

Quantum Programs as Kleisli Maps

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Furber and Jacobs have shown in their study of quantum computation that the category of commutative C^* -algebras and *PU-maps* (positive linear maps which preserve the unit) is isomorphic to the Kleisli category of a comonad on the category of commutative C^* -algebras with *MIU-maps* (linear maps which preserve multiplication, involution and unit). [3]

In this paper, we prove a non-commutative variant of this result: the category of C^* -algebras and PU-maps is isomorphic to the Kleisli category of a comonad on the subcategory of MIU-maps.

A variation on this result has been used to construct a model of Selinger and Valiron’s quantum lambda calculus using von Neumann algebras. [1]

The semantics of a non-deterministic program that takes two bits and returns three bits can be described as a multimap (= binary relation) from $\{0, 1\}^2$ to $\{0, 1\}^3$. Similarly, a program that takes two qubits and returns three qubits can be modelled as a positive linear unit-preserving map from $M_2 \otimes M_2 \otimes M_2$ to $M_2 \otimes M_2$, where M_2 is the C^* -algebra of 2×2 -matrices over \mathbb{C} .

More generally, the category $\mathbf{Set}_{\text{multi}}$ of multimaps between sets models non-deterministic programs (running on an ordinary computer), while the opposite of the category \mathbf{C}_{PU}^* of *PU-maps* (positive linear unit-preserving maps) between C^* -algebras models programs running on a quantum computer. (When we write “ C^* -algebra” we always mean “ C^* -algebra with unit”.)

A multimap from $\{0, 1\}^2$ to $\{0, 1\}^3$ is simply a map from $\{0, 1\}^2$ to $\mathcal{P}(\{0, 1\}^3)$. In the same line $\mathbf{Set}_{\text{multi}}$ is (isomorphic to) the Kleisli category of the powerset monad \mathcal{P} on \mathbf{Set} . What about \mathbf{C}_{PU}^* ?

We will show that there is a monad Ω on $(\mathbf{C}_{\text{MIU}}^*)^{\text{op}}$, the opposite of the category $\mathbf{C}_{\text{MIU}}^*$ of C^* -algebras and *MIU-maps* (linear maps that preserve the multiplication, involution and unit), such that $(\mathbf{C}_{\text{PU}}^*)^{\text{op}}$ is isomorphic to the Kleisli category of Ω . We say that $(\mathbf{C}_{\text{PU}}^*)^{\text{op}}$ is *Kleislian* over $(\mathbf{C}_{\text{MIU}}^*)^{\text{op}}$. So in the same way we add non-determinism to \mathbf{Set} by the powerset monad \mathcal{P} yielding $\mathbf{Set}_{\text{multi}}$, we can obtain $(\mathbf{C}_{\text{PU}}^*)^{\text{op}}$ from $(\mathbf{C}_{\text{MIU}}^*)^{\text{op}}$ by a monad Ω .

Let us spend some words on how we obtain this monad Ω . Note that since every positive element of a C^* -algebra \mathcal{A} is of the form a^*a for some $a \in \mathcal{A}$ any MIU-map will be positive. Thus $\mathbf{C}_{\text{MIU}}^*$ is a subcategory of \mathbf{C}_{PU}^* . Let $U : \mathbf{C}_{\text{MIU}}^* \rightarrow \mathbf{C}_{\text{PU}}^*$ be the embedding.

In Section 1 we will prove that U has a left adjoint $F : \mathbf{C}_{\text{PU}}^* \rightarrow \mathbf{C}_{\text{MIU}}^*$, see Theorem 5. This adjunction gives us a comonad $\Omega := FU$ on $\mathbf{C}_{\text{MIU}}^*$ (which is a monad on $(\mathbf{C}_{\text{MIU}}^*)^{\text{op}}$) with the same counit as the adjunction. The comultiplication δ is given by $\delta_{\mathcal{A}} = F\eta_{U\mathcal{A}}$ for every object \mathcal{A} from $\mathbf{C}_{\text{MIU}}^*$ where η is the unit of the adjunction between F and U .

