A Bestiary of Sets and Relations

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Building on established literature and recent developments in the graph-theoretic characterisation of its CPM category, we provide a treatment of pure state and mixed state quantum mechanics in the category fRel of finite sets and relations. On the way, we highlight the wealth of exotic beasts that hide amongst the extensive operational and structural similarities that the theory shares with more traditional arenas of categorical quantum mechanics, such as the category fdHilb. We conclude our journey by proving that fRel is local, but not without some unexpected twists.

1 Introduction

The Categorical Quantum Mechanics programme [3] [8] [17] is concerned with the understanding, through the language of dagger symmetric monoidal categories, of the structural and operational features of quantum theory. The investigation of classical-quantum duality goes through the definition of classical structures, [9] [11] a.k.a. special commutative †-Frobenius algebras, which play a central role as the abstract incarnation of non-degenerate observables, and lie at the very heart of the paradigm. The dagger allows for an abstract definition of unitarity, while the CPM construction from [21] can be leveraged to rigorously define [12] [8] [10] discarding maps, mixed states, decoherence and measurements.

The extreme versatility of the approach has given birth, in the years, to many a toy model of quantum theory, a particularly interesting one being the category fRel of finite sets and relations The apparent classicality-by-construction of fRel contrasts starkly with the presence of many trademarks of quantum theory: superposition, entanglement, plenty of classical structures and phases. The full characterisation of classical structures in terms of abelian groupoids is known [19] [13] (and generalised in [16] to arbitrary groupoids and special †-Frobenius algebras), and provides a stimulating playground in which to stress-test several operational features [22] [14] [6] [5] [9] taken for granted in more the traditional arenas of quantum mechanics. fRel has recently found application in Natural Language Processing [20], where it relates to Montague Semantics; also in the context of NLP, its CPM category has been shown [18] to have a particularly handy graph-theoretic characterisation.

In this work, we give an overview of fRel as a model of categorical quantum mechanics: comparing and contrasting with the category fdHilb of Hilbert spaces and linear maps, we highlight the many similarities with the traditional framework, and a number of fairly traits unique to fRel. Using the newly developed [18] graph-theoretic characterisation of CPM[fRel], we explore the exotic landscape of mixed state quantum mechanics in fRel: we characterise decoherence maps and demolition measurements, and we manage to show that, despite the significant differences from traditional frameworks, the theory is local. This proof of locality, done with respect to operationally defined demolition measurements, extends and completes the one presented in [1], which only applies to measurements valued in the discrete classical structure (the only one with enough classical points). In order to improve the flow, we have omitted the proofs of results well established in the literature provided and/or straightforward to check if necessary.

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2 Pure state quantum mechanics in fRel

The category fRel is defined as having finite sets as its objects, relations $R \subseteq X \times Y$ as morphisms $X \to Y$ and relational composition. As fdHilb, it is a † symmetric monoidal category (henceforth †-SMC), with the cartesian product $\times$ of sets/relations as tensor, the singleton set $1 := \{\star\}$ as tensor unit and a dagger defined by:

$$R^\dagger := \{(y,x) \mid (x,y) \in R\} \tag{2.1}$$

The scalars of fRel are $\{\bot, \top\}$, with the tensor and unitors inducing multiplication $\times$ (or equivalently inf) on them. As fdHilb, the category fRel is enriched over finite monoids, with the tensor distributing over a superposition operation $\vee$, the union of relations.[1] The scalars form a semiring $(\{\bot,\top\}, \vee, \times)$, which opens up the doors for the application of methods from sheaf-theoretic non-locality. [2]

Morphisms $R : X \to Y$, seen as subsets $R \subseteq X \times Y$, form a sup-semilattice under $\vee$.[2] This applies in particular to states, which can be seen as subsets $\psi \subseteq X$: their hierarchical superposition structure, with $|X\rangle$ as maximum and the elements $|x\rangle$ for $x \in X$ as atoms, is the first significant difference from fdHilb. From now on, we will denote the elements of X by $x \in X$, and the states/subsets of X equivalently by $\psi \subseteq X$ or $|\psi\rangle : 1 \to X$.

Just as in fdHilb, the tensor of fRel is not a cartesian tensor in the categorical sense:[3] it is sufficient to observe that fRel has lots of entangled states (letting X and Y have n and m elements respectively, there are $2^{n+m-2}+1$ separable states of $X \times Y$, out of 2^{nm} states). The central role in CQM, however, is played by classical structures, rather than states, and we now move to their characterisation in fRel as abelian groupoids.

Definition 2.1. An **abelian groupoid** on a finite set X is a disjoint family $(G_\lambda, +_\lambda, 0_\lambda)_{\lambda \in \Lambda}$ of abelian groups such that $\vee_{\lambda \in \Lambda} G_\lambda = X$.

We will use notation $\oplus_{\lambda \in \Lambda} G_\lambda$ to denote one such abelian groupoid. Notice that any groupoid induces a partition of X into states $G_\lambda \subseteq X$: when one such groupoid (and hence partition) is understood, we will label the elements of X by g_λ, with λ ranging over Λ and g_λ ranging over G_λ. Now we can turn our attention to classical structures.

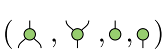
Theorem 2.2. *Classical structures in* fRel *coincide with abelian groupoids, i.e. if*  *is a classical structure on X, then there exists a unique abelian groupoid* $\oplus_{\lambda \in \Lambda} G_\lambda$ *such that:*

 $$= \left\{ ((g_\lambda, g'_\lambda), g_\lambda + g'_\lambda) \,\middle|\, g, g' \in G_\lambda, \lambda \in \Lambda \right\} \tag{2.2}$$

 $$= \{(\star, 0_\lambda) \mid \lambda \in \Lambda\} \tag{2.3}$$

 $$= \left\{ (g_\lambda, (g'_\lambda, g''_\lambda)) \,\middle|\, g, g', g'' \in G_\lambda, \lambda \in \Lambda \textit{ and } g'_\lambda + g''_\lambda = g_\lambda \right\} \tag{2.4}$$

 $$= \{(0_\lambda, \star) \mid \lambda \in \Lambda\} \tag{2.5}$$

Furthermore, the classical points of  *are exactly the states* $|G_\lambda\rangle : 1 \to X$*. The phases of*  *are exactly the states in the form* $\{(\star, g_\lambda) \mid \lambda \in \lambda\}$ *with* $g_\lambda \in G_\lambda$ *for all* λ *(so there are a lot more phases than classical points).*

[1] We will denote union and intersections of subets/relations by $\vee$ and $\wedge$, to avoid confusion with the cup $\cup$ and cap $\cap$ of the compact structure defined later. It also bodes well with potential generalisations from booleans to semirings or locales.

[2] In fact, they form a complete distributive boolean lattice, with intersection of relations $\wedge$ and complement of relations $\neg$.

[3] Despite being called the *cartesian product* of relations.

Proof. Proof of the classical structure / abelian groupoid connection is originally presented in [19]. The classical points are classified in [13]. To the best of our knowledge, the phase structure in fRel is first discussed in [15]. □

We see that  is a partial function acting as the group operation of G_λ whenever its two arguments both belong to the same group G_λ, and being undefined otherwise. In the case of strongly complementary structures things get particularly interesting: we present the result here (from [13]) for sake of completeness, but we will not concern ourselves with strong complementarity any further in this work, even though it is possible to use strong complementarity to implement Fourier transforms in fRel (as shown in [14]).

Theorem 2.3. *Let*  *be a pair of strongly complementary classical structures*[4] *in* fRel*. Then there exist unique groups* G, H *such that* $\color{green}{\bullet}$ *corresponds to the groupoid* $\oplus_{h\in H} G_h$ *with* $G_h \cong G$ *for all* $h \in H$ *and* $\color{red}{\bullet}$ *corresponds to the groupoid* $\oplus_{g\in G} H_g$ *with* $H_g \cong H$ *for all* $g \in G$.

By **morphisms of classical structures** we will mean homomorphisms of the comultiplications and counits. In any †-SMC, they act as functions on the sets of classical points, and just as in fdHilb there is a canonical way of seeing functions of the sets of classical points as morphisms of classical structures in fRel.

