$ and let $u_1, \dots, u_n \in \mathbf{R}^{(\mathscr{Q})}$. Then we have*

$$\mathrm{com}(u_1, \dots, u_n) = \mathscr{P}\{\psi \in \mathscr{H} \mid [A,B]\psi = 0 \text{ for all } A, B \in \{\hat{u}_1, \dots, \hat{u}_n\}''\}.$$

Although we cannot find an observational proposition $\underline{\vee}(X_1, \dots, X_n)$ satisfying Eq. (1) for infinite observables $X_1, \dots, X_n \in \mathscr{O}(\mathscr{H})$, we can introduce a new atomic observational propositions $\underline{\vee}(X_1, \dots, X_n)$ with Eq. (1) for all $X_1, \dots, X_n \in \mathscr{O}(\mathscr{H})$. We introduce the following additional rule for formation of observational propositions:

(R3) For any $X_1, \dots, X_n \in \mathscr{O}(\mathscr{H})$ and $x_1, \dots, x_n \in \mathbf{R}$, the expression $\underline{\vee}(X_1, \dots, X_n)$ is an observational proposition.

Moreover, we introduce the following additional rule for projection-valued truth values:

(T4) $[\![\underline{\vee}(X_1, \dots, X_n)]\!]_o = \mathscr{P}\{\psi \in \mathscr{H} \mid [A,B]\psi = 0 \text{ for all } A, B \in \{\hat{u}_1, \dots, \hat{u}_n\}''\}$.

From Theorem 9, Eq. (1) holds for any $X_1, \dots, X_n \in \mathscr{O}(\mathscr{H})$ under (T4). Thus, we naturally extend the notion of joint determinateness to arbitrary observables. We say that observables $X_1, \dots, X_n \in \mathscr{O}(\mathscr{H})$ are *jointly determinate* in a state ρ if $\Pr\{\underline{\vee}(X_1, \dots, X_n) \| \rho\} = 1$, or equivalently if $\mathrm{Tr}[\mathrm{com}(X_1, \dots, X_n)\rho] = 1$. It is easy to see that this condition is equivalent to that $[A,B]\rho = 0$ for all $A, B \in \{\tilde{X}_1, \dots, \tilde{X}_n\}''$.

A probability distribution function $F(x_1, \dots, x_n)$ on $\mathbf{R}^n$, is called a *joint probability distribution function* of $X_1, \dots, X_n \in \mathscr{O}(\mathscr{H})$ in $\rho \in \mathscr{S}(\mathscr{H})$ if

$$F(x_1, \dots, x_n) = \Pr\{X_1 \le x_1 \wedge \cdots \wedge X_n \le x_n \| \rho\}.$$

A joint probability distribution F of $X_1, \dots, X_n$ in ρ is unique, if any, and denoted by $F_\rho^{X_1, \cdots X_n}(x_1, \dots, x_n)$.

Since the joint determinateness is considered to be the state-dependent notion of commutativity, it is expected that the joint determinateness is equivalent to the state-dependent existence of the joint probability distribution function, as shown below.

Theorem 10. *Observables $X_1,\ldots,X_n \in \mathscr{O}(\mathscr{H})$ are jointly determinate in a state ρ if and only if there exists a joint probability distribution function $F_\rho^{X_1,\cdots X_n}(x_1,\ldots,x_n)$ of $X_1,\ldots,X_n$ in ρ. In this case, for any polynomial $p(f_1(X_1),\ldots,f_n(X_n))$ of observables $f_1(X_1),\ldots,f_n(X_n)$, where $f_1,\ldots,f_n$ are bounded Borel functions, we have*

$$\mathrm{Tr}[p(f_1(X_1),\ldots,f_n(X_n))\rho] = \int\cdots\int_{\mathbf{R}^n} p(f_1(x_1),\ldots,f_n(x_n))F_\rho^{X_1,\cdots X_n}(dx_1,\ldots,dx_n).$$

8 Quantum equality

For any finite observables X,Y, we define the observational proposition $X=Y$ by

$$X=Y := \bigvee_{x\in\mathrm{Sp}(X)} X=x \wedge Y=x.$$

We say that observables X and Y *are equal in a state* ρ if $X=Y$ holds in ρ. In this case, we shall write $X=_\rho Y$. In general, we say that observables X and Y *are equal in a state ρ with probability* $\Pr\{X=Y\|\rho\}$. Then, we have the following [12].