In Section 2 we will prove that $(\mathbf{C}_{\text{PU}}^*)^{\text{op}}$ is isomorphic to $\mathcal{K}\ell(FU)$ if FU is considered a monad on $(\mathbf{C}_{\text{MIU}}^*)^{\text{op}}$. In fact, we will prove that the *comparison functor* $L : \mathcal{K}\ell(FU) \rightarrow (\mathbf{C}_{\text{PU}}^*)^{\text{op}}$ (which sends a MIU-map $f : FU\mathcal{A} \rightarrow \mathcal{B}$ to $Uf \circ \eta_{U\mathcal{A}} : U\mathcal{A} \rightarrow U\mathcal{B}$) is an isomorphism, see Corollary 10.

The method used to show that $(\mathbf{C}_{\text{PU}}^*)^{\text{op}}$ is Kleislian over $(\mathbf{C}_{\text{MIU}}^*)^{\text{op}}$ is quite general and it will be obvious that many variations on $(\mathbf{C}_{\text{PU}}^*)^{\text{op}}$ will be Kleislian over $(\mathbf{C}_{\text{MIU}}^*)^{\text{op}}$ as well, such as the opposite of the category of subunital completely positive linear maps between C^* -algebras. The flip-side of this generality is that we discover precious little about the monad Ω which leaves room for future inquiry (see Section 3).

We will also see that the opposite $(\mathbf{W}_{\text{NCPsU}}^*)^{\text{op}}$ of the category of normal completely positive subunital maps between von Neumann algebras is Kleislian over the subcategory $(\mathbf{W}_{\text{NMIU}}^*)^{\text{op}}$ of normal unital $*$ -homomorphisms. This fact is used in [1] to construct an adequate model of Selinger and Valiron’s quantum lambda calculus using von Neumann algebras.

1 The Left Adjoint

In Theorem 5 we will show that U has a left adjoint, $F: \mathbf{C}_{\text{MIU}}^* \rightarrow \mathbf{C}_{\text{PU}}^*$, using a quite general method. As a result we do not get any “concrete” information about F in the sense that while we will learn that for every C^* -algebra \mathcal{A} there exists an arrow $\rho: \mathcal{A} \rightarrow UF\mathcal{A}$ which is initial from \mathcal{A} to U we will learn nothing more about ρ than this. Nevertheless, for some (very) basic C^* -algebras \mathcal{A} we can describe $F\mathcal{A}$ directly, as is shown below in Example 1–3.

Example 1. Let us start easy: \mathbb{C} will be mapped to itself by F , that is:
the identity $\rho: \mathbb{C} \rightarrow U\mathbb{C}$ is an initial arrow from \mathbb{C} to $U(-)$.

Indeed, let \mathcal{A} be a C^* -algebra and let $\sigma: \mathbb{C} \rightarrow U\mathcal{A}$ be a PU-map. Then σ must be given by $\sigma(\lambda) = \lambda \cdot 1$ for $\lambda \in \mathbb{C}$, where 1 is the identity of \mathcal{A} . Thus σ is a MIU-map as well. Hence there is a unique MIU-map $\hat{\sigma}: \mathbb{C} \rightarrow \mathcal{A}$ (namely $\hat{\sigma} = \sigma$) such that $\hat{\sigma} \circ \rho = \sigma$. (\mathbb{C} is initial in both $\mathbf{C}_{\text{MIU}}^*$ and \mathbf{C}_{PU}^* .)

Example 2. The image of \mathbb{C}^2 under F will be the C^* -algebra $C[0, 1]$ of continuous functions from $[0, 1]$ to \mathbb{C} . As will become clear below, this is very much related to the familiar functional calculus for C^* -algebras: given an element a of a C^* -algebra \mathcal{A} with $0 \leq a \leq 1$ and $f \in C[0, 1]$ we can make sense of “ $f(a)$ ”, as an element of \mathcal{A} .