Given a pair of classical structures on sets X and Y corresponding to groupoids $\oplus_{\lambda\in\Lambda} G_\lambda$ and $\oplus_{\gamma\in\Gamma} H_\gamma$, and a partial function $f : \Lambda \rightharpoonup \Gamma$ of sets, we can construct the following morphism $R_f : X \to Y$ in fRel which is a morphism of the given classical structures and acts as the required partial function on the classical points:

$$R_f := \bigvee_{\lambda \in \mathrm{dom} f} G_\lambda \times H_{f(\lambda)} \tag{2.6}$$

This is the equivalent of the following morphism in fdHilb:

$$R_f := \sum_{\lambda \in \mathrm{dom} f} |f(\lambda)\rangle\langle\lambda| \tag{2.7}$$

So in fRel, exactly as in fdHilb, there is a natural way of doing classical computation by fixing classical structures and using the R_f above to construct the required morphisms. But unlike fdHilb, fRel has a lot more morphisms of classical structures than that. For example, if $f : \Lambda \xrightarrow{\mathrm{Set}} \Gamma$ is as before and $\Phi_\lambda : G_\lambda \xrightarrow{\mathrm{Ab}} H_{f(\lambda)}$ is a family of group isomorphisms, then the following relation (which is a partial function $X \to Y$) acts exactly as the relation R_f (which is, in general, not a function at all) on the desired classical points:

$$g_\lambda \mapsto \Phi_\lambda(g_\lambda) \tag{2.8}$$

This is a consequence of another, more fundamental difference between fRel and fdHilb: most classical structures do not have enough classical points. In fact there is a unique classical structure on each set X that does: it is the the **discrete structure**, given by the discrete groupoid $\oplus_{x\in X} 0_x$ and having the singletons $|x\rangle := \{x\}$ of X as its classical points. Indeed, the classical points of the discrete structure yield a resolution of the identity:

$$\bigvee_{x\in X} |x\rangle\langle x| = \{(x,x) \mid x \in X\} = id_X \tag{2.9}$$

[4]From now on we will use a dot of the structure colour to denote a classical structure.

When the classical structures on X and Y are the discrete structures, Equation 2.6 provides the usual embedding of the category of finite sets and partial functions in fRel:

$$R_f = \{(x, f(x)) \,|\, x \in \mathrm{dom}\, f\} \tag{2.10}$$

In any †-SMC, any classical structure on some system X can be used to induce a cup $\cup_X$ and cap $\cap_X$ on X: the discrete structures in fRel induce a natural family of cups and caps, and hence a compact-closed structure for fRel.

The cup on X is given by the relation $\cup_X := \{(\star, (x,x)) \,|\, x \in X\}$, while the cap is the partial map $\cap_X := X \times X \rightharpoonup 1$ sending (x,x) to $\star$ for all $x \in X$ and undefined everywhere else. The resulting conjugation is trivial, i.e. $R^\star = R$, and transposition coincides with the dagger, i.e. $R^T = R^\dagger$. This is somewhat different from fdHilb, where the traditional compact closed structure is not induced by any specific classical structure.

Finally, a central role in pure state quantum mechanics is also played by isometries and unitaries. Recall that an isometry in any †-SMC is a morphism $f : X \to Y$ such that $f^\dagger \circ f = id_X$, and a unitary is a morphism U such that both U and $U^\dagger$ are isometries. In fdHilb, unitaries coincide with orthonormal change-of-basis transformations, i.e. bijective classical maps. This results in the following (straightforward) lemma.

Lemma 2.4. *If $f : X \to Y$ is a morphism in* fdHilb*, then f is an isometry if and only if there are classical structures on X and Y making f into an injective classical map. Furthermore, f is a unitary if and only if it is a bijective classical map.*[5]

Proof. Classical structures (special commutative †-Frobenius algebras) in fdHilb correspond to orthonormal bases by [11]: let's fix the orthonormal basis corresponding to a classical structure on X and consider the matrix of f in that basis.

Then f is an isometry if and only if all column vectors are orthonormal, and any orthonormal basis including the column vectors as a subset will give a classical structure on Y making f an injective classical map.

Furthermore, f is unitary if and only if the column vectors form an orthonormal basis, corresponding to a unique classical structure on Y. □

In fRel, one could hope for unitaries that are isomorphisms between arbitrary classical structures (as it happens in fdHilb), giving rise to a non-trivial interplay. Unfortunately, the condition of isometry/unitarity turns out to be a lot more restrictive in fRel than it it in fdHilb, as the following lemma summarises.

Lemma 2.5. *If $f : X \to Y$ is a morphism in* fRel*, then f is an isometry if and only $f^\dagger$ is a surjective partial function. Equivalently, f is an isometry if and only if it is an injective classical map with respect to the discrete structure on X and some classical structure on Y. Furthermore, f is a unitary if and only if it is a bijection.*[6]

Proof. Let $f : X \to Y$ be a morphism in fRel, i.e. a relation $f \subseteq X \times Y$. The condition $f^\dagger \circ f = id_X$ amounts to the following equation:

$$\bigvee_{x \in \mathrm{dom}\, f} \{(x,z) \,|\, (x,y) \in f \text{ and } (z,y) \in f\} = \{(x,x) \,|\, x \in X\} \tag{2.11}$$

[5] In fact, if f is an isometry then fixing any classical structure on X means there is a classical structure on Y making f into an injective classical map. If f is furthermore a unitary, the structure on Y is also unique.

[6] Note that this is NOT the same as a bijective classical map between classical structures.

This is true if and only if both (i) $\text{dom} f = X$ (i.e. $f^\dagger$ is surjective) and (ii) $(x,y) \in f$ and $(z,y) \in f$ imply $x = z$ (i.e. $f^\dagger$ is a partial function). Furthermore f is unitary if and only if both f and $f^\dagger$ are surjective partial functions, which happens if and only if f is a bijection. If $f^\dagger$ is surjective partial function, then f can always be written in the following form, where $f^{-1}(x)$ are disjoint subsets of Y for all $x \in X$:

$$\bigvee_{x \in X} |f^{-1}(x)\rangle\langle x| \tag{2.12}$$

Then any classical structure on Y including all $|f^{-1}(x)\rangle$ amongst its classical points will make f into a classical injection from the discrete structure on X. □

Thus unitaries in fRel are exactly the bijective classical maps between discrete structures: pure state quantum mechanics in fRel suddenly becomes quite boring. Let's now proceed to mixed state quantum mechanics in CPM[fRel], in the hope that the peculiar measurement structure, resulting from the general lack of enough classical points, will spice things up.

Remark 2.6. One last point on the relationship between classical structures in fRel, before moving on to mixed state quantum mechanics. Classical structures on some space X in any †-SMC with a distributive superposition operation can be given a preorder by defining $\circ \leq \bullet$ if and only if classical points of $\bullet$ can be expressed as superpositions of classical points of $\circ$ (possibly multiplied by scalars).

In fdHilb, the preorder is an equivalence relation with a single equivalence class, as all structures have enough classical points, but in fRel this is not so. Recall that the classical structure induced by groupoid $\oplus_{\lambda \in \Lambda} G_\lambda$ on a set X yields a partition $(G_\lambda)_{\lambda \in \Lambda}$ of X: thus there is a surjective map of classical structures onto partitions, where classical structures corresponding to the same partition are exactly those having the same classical points.

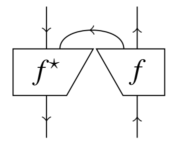
The surjection exactly quotients away the equivalence classes in the preorder, which therefore descends to the partial order (in fact a lattice) on partitions of X: this is given by the *refinement* partial order, with the partition into singletons as a minimum and the partition with only X as the maximum. The discrete structure is the only one mapping to the singleton partition, and is therefore the unique minimum for the preorder on classical structures.

3 Mixed state quantum mechanics in fRel

The fundamental observation of categorical quantum mechanics is that there are only a few ingredients needed for an abstract, operational characterisation of pure state quantum theory: states, a dagger for inner products, a (symmetric) tensor for joint systems, classical structures for classical computation, unitaries for dynamics, an optional enrichment of morphisms (with appropriate distributivity law for the tensor) giving some notion of superposition.

But an equally fundamental aspect of quantum theory, not immediately captured by this framework, is given by measurements and mixed states: in order to introduce them in fRel, we turn our attention to the associated CPM category CPM[fRel].

The CPM category CPM[fRel] has the same objects of fRel, and morphisms $R : X \xrightarrow{CPM} Y$ in CPM[fRel] are exactly the morphisms $R : X \times X \to Y \times Y$ in fRel taking the following form:



(3.1)

If $f : X \to Z \times Y$, then we denote $f^\star = (f^\dagger)^T = f$ in Diagram 3.1 as a morphism $f : X \to Y \times Z$ to keep the picture symmetric and avoid wire-crossing. We shall refer to morphisms in CPM[fRel] as **CPM maps**, to CPM maps $1 \stackrel{CPM}{\longrightarrow} X$ as **mixed states** in X, and to CPM morphisms with no cap involved (i.e. with $f : X \to 1 \times Y$) as **pure maps** (or **pure states**, if $X = 1$).

In particular, the caps are CPM maps $\cap_X : X \stackrel{CPM}{\longrightarrow} 1$ and the cups are mixed states $\cup_X : 1 \stackrel{CPM}{\longrightarrow} X$. Explicitly they are defined to be the following relations:

$$\begin{aligned} \cap_X &:= \{((x,x),\star) \mid x \in X\} \\ \cup_X &:= \{(\star,(x,x)) \mid x \in X\} \end{aligned} \tag{3.2}$$

We will refer to $\cap_X$ as the **discarding map** on X, to post-composition with $\cap_X$ as **tracing out** X, and to $\cup_X$ as the **totally mixed state** in X.