Theorem 11. *Finite observables X and Y are equal in a vector state ψ if and only if the state ψ is a superposition of common eigenvectors of X and Y with common eigenvalues.*

The state-dependent equality is characterized by the equality in quantum set theory as follows.

Theorem 12. *For any finite observables $X,Y \in \mathscr{O}_\omega(\mathscr{H})$, we have*

$$[\![X=Y]\!]_o = [\![\tilde{X}=\tilde{Y}]\!]. \tag{2}$$

Under the Takeuti correspondence, the truth values of equality between reals are characterized as follows.

Theorem 13. *Let $\mathscr{Q}$ be a logic on $\mathscr{H}$ and let $u,v \in \mathbf{R}^{(\mathscr{Q})}$. Then we have*

$$[\![u=v]\!] = \mathscr{P}\{\psi\in\mathscr{H} \mid E^{\hat{u}}(\lambda)\psi = E^{\hat{v}}(\lambda)\psi \text{ for all } \lambda\in\mathbf{R}\}.$$

We cannot find an observational proposition $X=Y$ satisfying Eq. (2) for infinite observables $X,Y \in \mathscr{O}(\mathscr{H})$. We introduce a new atomic observational propositions $X=Y$ with Eq. (2) for all $X,Y\in\mathscr{O}(\mathscr{H})$ by the following additional rules for formation of observational propositions and for projection-valued truth values:

(R4) For any $X,Y\in\mathscr{O}(\mathscr{H})$ and $x,y\in\mathbf{R}$, the expression $X=Y$ is an observational proposition.

(T5) $[\![X=Y]\!]_o = \mathscr{P}\{\psi\in\mathscr{H} \mid E^X(\lambda)\psi = E^Y(\lambda)\psi \text{ for all } \lambda\in\mathbf{R}\}$.

Note that from Theorem 13, Eq. (2) holds for any $X,Y\in\mathscr{O}(\mathscr{H})$. We say that observables X and Y are *equal* in a state ρ if $\Pr\{X=Y\|\rho\}=1$, or equivalently if $\mathrm{Tr}[[\![X=Y]\!]_o\rho]=1$. It is easy to see that this condition is equivalent to that $E^X(\lambda)\rho = E^Y(\lambda)\rho$ for all $\lambda\in\mathbf{R}$. Thus, we naturally extend the state-dependent notion of equality to arbitrary observables.

Theorem 14. *For any observables $X,Y\in\mathscr{O}_\omega(\mathscr{H})$ and $\rho\in\mathscr{S}(\mathscr{H})$, we have $X=_\rho Y$ if and only if there exists a joint probability distribution function $F_\rho^{X,Y}(x,y)$ of X,Y in ρ and it satisfies*

$$\iint_\Delta F_\rho^{X,Y}(x,y) = 1,$$

where Δ is the diagonal set $\Delta=\{(x,y)\in\mathbf{R}^2 \mid x=y\}$.

Let $\phi(X_1,\dots,X_n)$ be an observational proposition that includes symbols for observables only from the list $X_1,\dots,X_n$. Then, $\phi(X_1,\dots,X_n)$ is said to be *contextually well-formed* in a state ρ if $X_1,\dots,X_n$ are jointly determinate in ρ. The following theorem is an easy consequence from the transfer principle in quantum set theory [11], and shows that for well-formed observational propositions $\phi(X_1,\dots,X_n)$ for a fixed family $X_1,\dots,X_n$ of observables, the projection-valued truth value assignments satisfy Boolean inference rules and the probability assignments satisfy rules for calculus of classical probability.

Theorem 15. *If $\phi(X_1,\dots,X_n)$ is a tautology in classical logic, then we have*

$$[\![\underline{\vee}(X_1,\dots,X_n)]\!]_o \le [\![\phi(X_1,\dots,X_n)]\!]_o.$$

Moreover, if $\phi(X_1,\dots,X_n)$ is contextually well-formed in a state ρ, then $\phi(X_1,\dots,X_n)$ holds in ρ.