The map $\rho: \mathbb{C}^2 \rightarrow UC[0, 1]$ given by, for $\lambda, \mu \in \mathbb{C}$, $x \in [0, 1]$,

$$\rho(\lambda, \mu)(x) = \lambda x + \mu(1 - x)$$

is an initial arrow from \mathbb{C}^2 to U .

Let $\sigma: \mathbb{C}^2 \rightarrow U\mathcal{A}$ be a PU-map. We must show that there is a unique MIU-map $\bar{\sigma}: C[0, 1] \rightarrow \mathcal{A}$ such that $\sigma = \bar{\sigma} \circ \rho$.

Writing $a := \sigma(1, 0)$, we have $\sigma(\lambda, \mu) = \lambda a + \mu(1 - a)$ for all $\lambda, \mu \in \mathbb{C}$. Note that $(0, 0) \leq (1, 0) \leq (1, 1)$ and thus $0 \leq a \leq 1$. Let $C^*(a)$ be the C^* -subalgebra of \mathcal{A} generated by a . Then $C^*(a)$ is commutative since a is positive (and thus normal). Given a MIU-map $\omega: C^*(a) \rightarrow \mathbb{C}$ we have $\omega(a) \in [0, 1]$ since $0 \leq a \leq 1$. Thus $\omega \mapsto \omega(a)$ gives a map $j: \Sigma C^*(a) \rightarrow [0, 1]$, where $\Sigma C^*(a)$ is the spectrum of $C^*(a)$, that is, $\Sigma C^*(a)$ is the set of MIU-maps from $C^*(a)$ to \mathbb{C} with the topology of pointwise convergence. (By the way, the image of j is the spectrum of the *element* a .) The map j is continuous since the topology on $\Sigma C^*(a)$ is induced by the product topology on $\mathbb{C}^{C^*(a)}$. Thus the assignment $h \mapsto h \circ j$ gives a MIU-map $Cj: C[0, 1] \rightarrow C\Sigma C^*(a)$. By Gelfand’s representation theorem there is a MIU-isomorphism

$$\gamma: C^*(a) \longrightarrow C\Sigma C^*(a)$$

given by $\gamma(b)(\omega) = \omega(b)$ for all $b \in C^*(a)$ and $\omega \in \Sigma C^*(a)$. Now, define

$$\bar{\sigma} := \gamma^{-1} \circ Cj: C[0, 1] \longrightarrow C^*(a) \hookrightarrow \mathcal{A}.$$

(In the language of the functional calculus, $\bar{\sigma}$ maps f to $f(a)$.) We claim that $\bar{\sigma} \circ \rho = \sigma$. It suffices to

show that $Cj \circ \rho \equiv \gamma \circ \bar{\sigma} \circ \rho = \gamma \circ \sigma$. Let $\lambda, \mu \in \mathbb{C}$ and $\omega \in \Sigma C^*(a)$ be given. We have

$$\begin{aligned}
(Cj \circ \rho)(\lambda, \mu)(\omega) &= (Cj)(\rho(\lambda, \mu))(\omega) \\
&= \rho(\lambda, \mu)(j(\omega)) && \text{by def. of } Cj \\
&= \lambda j(\omega) + \mu(1 - j(\omega)) && \text{by def. of } \rho \\
&= \lambda \omega(a) + \mu(1 - \omega(a)) && \text{by def. of } j \\
&= \omega(\lambda a + \mu(1 - a)) && \text{as } \omega \text{ is a MIU-map} \\
&= \omega(\sigma(\lambda, \mu)) && \text{by choice of } a \\
&= \gamma(\sigma(\lambda, \mu))(\omega). && \text{by def. of } \gamma \\
&= (\gamma \circ \sigma)(\lambda, \mu)(\omega).
\end{aligned}$$

It remains to be shown that $\bar{\sigma}$ is the only MIU-map $\tau: C[0, 1] \rightarrow \mathcal{A}$ such that $U\tau \circ \rho = \sigma$. Let τ be such a map; we prove that $\tau = \bar{\sigma}$. By assumption τ and $\bar{\sigma}$ agree on the elements $f \in C[0, 1]$ of the form

$$f(x) = \lambda x + \mu(1 - x).$$

In particular, $\bar{\sigma}$ and τ agree on the map $h: [0, 1] \rightarrow \mathbb{C}$ given by $h(x) = x$.