Then general CPM maps are exactly obtained by tracing out some factor of the codomain of some pure map, or equivalently by applying some map to a totally mixed state in some factor of the domain: the conceptual importance of this observation comes from the following theorem (proven in [21] and given operational interpretation in [12]).

Theorem 3.1. *There is a faithful functor $I : \mathscr{C} \rightarrowtail$ CPM[$\mathscr{C}$] of †-SMCs from any compact-closed †-SMC (with enough states) to the associated CPM Category* CPM[$\mathscr{C}$]*, itself a compact-closed †-SMC, bijective on objects and mapping morphisms $f : X \to Y$ in $\mathscr{C}$ to the respective pure maps given by $f : X \to 1 \times Y$ in the notation of Diagram 3.1. Thus the CPM construction can be seen, operationally, as the abstract process theory obtained from $\mathscr{C}$ by adding discarding maps (and/or totally mixed states).*

In this setting, tracing out systems is interpreted as complete erasure of information about them, so it is no surprise that information-theoretic considerations come into play. Indeed, we will be interested in a particular class of CPM maps and mixed states: we say that a CPM map $R : X \stackrel{CPM}{\longrightarrow} Y$ (or mixed state, if $X = 1$) is **causal** if $\cap_Y \cdot R = \cap_X$.

Causal CPM maps are one of the reason isometries (and unitaries in particular) in pure state quantum mechanics are so interesting: a CPM map in the form of Diagram 3.1 is causal if and only if $f : X \to Z \times Y$ is an isometry.[7]

In order to proceed with our investigation of CPM[fRel], it is time to introduce the characterisation of mixed states and CPM maps in terms of graphs. Although the material here is fruit of the author's own work, full credit for the original characterisation goes to [18], which contains all the details.

If $\rho : 1 \stackrel{CPM}{\longrightarrow} X$ is a mixed state, then it is straightforward to check that, seen as a relation $\rho \subseteq X \times X$, it has the following properties:

(i) ρ is a symmetric relation, i.e. if $(x,y) \in \rho$ then $(y,x) \in \rho$

(ii) ρ is reflexive on all elements appearing in it, i.e. if $(x,y) \in \rho$ then $(x,x) \in \rho$ and $(y,y) \in \rho$

[7] Observe that the cap on $Z \times Y$ is the product of the caps on Z and on Y.

Conversely, every relation with those properties is a mixed state. Furthermore, causal mixed states are exactly those with $\rho \neq \emptyset$. As a consequence of this characterisation, we can identify mixed states in X with subgraphs of the complete graph K_X with X as set of nodes. The subgraph $\mathscr{G}_\rho \leq \mathrm{K}_X$ corresponding to a mixed state ρ in X is defined to have:

(i) nodes $\bullet x$ specified by the pairs $(x,x) \in \rho$ (corresponding to a subset of X)

(ii) edges $x \bullet - \bullet\, y$ specified by the pairs $(x,y) \in \rho$ with $x \neq y$

Furthermore, causal states correspond to the non-empty subgraphs. The graph characterisation of CPM maps can be then obtained using compact closure. The subgraph $\mathscr{G}_R \leq \mathrm{K}_{X\times Y}$ corresponding to a CPM map $R : X \stackrel{CPM}{\longrightarrow} Y$ is seen to have:

(i) nodes $\bullet[x,y]$ specified by those pairs $x \in X$ and $y \in Y$ such that $\langle y|\langle y|R|x\rangle|x\rangle = 1$, where R is seen as a fRel morphism $R : X \times X \to Y \times Y$

(ii) edges $[x,y] \bullet - \bullet\, [x',y']$ specified by quadruplets $x,x' \in X$ and $y,y' \in Y$ with $\langle y|\langle y'|R|x\rangle|x'\rangle = 1$, where R is seen as an fRel morphism $R : X \times X \to Y \times Y$

Given a CPM morphism $R : X \stackrel{CPM}{\longrightarrow} Y$ and a mixed state ρ in X, it is interesting to characterise the subgraph $\mathscr{G}_{R\cdot\rho} \leq \mathrm{K}_Y$ of the mixed state $R\cdot\rho$ in terms of the subgraphs $\mathscr{G}_R \leq \mathrm{K}_{X\times Y}$ and $\mathscr{G}_\rho \leq \mathrm{K}_X$ of R and ρ:

(i) if $\bullet x$ is a node of $\mathscr{G}_\rho$ and $\bullet[x,y]$ is a node of $\mathscr{G}_R$, then $\bullet y$ is a node of $\mathscr{G}_{R\cdot\rho}$

(ii) if $x \bullet - \bullet\, x'$ is an edge of $\mathscr{G}_\rho$ and $[x,y] \bullet - \bullet\, [x',y']$ is an edge of $\mathscr{G}_R$, then $y \bullet - \bullet\, y'$ is an edge of $\mathscr{G}_{R\cdot\rho}$

A generalisation of the argument above can be used to characterise the subgraph $\mathscr{G}_{S\cdot R} \leq \mathrm{K}_{X\times Z}$ of the composition of two CPM maps $R : X \stackrel{CPM}{\longrightarrow} Y$ and $S : Y \stackrel{CPM}{\longrightarrow} Z$:

(i) if $\bullet[x,y]$ is a node of $\mathscr{G}_R$ and $\bullet[y,z]$ is a node of $\mathscr{G}_R$, then $\bullet[x,z]$ is a node of $\mathscr{G}_{S\cdot R}$

(ii) if $[x,y] \bullet - \bullet\, [x',y']$ is an edge of $\mathscr{G}_R$ and $[y,z] \bullet - \bullet\, [y',z']$ is an edge of $\mathscr{G}_R$, then $[x,z] \bullet - \bullet\, [x',z']$ is an edge of $\mathscr{G}_{S\cdot R}$

A few examples of CPM maps and mixed states will give us a hands-on understanding of this graph-theoretic characterisation.

1. If ρ is a pure state in X, corresponding to a fRel state $S \subseteq X$, then $\mathscr{G}_\rho \leq \mathrm{K}_X$ is the clique on S, because $\rho = \{(s,s') \mid s,s' \in S\}$; conversely, all cliques are pure states.

2. The pure map $id_X : X \stackrel{CPM}{\longrightarrow} X$, seen as the morphism $id_{X\times X} : X \times X \to X \times X$ in fRel, satisfies $\langle y|\langle y'|id_{X\times X}|x\rangle|x'\rangle = 1$ if and only if $x = y$ and $x' = y'$; as a consequence, the subgraph $\mathscr{G}_{id_X} \leq \mathrm{K}_{X\times X}$ is the clique on the diagonal $\Delta_X := \{(x,x) \mid x \in X\} \subset X \times X$.

3. The discarding map $\cap_X : X \stackrel{CPM}{\longrightarrow} 1$ corresponds to the discrete graph with node set $X \times 1 \cong X$ and no edges. Similarly, the totally mixed state in X corresponds to the discrete graph with node set X.

4. If G is a subgraph of $\mathrm{K}_{X\times Y}$, define $\pi_X\, G$ to be the projection of G on X, and $\pi_Y\, G$ to be the projection on Y. The subgraph $\mathscr{G}_{\cap_Y\cdot R} \leq \mathrm{K}_{X\times 1}$ of the composite $\cap_Y \cdot R$ for some CPM map $R : X \stackrel{CPM}{\longrightarrow} Y$ has exactly the same nodes as $\pi_X\, \mathscr{G}_R$, but only those edges $x \bullet - \bullet\, x'$ such that $[x,y] \bullet - \bullet\, [x',y]$ is an edge in $\mathscr{G}_R$ for some $y \in Y$. Causal maps R are then those for which $\pi_X\, \mathscr{G}_R$ covers all elements of X, and such that no Y-constant edges exist.

As subgraphs of the complete graph K_X, the mixed states in X come with a boolean lattice structure given by the subgraph (or, equivalently, subset) relation $\subseteq$, and in particular with graph union $\vee$ (or, equivalently, subset union). We can define a notion of **purity** as the partial order $\preceq$ obtained by restricting

$\subseteq$ to graphs with the same node-set: this has discrete subgraphs as its minima, and cliques (pure states) as its maxima.

The CPM category CPM[fdHilb] does not inherit the enriched structure of fdHilb, but it has an operation of convex combination which preserves causality. The CPM category CPM[fRel], on the other hand, turns out to be closed under the superposition operation $\vee$ from fRel, in the form of union of graphs and preserving causality: in CPM[fRel], we shall refer to this as **convex combination**. In CPM[fdHilb], non-trivial convex combination of non-pure states, or of distinct pure states, is never pure. In CPM[fRel], on the other hand, convex combination of non-pure states can yield a pure state.