9 Measurements of observables

A *measuring process* for $\mathscr{H}$ is defined to be a quadruple $(\mathscr{K},\sigma,U,M)$ consisting of a Hilbert space $\mathscr{K}$, a state σ on $\mathscr{K}$, a unitary operator U on $\mathscr{H}\otimes\mathscr{K}$, and an observable M on $\mathscr{K}$ [8]. A measuring process $\mathbf{M}(\mathbf{x})=(\mathscr{K},\sigma,U,M)$ with *output variable* $\mathbf{x}$ describes a measurement carried out by an interaction, called the *measuring interaction*, from time 0 to time Δt between the measured system $\mathbf{S}$ described by $\mathscr{H}$ and the *probe* system $\mathbf{P}$ described by $\mathscr{K}$ that is prepared in the state σ at time 0. The outcome of this measurement is obtained by measuring the observable M, called the *meter observable*, in the probe at time Δt. The unitary operator U describes the time evolution of $\mathbf{S}+\mathbf{P}$ from time 0 to Δt. We shall write $M(0)=1\otimes M$, $M(\Delta t)=U^\dagger M(0)U$, $X(0)=X\otimes 1$, and $X(\Delta t)=U^\dagger X(0)U$ for any observable $X\in\mathscr{O}(\mathscr{H})$. We can use the probabilistic interpretation for the system $\mathbf{S}+\mathbf{P}$. The *output distribution* $\Pr\{\mathbf{x}\le x\|\rho\}$, the probability distribution function of the output variable $\mathbf{x}$ of this measurement on input state $\rho\in\mathscr{S}(\mathscr{H})$, is naturally defined as

$$\Pr\{\mathbf{x}\le x\|\rho\}=\Pr\{M(\Delta t)\le x\|\rho\otimes\sigma\}=\mathrm{Tr}[E^{M(\Delta t)}(x)\rho\otimes\sigma].$$

The POVM of the measuring process $\mathbf{M}(\mathbf{x})$ is defined by

$$\Pi(x)=\mathrm{Tr}_{\mathscr{K}}[E^{M(\Delta t)}(x)(I\otimes\sigma)].$$

Then, we have

(P1) $\lim_{x\to-\infty}\Pi(x)=0$, $\lim_{x\to+\infty}\Pi(x)=1$, and $\lim_{x_0\le x\to x_0}\Pi(x)=\Pi(x_0)$,

(P2) $\Pi(x')\le\Pi(x'')$ for $x'\le x''$,

(P3) $\Pr\{\mathbf{x}\le x\|\rho\}=\mathrm{Tr}[\Pi(x)\rho]$.

Conversely, it has been proved in Ref. [8] that for every $\{\Pi(x)\}_{x\in\mathbf{R}}$ satisfying (P1), and (P2), there is a measuring process $(\mathscr{K},\sigma,U,M)$ satisfying (P3).

Let $A\in\mathscr{O}(\mathscr{H})$ and $\rho\in\mathscr{S}(\mathscr{H})$. A measuring process $\mathbf{M}(\mathbf{x})=(\mathscr{K},\sigma,U,M)$ with the POVM $\Pi(x)$ is said to *measure* A in ρ if $A(0)=_{\rho\otimes\sigma}M(\Delta t)$, and *weakly measure* A in ρ if $\mathrm{Tr}[\Pi(x)E^A(y)\rho]=\mathrm{Tr}[E^A(\min\{x,y\})\rho]$ for any $x,y\in\mathbf{R}$. A measuring process $\mathbf{M}(\mathbf{x})$ is said to *satisfy the Born statistical formula* (BSF) for A in ρ if it satisfies $\Pr\{\mathbf{x}\le x\|\rho\}=\mathrm{Tr}[E^A(x)\rho]$ for all $x\in\mathbf{R}$. The following theorem characterizes measurements of an observable in a given state.

Theorem 16. *Let $\mathbf{M}(\mathbf{x})=(\mathscr{K},\sigma,U,M)$ be a measuring process for $\mathscr{H}$ with the POVM $\Pi(x)$. For any observable $A\in\mathscr{O}(\mathscr{H})$ and any state $\rho\in\mathscr{S}(\mathscr{H})$, the following conditions are all equivalent.*

(i) $\mathbf{M}(\mathbf{x})$ *measures A in ρ.*

(ii) $\mathbf{M}(\mathbf{x})$ *weakly measures A in ρ.*

(iii) $\mathbf{M}(\mathbf{x})$ *satisfies the BSF for A in any vector state $\psi \in \mathscr{C}(A,\rho)$.*

Theorem 17. *Let $\mathbf{M}(\mathbf{x}) = (\mathscr{K},\sigma,U,M)$ be a measuring process for $\mathscr{H}$ with the POVM $\Pi(x)$. Then, $\mathbf{M}(\mathbf{x})$ measures $A \in \mathscr{O}(\mathscr{H})$ in any $\rho \in \mathscr{S}(\mathscr{H})$ if and only if $\Pi(x) = E^A(x)$ for all $x \in \mathbf{R}$.*