Now, since $\bar{\sigma}$ and τ are MIU-maps and h generates the C^* -algebra $C[0, 1]$ (this is Weierstrass's theorem), it follows that $\bar{\sigma} = \tau$.

Example 3. The image of \mathbb{C}^3 under F will not be commutative, or more formally:

If $\rho: \mathbb{C}^3 \rightarrow U\mathcal{B}$ is an initial map from \mathbb{C}^3 to U , then \mathcal{B} is not commutative.

Suppose that \mathcal{B} is commutative towards contradiction. Let \mathcal{A} be a C^* -algebra in which there are positive a_1, a_2, a_3 such that $a_1 a_2 \neq a_2 a_1$ and $a_1 + a_2 + a_3 = 1$.

(For example, we can take \mathcal{A} to be the set of linear operators on \mathbb{C}^2 and let

$$a_1 := 1/2 P_1 \quad a_2 := 1/2 P_+ \quad a_3 := I - 1/2 P_1 - 1/2 P_+$$

where P_1 denotes the orthogonal projection onto $\{(0, x): x \in \mathbb{C}\}$ and P_+ is the orthogonal projection onto $\{(x, x): x \in \mathbb{C}\}$.)

Define $f: \mathbb{C}^3 \rightarrow \mathcal{A}$ by, for all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$,

$$f(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3.$$

Then it is not hard to see that f a PU-map. So as \mathcal{B} is the initial arrow from \mathbb{C}^3 to U there is a (unique) MIU-map $\bar{f}: \mathcal{B} \rightarrow \mathcal{A}$ such that $\bar{f} \circ \rho = f$. We have

$$\begin{aligned}
a_1 \cdot a_2 &= f(1, 0, 0) \cdot f(0, 1, 0) \\
&= \bar{f}(\rho(1, 0, 0)) \cdot \bar{f}(\rho(0, 1, 0)) \\
&= \bar{f}(\rho(1, 0, 0) \cdot \rho(0, 1, 0)) \\
&= \bar{f}(\rho(0, 1, 0) \cdot \rho(1, 0, 0)) && \text{because } \mathcal{B} \text{ is commutative} \\
&= \bar{f}(\rho(0, 1, 0)) \cdot \bar{f}(\rho(1, 0, 0)) \\
&= a_2 \cdot a_1.
\end{aligned}$$

This contradicts $a_1 \cdot a_2 \neq a_2 \cdot a_1$. Hence \mathcal{B} is not commutative.

Remark 4. Before we prove that the embedding $\mathbf{C}_{\text{MIU}}^* \rightarrow \mathbf{C}_{\text{PU}}^*$ has a left adjoint F (see Theorem 5) let us compare what we already know about F with the commutative case. Let $\mathbf{CC}_{\text{MIU}}^*$ denote the category of MIU-maps between commutative C^* -algebras and let $\mathbf{CC}_{\text{PU}}^*$ denote the category of PU-maps between commutative C^* -algebras. From the work in [3] it follows that the embedding $\mathbf{CC}_{\text{MIU}}^* \rightarrow \mathbf{CC}_{\text{PU}}^*$ has a left adjoint F' and moreover that $F'\mathcal{A} = C\text{Stat}\mathcal{A}$, where $\text{Stat}\mathcal{A}$ is the topological space of PU-maps from \mathcal{A} to \mathbb{C} with pointwise convergence and $C\text{Stat}\mathcal{A}$ is the C^* -algebra of continuous functions from $\text{Stat}\mathcal{A}$ to \mathbb{C} .