Lemma 3.2. *Let* $P : 1 \xrightarrow{CPM} X$ *be a pure state in* CPM[fRel] *corresponding to a subset of* X *with* n *elements. For any* $m = 2, ..., \frac{n(n-1)}{2}$, P *can be expressed as a convex combination of* m *non-pure states* $(\rho_j)_{j=1,...,m}$. *Furthermore, if* $\rho : 1 \xrightarrow{CPM} X$ *is any mixed state, then there exists another mixed state* $\rho' : 1 \xrightarrow{CPM} X$ *with* $\mathscr{G}_\rho$ *and* $\mathscr{G}_{\rho'}$ *having the same node set and* $\rho \vee \rho'$ *a pure state.*

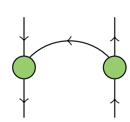
Proof. We prove this of the case $m = \frac{n(n-1)}{2}$, and the other cases follow easily. Let $D \preceq P$ be the state with discrete graph on the same node set of P, and $D \prec \rho_1, ..., \rho_m$ are all the states with graphs having the same node set of P and exactly 1 edge (i.e. 1 edge away from being a purity minimum in the sense defined above). Then $\vee_{j=1,...,n} \rho_j = P$. Furthermore, if ρ is any mixed state, let $G'_\rho \leq \mathrm{K}_X$ be the complement subgraph to G_ρ, with the same nodes as G_ρ and such that an edge is in G'_ρ if and only if it isn't in G_ρ. Then $G_\rho \vee G\rho' = \mathrm{K}_X$ and hence $\rho \vee \rho'$ is pure. Also, ρ' is not pure unless G_ρ is a discrete subgraph. □

4 Decoherence and measurements

The definition of measurements and the treatment of the ensuing classical data is a tricky subject in quantum theory. A rigorous formalisation for fdHilb can be achieved by working in CPM[fdHilb] and considering a certain family of CPM maps, the decoherence maps for each classical structure on a system. Given a classical structure $\circ$ on some Hilbert space $\mathscr{H}$, the associated decoherence map is a CPM map dec($\circ$) with the following property: if ρ is any mixed state in $\mathscr{H}$, then dec($\circ$) $\cdot\, \rho$ is always a convex combination $\sum_j p_j |j\rangle\langle j|$ of $\circ$-classical points.

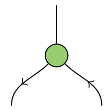
Because of this property, the result of $\circ$-decoherence can always be interpreted as probabilistic $\circ$-classical data. Decoherence maps can be defined in the CPM category associated with any abstract process theory in the following way.

Definition 4.1. Let $\circ$ be a classical structure on some space X in some compact closed †-SMC $\mathscr{C}$ Then the $\circ$**-decoherence** map dec($\circ$) is defined to be the following causal morphism of CPM[$\mathscr{C}$]:



(4.1)

In CPM[fdHilb], the result of a $\circ$-decoherence map is always a convex combination $\sum_j p_j |j\rangle\langle j|$ of $\circ$-classical points: as long as any further operation on the result is $\circ$-classical (i.e. only pure CPM maps coming from classical endomorphisms of $\circ$ are allowed), the entire process is equivalent to the state $\sum_j p_j |j\rangle$ being acted upon in fdHilb by the same classical endomorphisms. This leads to a so-called quantum-classical formalism, where the decoherence map from Diagram 4.1 is replaced with the following map, and it is understood that any operation following it[8] will be ∘-classical:



(4.2)

Unfortunately, in CPM[fRel] things are not this simple: the ∘-decoherence map applied to a mixed state does not in general return a convex combination of ∘-classical points. The short-cut summarised by the map in Diagram 4.2 does not work, and we have to work entirely in the CPM category if we want to make proper sense of measurements and decoherence in fRel.

Definition 4.2. Let ∘ be a classical structure in fRel on some set X, associated with the abelian groupoid $\oplus_{\lambda \in \Lambda} G_\lambda$, and let $d_\lambda \in G_\lambda$. We define the **orbit subgraph** for d_λ as the subgraph of K_X with nodes $\bullet g_\lambda$ for all $g_\lambda \in G_\lambda$ and edges $g_\lambda \bullet\!-\!\bullet (g_\lambda + d_\lambda)$ for all $g_\lambda \in G_\lambda$. Intuitively, the orbit subgraph traces the orbit of d_λ under the right regular action $g_\lambda \mapsto g_\lambda + d_\lambda$.

Theorem 4.3. *Let* ∘ *be a classical structure in* fRel *on some set X, associated with the abelian groupoid* $\oplus_{\lambda \in \Lambda} G_\lambda$. *Let* $\sigma = \mathrm{dec}(\circ) \cdot \rho$ *be the mixed state resulting from decohering some state* ρ. *Then the subgraph* $\mathscr{G}_\sigma \leq \mathrm{K}_X$ *is the union of the orbit subgraphs for* $d_\lambda = g_\lambda - g'_\lambda$, *for all* d_λ *such that:*

$$\exists g_\lambda, g'_\lambda \in G_\lambda \text{ s.t. } g_\lambda \bullet\!-\!\bullet g'_\lambda \text{ appears in } \mathscr{G}_\rho \tag{4.3}$$

An example graph for the abelian groupoid $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ *on a 5-element set is given in 4.4 below.*

Proof. We see the decoherence map as a morphism $X \times X \to X \times X$ in fRel, and try to evaluate the composition $\langle h'_{\tilde{\gamma}}|\langle h_\gamma|\,\mathrm{dec}(\circ)|g'_{\tilde{\lambda}}\rangle|g_\lambda\rangle$. Consider the fRel morphism $X \to X$ obtained by composing the state $|g_\lambda\rangle$ and effect $\langle h'_{\tilde{\gamma}}|$ to the bottom right and top left wires of figure 4.1 respectively:

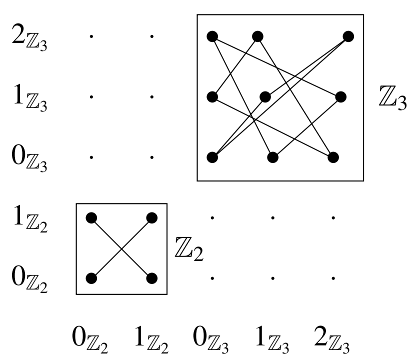
a. on the RHS, we have  $\cdot |g_\lambda\rangle = \vee_{h_\lambda \in G_\lambda} |g_\lambda - h_\lambda\rangle|h_\lambda\rangle$

b. we now have $|g_\lambda - h_\lambda\rangle$ going on the central wire and $|h_\lambda\rangle$ going up the top right wire

c. on the LHS, we have  $\cdot (|g_\lambda - h_\lambda\rangle|h'_{\tilde{\gamma}}\rangle) = \delta_{\lambda\tilde{\gamma}}|g_\lambda - h_\lambda + h'_\lambda\rangle$

d. we now have $\delta_{\lambda\tilde{\gamma}}|g_\lambda - h_\lambda + h'_\lambda\rangle$ going down the bottom left wire

We conclude that the decoherence map, seen as a morphism in fRel, yields the scalar 1, when applied to $|g'_{\tilde{\lambda}}\rangle|g_\lambda\rangle$ and evaluated against effects $\langle h'_{\tilde{\gamma}}|\langle h_\gamma|$, if and only if $\lambda = \tilde{\lambda} = \gamma = \tilde{\gamma}$ and $g'_\lambda = g_\lambda - h_\lambda + h'_\lambda$. Thus the decoherence map, seen as a CPM morphism, sends a node $\bullet g_\lambda$ to the node set $\{h_\lambda \in G_\lambda\}$, and an edge $g_\lambda \bullet\!-\!\bullet g'_{\tilde{\lambda}}$ to the orbit subgraph for $d_\lambda := g_\lambda - g'_\lambda$ if $\lambda = \tilde{\lambda}$, and to no edge otherwise. □



(4.4)

[8] Where categorical composition is read bottom to top.

As a consequence of Theorem 4.3, $\sigma = \mathrm{dec}(\circ) \cdot \rho$ can be written as:

$$\sigma = \bigvee_{\lambda \in \Lambda'} \tau_\lambda \quad \text{for some } \Lambda' \subseteq \Lambda \text{ and some } \tau_\lambda \preceq |G_\lambda\rangle\langle G_\lambda| \tag{4.5}$$

The mixed state σ is a convex combination of $\circ$-classical points if only if $\tau_\lambda = |G_\lambda\rangle\langle G_\lambda|$ for all $\lambda \in \Lambda'$, which is not in general the case: this is enough to call into question the assumption that decoherence maps provide, in general, a suitable quantum-classical interface (an assumption which, for example, is built into the conceptual backbone of the CP* construction [4]).

However, this may in principle just be an issue with the specific form 4.2 chosen for decoherence maps, and there could be some different morphism which works for classical structures with not enough classical points: as it turns out, there isn't one.