10 Simultaneous measurability

For any measuring process $\mathbf{M}(\mathbf{x}) = (\mathscr{K},\sigma,U,M)$ and a real-valued Borel function f, the measuring process $\mathbf{M}(f(\mathbf{x}))$ with output variable $f(\mathbf{x})$ is defined by $\mathbf{M}(f(\mathbf{x})) = (\mathscr{K},\sigma,U,f(M))$. Observables A,B are said to be *simultaneously measurable* in a state $\rho \in \mathscr{S}(\mathscr{H})$ by $\mathbf{M}(\mathbf{x})$ if there are Borel functions f,g such that $\mathbf{M}(f(\mathbf{x}))$ and $\mathbf{M}(g(\mathbf{x}))$ measure A and B in ρ, respectively. Observables A,B are said to be *simultaneously measurable* in ρ if there is a measuring process $\mathbf{M}(\mathbf{x})$ such that A and B are simultaneously measurable in ρ by $\mathbf{M}(\mathbf{x})$.

The cyclic subspace $\mathscr{C}(A,B,\rho)$ of $\mathscr{H}$ generated by A,B and ρ is defined by

$$\mathscr{C}(A,B,\rho) = (\{A,B\}''\rho\mathscr{H})^{\perp\perp}.$$

We define $\mathscr{C}(A,\rho) = \mathscr{C}(A,1,\rho)$ and $\mathscr{C}(B,\rho) = \mathscr{C}(1,B,\rho)$.

The simultaneous measurability and the commutativity are not equivalent notion under the state-dependent formulation, as the following theorem clarifies.

Theorem 18. (i) *Two observables $A,B \in \mathscr{O}(\mathscr{H})$ are jointly determinate in a state $\rho \in \mathscr{S}(\mathscr{H})$ if and only if there exists a POVM $\Pi(x,y)$ on $\mathbf{R}^2$ satisfying*

$$\begin{aligned}\lim_{y\to+\infty}\Pi(x,y) &= E^A(x) \quad \text{on } \mathscr{C}(A,B,\rho) \text{ for all } x\in\mathbf{R},\\ \lim_{x\to+\infty}\Pi(x,y) &= E^B(y) \quad \text{on } \mathscr{C}(A,B,\rho) \text{ for all } y\in\mathbf{R}.\end{aligned}$$

(ii) *Two observables $A,B \in \mathscr{O}(\mathscr{H})$ are simultaneously measurable in a state $\rho \in \mathscr{S}(\mathscr{H})$ if and only if there exists a POVM $\Pi(x,y)$ on $\mathbf{R}^2$ satisfying*

$$\lim_{y\to+\infty}\Pi(x,y) = E^A(x) \quad \text{on } \mathscr{C}(A,\rho) \text{ for all } x\in\mathbf{R},$$
$$\lim_{x\to+\infty}\Pi(x,y) = E^B(y) \quad \text{on } \mathscr{C}(B,\rho) \text{ for all } y\in\mathbf{R}.$$

11 Conclusion

To formulate the standard probabilistic interpretation of quantum theory, we have introduced the language of observational propositions with rules (R1) and (R2) for well-formed formulas constructed from atomic formulas of the form $X \le x$, rules (T1), (T2), and (T3) for projection-valued truth value assignment, and rule (P1) for probability assignment. Then, the standard probabilistic interpretation gives the statistical predictions for standard observational propositions formulated by (W1), which concern only commuting family of observables. The Born statistical formula is naturally derived in this way. We have extended the standard interpretation by introducing two types of atomic formulas $\mathrm{com}(X_1,\dots,X_n)$ for joint determinateness and $X = Y$ for equality. To extended observational propositions formed through rules (R1), ..., (R4), the projection-valued truth values are assigned by rule (T1), ..., (T5), and the probability assignments are given by rule (P1). Then, we can naturally extend the standard interpretation to a general and state-dependent interpretation for observational propositions including the relations of joint determinateness and equality. Quantum set theory ensures that any contextually well-formed formula provable in ZFC has probability assignment to be 1. This extends the classical inference for quantum theoretical predictions from commuting observables to jointly determinate observables. We apply this new interpretation to construct a theory of measurement of observables and a theory of simultaneous measurement in the state-dependent approach, to which the standard interpretation cannot apply. We have reported only basic formulations here, but further development in this approach will be reported elsewhere.

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