Let $x \in [0, 1]$. Then the assignment $(\lambda, \mu) \mapsto x\lambda + (1-x)\mu$ gives a PU-map $\bar{x}: \mathbb{C}^2 \rightarrow \mathbb{C}$. It is not hard to see that $x \mapsto \bar{x}$ gives an isomorphism from $[0, 1]$ to $\text{Stat}\mathbb{C}^2$. Thus $F'\mathbb{C}^2 \cong C[0, 1]$. Hence on \mathbb{C}^2 the functor F and its commutative variant F' agree (see Example 2). However, on \mathbb{C}^3 the functors F and F' differ. Indeed, $F'\mathbb{C}^3$ is commutative while $F\mathbb{C}^3$ is not (see Example 3).

$$\begin{array}{ccc}
 & \xleftarrow{F'} & \\
 \mathbf{CC}_{\text{MIU}}^* & \perp & \mathbf{CC}_{\text{PU}}^* \\
 & \xrightarrow{F} & \\
 \downarrow & & \downarrow \\
 \mathbf{C}_{\text{MIU}}^* & \perp & \mathbf{C}_{\text{PU}}^* \\
 & \xrightarrow{F} &
 \end{array}$$

Roughly summarised: while in the diagram above the right adjoints commute with the vertical embeddings, the left adjoints do not.

Theorem 5. *The embedding $U: \mathbf{C}_{\text{MIU}}^* \rightarrow \mathbf{C}_{\text{PU}}^*$ has a left adjoint.*

Proof. By Freyd's Adjoint Functor Theorem (see Theorem V.6.1 of [6]) and the fact that all limits can be formed using only products and equalisers (see Theorem V.2.1 and Exercise V.4.2 of [6]) it suffices to prove the following.

- (i) The category $\mathbf{C}_{\text{MIU}}^*$ has all small products and equalisers.
- (ii) The functor $U: \mathbf{C}_{\text{MIU}}^* \rightarrow \mathbf{C}_{\text{PU}}^*$ preserves small products and equalisers.
- (iii) *Solution Set Condition.* For every C^* -algebra \mathcal{A} there is a set I and for each $i \in I$ a PU-map $f_i: \mathcal{A} \rightarrow \mathcal{A}_i$ such that for any PU-map $f: \mathcal{A} \rightarrow \mathcal{B}$ there is an $i \in I$ and a MIU-map $h: \mathcal{A}_i \rightarrow \mathcal{B}$ such that $h \circ f_i = f$.

Conditions (i) and (ii) can be verified with routine so we will spend only a few words on them (and leave the details to the reader). To see that Condition (iii) holds requires a little more ingenuity and so we will give the proof in detail.

(Conditions (i) and (ii)) Let us first think about small products in $\mathbf{C}_{\text{MIU}}^*$ and \mathbf{C}_{PU}^* .

Let I be a set, and for each $i \in I$ let \mathcal{A}_i be a C^* -algebra.

It is not hard to see that cartesian product $\prod_{i \in I} \mathcal{A}_i$ is a $*$ -algebra when endowed with coordinate-wise operations (and it is in fact the product of the \mathcal{A}_i in the category of $*$ -algebras with MIU-maps, and with PU-maps).

However, $\prod_{i \in I} \mathcal{A}_i$ cannot be the product of the \mathcal{A}_i as C^* -algebras: there is not even a C^* -norm on $\prod_{i \in I} \mathcal{A}_i$ unless \mathcal{A}_i is trivial for all but finitely many $i \in I$. Indeed, if $\|-\|$ were a C^* -norm on $\prod_{i \in I} \mathcal{A}_i$, then we must have $\|\sigma(i)\| \leq \|\sigma\|$ for all $\sigma \in \prod_{i \in I} \mathcal{A}_i$ and $i \in I$, and so for any sequence i_0, i_1, \dots of distinct elements of I for which $\mathcal{A}_{i_0}, \mathcal{A}_{i_1}, \dots$ are non-trivial, and for every $\sigma \in \prod_{i \in I} \mathcal{A}_i$ with $\sigma(i_n) = n \cdot 1$ for all n , we have $n = \|\sigma(i_n)\| \leq \|\sigma\|$ for all n , so $\|\sigma\| = \infty$, which is not allowed.