Lemma 4.4. *Let $\circ$ be a classical structure in* fRel *on some set Z, associated with the abelian groupoid $\oplus_{\lambda \in \Lambda} G_\lambda$. If there is a CPM morphism $d : Z \stackrel{CPM}{\longrightarrow} Z$ which satisfies the following three properties, then $\circ$ is the discrete structure and $d = \mathrm{dec}(\circ)$:*

(a) $d(|G_\lambda\rangle\langle G_\lambda|) = |G_\lambda\rangle\langle G_\lambda|$ *for all* $\lambda \in \Lambda$

(b) For all CPM states $\rho : 1 \stackrel{CPM}{\longrightarrow} Z$, $d(\rho) = \vee_{\lambda \in \Lambda'_\rho} |G_\lambda\rangle\langle\lambda|$ *for some subset* $\Lambda'_\rho \subseteq \Lambda$

(c) d is a demolition measurement (see Definition 4.5 below)

Proof. Assume that one such d exists, then the following must hold:

(i) by requirement (a), $\mathscr{G}_d$ is a subgraph of $\bigvee_{\lambda \in \Lambda} \mathscr{G}_{|G_\lambda\rangle\langle G_\lambda|}$

(ii) in particular, by point (i) and requirement (b), if $g_\lambda \in G_\lambda$ then $d(|g_\lambda\rangle\langle g_\lambda|) = G_\lambda$

(iii) by point (ii), $\mathscr{G}_d$ must have edges in the form $[g_\lambda, h_\lambda] \bullet\!-\!\bullet [g_\lambda, h'_\lambda]$ for all $g_\lambda, h_\lambda, h'_\lambda \in G_\lambda$

(iv) by requirement (c), $d^\dagger$ must be causal, and in particular if $[g_\lambda, h_\lambda] \bullet\!-\!\bullet [g_\lambda, h'_\lambda]$ is an edge of $\mathscr{G}_d$ then $h_\lambda = h'_\lambda$

By points (iii) and (iv) above, we conclude that all G_λ are singletons, and therefore $\circ$ is the discrete structure. By requirement (c), d must be causal, and combining this with point (i) above we obtain that d must be the decoherence map for the discrete structure $\circ$. □

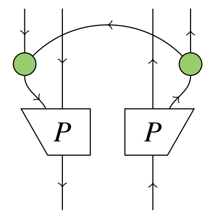
We conclude that the identification of decohered states in CPM[fRel] as possibilistic mixtures of classical data is not sound, and should be avoided unless further measures are in place (e.g. a suitable equivalence relation on CPM states and morphisms).

Having understood decoherence, it is finally time to introduce measurements in CPM[fRel] and tackle the fundamental question of locality. We set aside the issues with convex mixing and take a more traditional approach, using demolition measurements and evaluating against classical points (the equivalence relation we were talking about) to obtain a possibilistic empirical model. We will discuss other options for future work in the conclusions.

Definition 4.5 introduces non-demolition and demolition measurements in CPM[fRel], in accordance with the framework laid out in [10] and the upcoming [8]. Theorem 4.6 shows that the same results of a $\circ$-demolition measurement (with evaluation against $\circ$-classical points) can be obtained by decoherence in some other structure $\bullet$, followed by some classical manipulation of the results.

Definition 4.7 defines the desired class of empirical models for fRel: by virtue of Theorem 4.6, we only need to consider measurements given directly by decoherence maps. Theorem 4.8 finally shows that every measurement scenario has a local hidden variable model, settling once and for all that fRel is local.

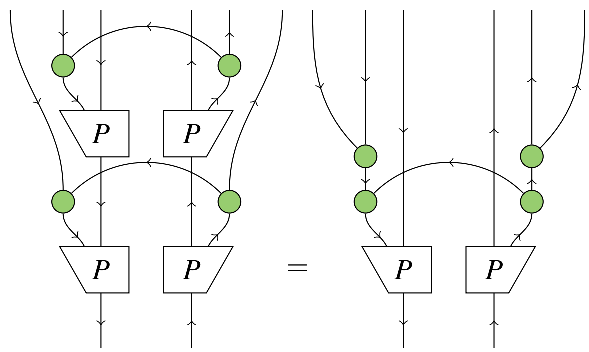
Definition 4.5. Let $\circ$ be a classical structure in fRel on some set Z, associated with the abelian groupoid $\bigoplus_{\lambda\in\Lambda} G_\lambda$. A $\circ$-valued **non-demolition measurement** on some set X is a causal CPM map $M : X \xrightarrow{CPM} X \times Z$ in the following form, and satisfying the idempotence and self-adjointness properties below:



(4.6)

Let $M_\lambda := \left(id_X \times \rho^\dagger_{G_\lambda}\right) \cdot M$ be M evaluated against the pure state $\rho_{G_\lambda} := |G_\lambda\rangle|G_\lambda\rangle$ in CPM[fRel] corresponding to the classical points G_λ of $\circ$. Then:

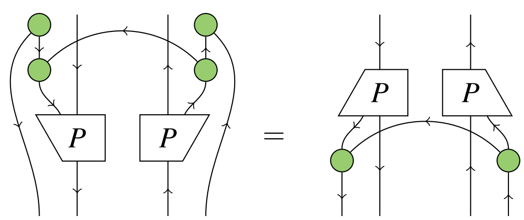
(a.) M is **idempotent** if it satisfies the following equation:



(4.7)

In particular, for all $\lambda \in \Lambda$ we have $M_\lambda \cdot M_\lambda = M_\lambda$

(b.) M is **self-adjoint** if it satisfies the following equation:



(4.8)

In particular, for all $\lambda \in \Lambda$ we have $M^\dagger_\lambda = M_\lambda$

A $\circ$-valued **demolition measurement** on X is a CPM map $\bar{M} : X \xrightarrow{CPM} Z$ in the form $\bar{M} = (\cap_X \times id_Z) \cdot M$ for some $\circ$-valued non-demolition measurement M. Thus demolition measurements are exactly the CPM maps obtained by tracing out the set X in a non-demolition measurement on X.

In particular the decoherence maps are demolition measurements: it turns out that, in fRel, they are the only measurements we ever need.

Theorem 4.6. *Let $\circ$ be a classical structure in* fRel *on some set Z, associated with the abelian groupoid $\bigoplus_{\lambda\in\Lambda} G_\lambda$. Let $\bar{M} : X \xrightarrow{CPM} Z$ be a $\circ$-valued demolition measurement. Let $\bar{M}_\lambda := \rho_{G_\lambda} \cdot \bar{M} : X \xrightarrow{CPM} 1$ be $\bar{M}$ evaluated against the $\circ$-classical point $|G_\lambda\rangle$ of $\circ$. Then there exist:*
(i) a classical structure $\bullet$ on X (for some abelian groupoid $\bigoplus_{\gamma\in\Gamma} H_\gamma$)

(ii) an endomorphism of the classical structures corresponding to some $f : \Gamma \xrightarrow{\text{Set}} \Lambda$ such that the following holds, where $D_\gamma : X \xrightarrow{CPM} 1$ is the $\bullet$-decoherence map evaluated against $|H_\gamma\rangle$:

$$\bar{M}_\lambda = \bigvee_{\gamma \text{ s.t. } f(\gamma)=\lambda} D_\gamma \tag{4.9}$$

Proof. The measurement M is a causal CPM map: thus the P map in Diagram **??** is an isometry, and hence by Lemma 2.5 it is a classical map in the following form, with $J_{s(x)}$ classical points of some classical structure on $X \times Y$ (given by some abelian groupoid $\oplus_{\delta \in \Delta} J_\delta$) and $s : X \xrightarrow{\text{Set}} \Delta$ a classical injection:

$$P = \bigvee_{x \in X} |J_{s(x)}\rangle\langle x| \tag{4.10}$$

Until further notice we work in fRel (i.e. with pure maps only). The subsets $(G_\lambda)_{\lambda \in \Lambda}$ are disjoint, and so are the subsets $(J_{s(x)})_{x \in X}$: thus the family of states $|D_{\lambda,x}\rangle := (id_X \times \langle G_\lambda|) \cdot |J_{s(x)}\rangle$ is also composed of disjoint subsets of X.

Note that the CPM map M_λ is pure, since the central wire of the $\circ$-decoherence map disappears once the decoherence is evaluated against a pure $\circ$-classical state: we can continue working in fRel, with $P_\lambda := (id_X \times \langle G_\lambda|) \cdot P$ in place of M_λ (which lives in CPM[fRel]). Given the $D_{\lambda,x}$ defined above, the map P_λ takes the following form:

$$P_\lambda = \bigvee_{x \in X} |D_{\lambda,x}\rangle\langle x| \tag{4.11}$$

Now M is idempotent/self-adjoint if and only if all M_λ are idempotent/self-adjoint, if and only if all P_λ are idempotent/self-adjoint (because the M_λ are pure maps). Let R_λ be the following relation on X:

$$xR_\lambda y \text{ if and only if } y \in D_{\lambda,x} \tag{4.12}$$

Self-adjointness of P_λ is equivalent to R_λ being symmetric, while idempotence of P_λ is equivalent to R_λ being transitive; since all $x \in X$ appear in R_λ, then by symmetry and transitivity R_λ is also reflexive. Thus R_λ is an equivalence relation on X for all λ.

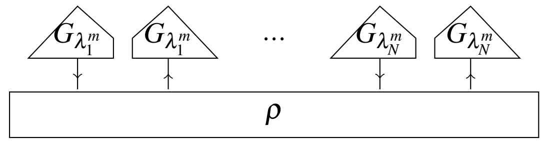
Now we go back to CPM[fRel]: the subgraph $\mathscr{G}_{\bar{M}_\lambda} \leq \mathrm{K}_X$ associated with the CPM map $\bar{M}_\lambda : X \xrightarrow{CPM} 1$ has edges $x \bullet\!-\!\bullet x'$ for all $x, x' \in X$ such that $xR_\lambda y$ and $x'R_\lambda y$ for some $y \in X$. But R_λ is an equivalence relation, so $\mathscr{G}_{\bar{M}_\lambda}$ is the union of the cliques on the equivalence classes of R_λ.

For each $\lambda \in \Lambda$, let $(H_{(\lambda,j)})_{j=1,\dots,n_\lambda}$ be any family of groups, each with element sets one of the n_λ equivalence classes of R_λ. Define $\Gamma := \{(\lambda, j) \mid \lambda \in \Lambda, j = 1, \dots, n_\lambda\}$, and let $\bullet$ be the classical structure on X associated with the abelian groupoid $\oplus_{\gamma \in \Gamma} H_\gamma$. Then each state $(\bar{M}_\lambda)^\dagger$ is a convex combination of $\bullet$-classical points.

Thus the results of the demolition measurement $\bar{M}$ can be reproduced by the $\bullet$-decoherence map, i.e. by testing against $\bullet$-classical points and then applying the function $f = (\lambda, j) \mapsto \lambda$ to reconstruct the $\bar{M}$ measurements result in terms of $\circ$-classical points. □

We have shown that the only demolition measurements we really need are the decoherence maps. But applying a decoherence map and then testing against a classical state is the same as testing directly against the classical state, so we can give the following, simpler definition of an empirical model, where mixed states are directly tested against classical points, with no demolition measurements in between.

Definition 4.7. Let ρ be a mixed state in $X_1 \times \ldots \times X_N$. For each $j = 1, \ldots, N$, let $(\bullet_j^m)_{m=1,\ldots,M}$ be a family of classical structures on X_j. Let $(\Lambda_j^m)_{jm}$ be sets indexing the classical points of the classical structures (without loss of generality, disjoint for different classical structures). Let $\Phi^m(\lambda_1^m, \ldots, \lambda_N^m) : \Lambda_1^m \times \ldots \times \Lambda_N^m \to \{\bot, \top\}$ be a boolean function defined by the scalar obtained evaluating ρ against the separable pure state $\rho_{\lambda_1^m} \times \ldots \times \rho_{\lambda_N^m}$ associated with the family of classical points $|G_{\lambda_j^m}\rangle_{j=1,\ldots,N}$, as shown in Equation 4.13. Then $(\Phi^m)_m$ is a **possibilistic empirical model** (based on mixed state ρ).

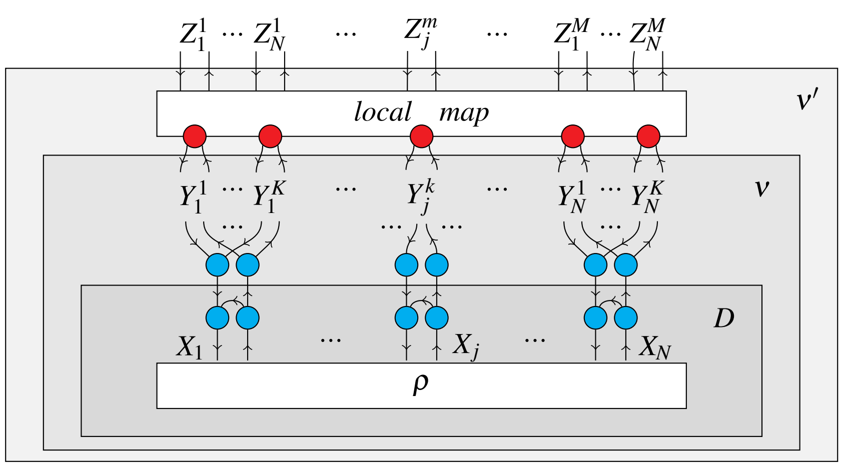
$\Phi^m(\lambda_1^m, \ldots, \lambda_N^m) \quad :=$  (4.13)

This definition of (possibilistic) empirical model squares with that given in [2], in the following way:

1. The commutative semiring is that of the booleans $(\{\bot, \top\}, \vee, \bot, \wedge, \top)$.
2. The set of measurements is $\left\{ \mathrm{dec}(\bullet)_j^m \,|\, m = 1, \ldots, M \text{ and } j = 1, \ldots, N \right\}$. As a consequence of Theorem 4.6, this is general enough to capture all measurements in CPM[fRel].[9][10]
3. The measurement contexts take the form $C_m := \left\{ \mathrm{dec}(\bullet)_j^m \mid j = 1, \ldots, N \right\}$, for $m = 1, \ldots, M$.
4. The sheaf of events is defined by $\mathscr{E}(C_m) := \prod_{j=1,\ldots,N} \Lambda_j^m$, with measurement-dependent outcomes.
5. The empirical model is the family $(\Phi^m : \mathscr{E}(C_m) \to \{\bot, \top\})_m$ of boolean-valued distributions.

Under the correspondence above, the local hidden variable we construct in Theorem 4.8 below take the form of those defined in [2] (further details can be found in the Appendix). In this sense, our last result should be interpreted as stating that fRel is **local**: every empirical model, obtained by considering demolition measurements as per Definition 4.5, admits a local hidden variable.

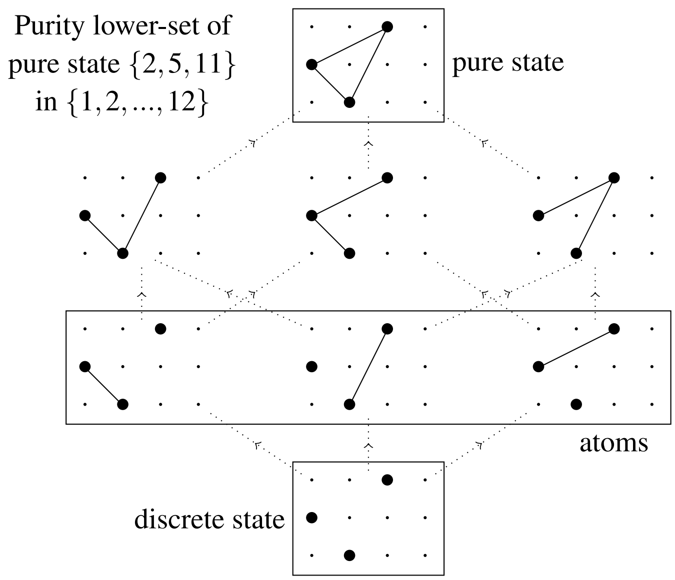
Theorem 4.8. *Every possibilistic empirical model* $(\Phi^m)_m$ *in* CPM[fRel] *constructed as in Definition 4.7 admits a local hidden variable* ν. *This is shown in Figure 4.14, and is obtained as follows:*
(i) the mixed state ρ *underlying the empirical model is decohered in the discrete structures on* $X_1, \ldots, X_N$*;*
(ii) the discrete classical data is appropriately copied to obtain a local hidden variable ν.

 (4.14)

[9] The classical function involved in Theorem 4.6 irrelevant when it comes to non-locality.

[10] To be more precise, all measurements according to the framework laid out in [10] and the upcoming [8], which is restricted to *commutative* special †-Frobenius Algebras.

Proof. The proof hinges on the following, somewhat puzzling fact: the purity of a mixed state cannot be observed in CPM[fRel]. To be more precise, let ρ, τ be mixed states and σ be any other mixed state such that $\tau \preceq \rho$: then $\sigma^\dagger \cdot \rho = \sigma^\dagger \cdot \tau$ (the proof of this can be found below). A picture of a *purity lower-set* in a 12-element set can be found below in 4.15: none of the 8 states presented there can be distinguished from the discrete state.



(4.15)

Now consider an empirical model $(\Phi^m)_m$, based on mixed state $\rho : 1 \xrightarrow{CPM} X_1 \times ... \times X_N$, for a measurement scenario $(({\color{red}\bullet}^m_j)_{j=1,...,N})_{m=1,...,M}$. Let $\oplus_{\lambda^m_j \in \Lambda^m_j} G_{\lambda^m_j}$ be the abelian groupoid associated with ${\color{red}\bullet}^m_j$. Let $D \preceq \rho$ be local purity minimum (i.e. $\mathscr{G}_D$ has the same nodes of $\mathscr{G}_\rho$ and no edge), and consider the empirical model $(\Psi^m)_m$, based on mixed state D instead of ρ and otherwise identical to $(\Phi^m)_m$ (i.e. again for the measurement scenario $(({\color{red}\bullet}^m_j)_{j=1,...,N})_{m=1,...,M}$).

Then, since the Φ^m and Ψ^m are functions are computed by considering scalars in the form $\sigma^\dagger \cdot \rho$ and $\sigma^\dagger \cdot D$ respectively (for the same σs), we have that the two empirical models coincide. Thus any empirical model can equivalently be obtained as the empirical model based on some state with discrete associated subgraph.

Elaborating a bit further, we conclude that $\Psi^m(\lambda^m_1, ..., \lambda^m_N) = 1$ if and only if$\bullet(g^m_1, ..., g^m_N) \in \mathscr{G}_D$ for some family $(g^m_j \in G_{\lambda^m_j})_{j=1,...,N}$. The state D can be concretely obtained from ρ by decohering each X_j component in the discrete structure ${\color{cyan}\bullet}$ on X_j. Furthermore the duplication map ${\color{cyan}\curlyvee}$: $X_j \to X_j \times X_j$ for the discrete structure sends the pure state $|(g^m_1, ..., g^m_j, ..., g^m_N)\rangle$ to $|(g^m_1, ..., g^m_j, g^m_j, ..., g^m_N)\rangle$.

Now consider the state $\nu' : 1 \xrightarrow{CPM} \prod_{m=1,...,M} \left(\prod_{j=1,...,N} Z^m_j\right)$ shown in Equation 4.14, where $Z^m_j := X_j$. Let $\equiv$ be the equivalence relation such that $(j,m) \equiv (j',m')$ if and only if ${\color{red}\bullet}^m_j$ and ${\color{red}\bullet}^{m'}_{j'}$ are the same classical structure, let K be the set of equivalence classes of $\equiv$ and $q : \{1,...,N\} \times \{1,...,M\} \twoheadrightarrow K$ be the associated quotient map (denote the structures by ${\color{red}\bullet}_k$).

Then ν' is obtained from a mixed state $\nu : 1 \xrightarrow{CPM} \prod_{j=1,...,N} \prod_{k \in K} Y^k_j$, where $Y^k_j := X_j$, by *multiplexing* each Y^k_j component in the ${\color{red}\bullet}_k$ structure and then connecting it to all the Z^m_j component wires with $q(j,m) = k$. The state ν, the **local hidden variable**, is obtained in Equation 4.14 by multiplexing each X_j component of D in the discrete structure and then connecting it to all the Y^k_j wires with $k \in K$.

Tracing out all components of ν' except for $(Z^m_j)_{j=1,...,N}$ yields a mixed state with discrete associated graph (since all $\bullet_k$ multiplexings have disappeared) and node set $\{\bullet(g_j)_j\}$. Evaluated against the pure separable state $\rho_{\lambda_1^m} \times ... \times \rho_{\lambda_N^m}$, this yields 1 if and only if $g_j \in G_{\lambda_j^m}$ for all $j = 1,...,N$, i.e. if and only if $\Phi^m(\lambda_1^m,...,\lambda_N^m) = 1$. Therefore ν is a local hidden variable for the empirical model $(\Phi^m)_m$. □

Proof. (Sub-lemma of Theorem 4.8) We have to show that, if ρ, σ are two mixed states $1 \xrightarrow{CPM} X$, then the scalar $\sigma^\dagger \cdot \rho$ only depends on the nodes of the subgraphs $\mathscr{G}_\rho, \mathscr{G}_\sigma \leq \mathrm{K}_X$. But $\mathscr{G}_{\sigma^\dagger \cdot \rho}$ is non-empty (equivalently, $\sigma^\dagger \cdot \rho = 1$) if and only if $\mathscr{G}_\rho$ and $\mathscr{G}_\sigma$ have some node in common. □

5 Conclusions

Building on established literature and recent developments in the graph-theoretic characterisation of CPM[fRel], we have provided an overview of pure state CQM in the category fRel of finite sets and relations, and explored mixed state CQM in the same setting. Comparing and contrasting with the category fdHilb of Hilbert spaces and linear maps, traditional arena of quantum theory, we have presented many solid parallelisms, and many puzzling features. Superposition is well-defined, but the basis of singletons plays a distinguished role; there are many classical structures, but the classical structure associated with singletons is strictly finer than all of the others. Unitarity is a very restrictive property, making pure state quantum mechanics in fRel not very interesting.

However, the existence of many classical structures without enough classical points, and many distinct classical structures for each set of classical points, makes mixed state quantum mechanics in fRel spicy and exotic. For example, convex mixing of pure (or non-pure) states can result in a pure state. But most importantly, decohering a state does not in general result in convex mixing of classical points: the interpretation of measurement results as possibilistic mixtures is ill-founded, and extra care is needed in the treatment of measurements and locality.

We gave the full characterisation of decoherence maps, and showed that they are the only demolition measurements needed if the results of measurements are, as traditionally done, tested against classical points. We showed that the degree of purity of a state, characterised in graph theoretic terms as the number of edges present, cannot be measured in our setup, and conclude that fRel is a local theory.

This latter result, however, is based on the assumption that measurements can be tested against classical points to obtain classical data: this is a legitimate assumption in fdHilb, where decoherence produces convex mixing of classical points, but it may turn not be in fRel. Thus we cannot exclude the existence of a more thorough categorical characterisation of the measurement process which would lead to more general empirical models and re-open the question of locality.

Furthermore, our definition is restricted to measurements associated to **commutative** special †-Frobenius algebras, and it is interesting to see whether the line of reasoning presented in this work will extend to the non-commutative case (which has already received attention in [16]). We hope that these questions will be settled in future work.

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6 Appendix - Local Hidden Variables

The framework for non-locality set by [2] consists of the following basic ingredients:

1. A commutative semiring $(R,+,0,\times,1)$, generalising the semiring $(\mathbb{R}^+,+,0,\times,1)$ in the definition of probabilities (e.g. the semiring of booleans $(\{\perp,\top\},\vee,0,\wedge,1)$, yielding *possibilistic* models).
2. A set $\mathscr{M}$ of **measurements**, and for each measurement $\mu \in \mathscr{M}$ a set O_μ of possible outcomes.
3. The **event sheaf** $\mathscr{E} : \mathscr{P}(\mathscr{M})^{op} \longrightarrow \text{Set}$, acting as $U \mapsto \prod_{\mu\in U} O_\mu$ on objects and as $(V \subseteq U) \mapsto \text{res}^U_V$ on morphisms, where res^U_V is the restriction map sending $s \in \prod_{\mu\in U} O_\mu$ to $s_{|V} \in \prod_{\mu\in V} O_\mu$. The event sheaf sends a subset U of measurements to their set of **joint outcomes**.
4. The **presheaf of distributions** $\mathscr{D}_R\mathscr{E}$, where $\mathscr{D}_R : \text{Set} \longrightarrow \text{Set}$ is the functor defined to send a set P to the set of R-distributions on P, i.e. those functions $d : P \to R$ which have finite support and satisfy the normalisation condition $\sum_{s\in P} d(s) = 1$. The presheaf of distributions then sends a set $U \subseteq \mathscr{M}$ of measurements to the set of R-distributions over the joint outcomes for the measurements in U. The action on morphisms is given by restriction:
$$d_{|V} := t \mapsto \sum_{s\in\mathscr{E}(U) \text{ with } \text{res}^U_V(s)=t} d(s) \tag{6.1}$$
5. The category $\mathscr{P}(\mathscr{M})$ comes with a Grothendieck topology[11] making $\mathscr{E}$ into a sheaf.
6. A cover $\mathfrak{U}$ for the Grothendieck topology on $\mathscr{P}(\mathscr{M})$, the **measurement cover**:[12] the sets $C \in \mathfrak{U}$ are the **measurement contexts**, stipulating all possible sets of **mutually compatible measurements**.
7. A compatible family $(e_C)_{C\in\mathfrak{U}}$ for the measurement cover with respect to the presheaf $\mathscr{D}_R\mathscr{E}$, i.e. a family of elements $e_C \in \mathscr{D}_R\mathscr{E}(C)$ such that $(e_C)_{|C\cap C'} = (e_{C'})_{|C\cap C'}$ for all $C, C' \in \mathfrak{U}$. This will be called a **(no-signalling) empirical model**, assigning an R-distribution to the joint outcomes of all sets of mutually compatible measurements.

The ingredients above form a **measurement scenario**. A **global section** for an empirical model $(e_C)_{C\in\mathscr{M}}$, which we shall also refer to as a **local hidden variable**, is an R-distribution $d \in \mathscr{D}_R\mathscr{E}(\mathscr{M})$ such that $e_C = d_{|C}$ for all $C \in \mathfrak{U}$. An empirical model is **local** if it admits a global section, i.e. if there is a R-distribution d on the joint outcomes of all measurements which, when restricted to the measurement context specified by $\mathfrak{U}$, yields the same R-distributions as the empirical model; an empirical model is **non-local** if no such R-distribution d exists.

This framework can be given the following (approximate) semantics in the context of CQM:

1. We consider a compact-closed[13] †-SMC $\mathscr{C}$ enriched over monoids, and let $(R,+,0,\otimes,1)$ be its semiring of scalars.
2. We consider measurements to be certain causal morphisms $\mu : X_\mu \overset{CPM}{\longrightarrow} X_\mu \otimes Z_\mu$ in CPM[$\mathscr{C}$], each satisfying idempotence and self-adjointness with respect to some classical structure $\circ_\mu$ on Z_μ (see [10] and the upcoming [8] for more details). We take some finite set $\mathscr{M}$ of them. For each

[11] In [2], $\mathscr{M}$ and all O_μ are finite, and the Gr. topology on $\mathscr{P}(\mathscr{M})$ is the one corresponding to the discrete topology on $\mathscr{M}$.

[12] In [2], $\mathfrak{U}$ is taken to be any family of subsets with $\cup\mathfrak{U} = \mathscr{M}$ and such that for any $C, C' \in \mathfrak{U}$ we have that $C \subseteq C'$ implies $C = C'$ (i.e. $\mathfrak{U}$ is an antichain).

[13] Strictly not necessary, as the CPM construction, the only reason the requirement of compact-closedness is there, can be replaced with the CP construction from [7].

measurement, we take the classical points $(G^{\mu}_{\lambda})_{\lambda \in \Lambda_\mu}$ of the associated classical structure $\circ_\mu$ as the possible outcomes, assuming that each classical structure involved has a finite set of classical points.

3. The event sheaf is defined as sending a set $U \subseteq \mathscr{M}$ of measurements to the following set:

$$\mathscr{E}(U) := \left\{ \bigotimes_{\mu \in U} G^{\mu}_{\lambda_\mu} \;\middle|\; (\lambda_\mu)_\mu \in \prod_{\mu \in U} \Lambda_\mu \right\} \tag{6.2}$$

4. The presheaf of distributions is defined as sending a set $U \subseteq \mathscr{M}$ of measurements to the set of all possible states $d : I \xrightarrow{CPM} \otimes_{\mu \in U} Z_\mu$ satisfying the following normalisation condition:

$$\left(\bigotimes_{\mu \in U} \cap_{Z_\mu} \right) \cdot d = 1 \tag{6.3}$$

i.e. such that d is causal. We see one such state d as a R-distribution on $\mathscr{E}(U)$ by defining:

$$d\big(\bigotimes_{\mu \in U} G^{\mu}_{\lambda_\mu}\big) := \left(\bigotimes_{\mu \in U} \rho^{\dagger}_{G^{\mu}_{\lambda_\mu}} \right) \cdot d \in R \tag{6.4}$$

where $\rho_{G^{\mu}_{\lambda_\mu}}$ is the pure state in CPM[$\mathscr{C}$] corresponding to pure state $|G^{\mu}_{\lambda_\mu}\rangle$ from $\mathscr{C}$. If for each $\mu \in \mathscr{M}$ we have that $id = \sum_{\lambda \in \Lambda_\mu} |G^{\mu}_{\lambda}\rangle\langle G^{\mu}_{\lambda}|$, then the normalisation condition of Equation 6.3 can equivalently be written as the more familiar:

$$\sum_{s \in \mathscr{E}(U)} d(s) = 1 \tag{6.5}$$

The restriction $d_{|V}$ of a distribution $d : I \xrightarrow{CPM} \otimes_{\mu \in U} Z_\mu$ for $V \subseteq U$ is given by:

$$d_{|V} := \left(\bigotimes_{\mu \in U} \xi_\mu \right) \cdot d, \text{ where } \xi_\mu = \begin{cases} id_{Z_\mu} & \text{if } \mu \in V \\ \cap_{Z_\mu} & \text{if } \mu \notin V \end{cases} \tag{6.6}$$

5. The topology on $\mathscr{P}(\mathscr{M})$ is as before.
6. A measurement cover $\mathfrak{U}$ is chosen as before.
7. A compatible family is chosen to be a family of states $e_C : I \xrightarrow{CPM} \otimes_{\mu \in C} Z_\mu$ for all $C \in \mathscr{M}$ satisfying the normalisation condition of Equation 6.3 and such that $(e_C)_{|C \cap C'} = (e_{C'})_{|C \cap C'}$ for all $C, C' \in \mathfrak{U}$, under the restriction operation defined in Equation 6.6.

We focus our attention to a particular kind of empirical models, based on measurements of some mixed state ρ. Let $\mathfrak{U}$ be a measurement cover, and name its distinct measurement contexts $C_1, ..., C_M$. Let each measurement context C_m consist of N measurements in the form:

$$\mu^{m}_{j} : X_j \xrightarrow{CPM} X_j \otimes Z^{m}_{j}, \text{ for } j = 1, ..., N \tag{6.7}$$

and let $\circ^{m}_{j}$ be the respective classical structures. Fix some causal state:

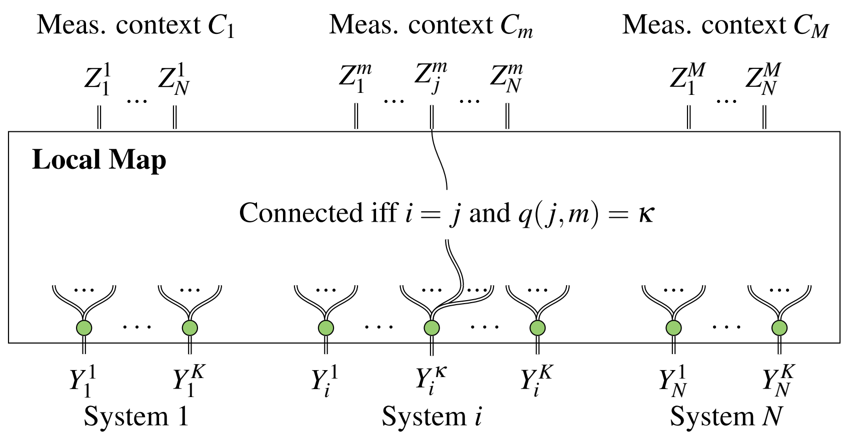
$$\rho : I \xrightarrow{CPM} \bigotimes_{i=1,...,N} X_i \tag{6.8}$$

Then an empirical model $e_m : I \xrightarrow{CPM} \otimes_{j=1,\ldots,N} Z_j^m$ can be defined as follows, for $m = 1,\ldots,M$:

$$e_{C_m} := \left(\bigotimes_{j=1}^{N} \left(\cap_{X_j} \otimes id_{Z_j^m} \right) \right) \cdot \rho \tag{6.9}$$

Local hidden variables are defined as distributions on the *set* of measurements, rather than on the family μ_j^m indexed by the measurement contexts. We define an equivalence relation by setting $(j,m) \equiv (j',m')$ if and only if $\mu_j^m = \mu_{j'}^{m'}$: we index the equivalence classes by $\kappa = 1,\ldots,K$ and let $q : \{1,\ldots,N\} \times \{1,\ldots,M\} \twoheadrightarrow \{1,\ldots,K\}$ be the quotient map. Then $\mu_j^m = \mu_{j'}^{m'}$ if and only if $q(j,m) = q(j',m')$: we define $Y_j^\kappa := Z_j^m$ and $\circ_\kappa := \circ_j^m$ for some (any) m such that $q(j,m) = \kappa$, where $\kappa = 1,\ldots,K$. The **local map** for the measurement scenario is then defined to be the map $(\otimes_i \otimes_\kappa Y_i^\kappa) \xrightarrow{CPM} \left(\otimes_m \otimes_j Z_j^m\right)$ given as follows (see figure 6.10 for a graphical definition):

a. we group the input wires in N groups of K wires
b. we group the output wires in M groups of N wires
c. each Y_i^κ input wire is connected to a $\circ_\kappa$ node
d. for all i, j and κ, m, the $\circ_\kappa$ node of each Y_i^κ input wire is connected to the Z_j^m output wire if and only if $i = j$ and $q(j,m) = \kappa$



(6.10)

In this context, we define a local hidden variable to be some state $\nu : I \xrightarrow{CPM} \otimes_i \otimes_\kappa Y_i^\kappa$ such that:

$$\left(\left(\bigotimes_{m=1}^{r-1} \bigotimes_{j=1}^{N} \cap_{Z_j^m} \right) \otimes \left(\bigotimes_{j=1}^{N} id_{Z_j^r} \right) \otimes \left(\bigotimes_{m=r+1}^{M} \bigotimes_{j=1}^{N} \cap_{Z_j^m} \right) \right) \cdot \text{localmap} \cdot \nu = e_{C_r} \tag{6.11}$$

for all $r = 1,\ldots,M$. In terms of R-distributions, this is the same as the definition from [2].