Nevertheless, the $*$ -subalgebra of $\prod_{i \in I} \mathcal{A}_i$ given by

$$\bigoplus_{i \in I} \mathcal{A}_i := \{ \sigma \in \prod_{i \in I} \mathcal{A}_i : \sup_{i \in I} \|\sigma(i)\| < +\infty \}$$

is a C^* -algebra with norm given by, for $\sigma \in \bigoplus_{i \in I} \mathcal{A}_i$,

$$\|\sigma\| = \sup_{i \in I} \|\sigma(i)\|.$$

We claim that $\bigoplus_{i \in I} \mathcal{A}_i$ is the product of the \mathcal{A}_i in \mathbf{C}_{PU}^* (and in $\mathbf{C}_{\text{MIU}}^*$).

Let \mathcal{C} be a C^* -algebra, and for each $i \in I$, let $f_i: \mathcal{C} \rightarrow \mathcal{A}_i$ be a PU-map. We must show that there is a unique PU-map $f: \mathcal{C} \rightarrow \bigoplus_{i \in I} \mathcal{A}_i$ such that $\pi_i \circ f = f_i$ for all $i \in I$ where $\pi_i: \bigoplus_{j \in I} \mathcal{A}_j \rightarrow \mathcal{A}_i$ is the i -th projection. It is clear that there is at most one such f , and it would satisfy for all $i \in I$, and $c \in \mathcal{C}$, $f(c)(i) = f_i(c)$.

To see that such map f exists is easy if we are able to prove that, for all $c \in \mathcal{C}$,

$$\sup_{i \in I} \|f_i(c)\| < +\infty. \quad (1)$$

Let $i \in I$ be given. We claim that $\|f_i(c)\| \leq \|c\|$ for any *positive* $c \in \mathcal{C}$. Indeed, we have $c \leq \|c\| \cdot 1$, and thus $f_i(c) \leq \|c\| \cdot f_i(1) = \|c\| \cdot 1$, and so $\|f_i(c)\| \leq \|c\|$. It follows that $\|f_i(c)\| \leq 4 \cdot \|c\|$ for any $c \in \mathcal{C}$ by writing $c = c_1 - c_2 + ic_3 - ic_4$ where $c_1, c_2, c_3, c_4 \in \mathcal{C}$ are all positive. (We even have $\|f(c)\| \leq \|c\|$ for all $c \in \mathcal{C}$, but this requires a bit more effort¹) Thus, we have $\sup_{i \in I} \|f_i(c)\| \leq 4\|c\| < +\infty$. Hence Statement (1) holds.

Thus $\bigoplus_{i \in I} \mathcal{A}_i$ is the product of the \mathcal{A}_i in \mathbf{C}_{PU}^* . It is easy to see that $\bigoplus_{i \in I} \mathcal{A}_i$ is the product of the \mathcal{A}_i in $\mathbf{C}_{\text{MIU}}^*$ as well. Hence $\mathbf{C}_{\text{MIU}}^*$ has all small products (as does \mathbf{C}_{PU}^*) and $U: \mathbf{C}_{\text{MIU}}^* \rightarrow \mathbf{C}_{\text{PU}}^*$ preserves small products.

Let us think about equalisers in $\mathbf{C}_{\text{MIU}}^*$ and \mathbf{C}_{PU}^* . Let \mathcal{A} and \mathcal{B} be C^* -algebras and let $f, g: \mathcal{A} \rightarrow \mathcal{B}$ be MIU-maps. We must prove that f and g have an equaliser $e: \mathcal{E} \rightarrow \mathcal{A}$ in $\mathbf{C}_{\text{MIU}}^*$, and that e is the equaliser of f and g in \mathbf{C}_{PU}^* as well.

Since f and g are MIU-maps (and hence continuous), it is not hard to see that

$$\mathcal{E} := \{ a \in \mathcal{A} : f(a) = g(a) \}$$

is a C^* -subalgebra of \mathcal{A} . We claim that the inclusion $e: \mathcal{E} \rightarrow \mathcal{A}$ is the equaliser of f, g in \mathbf{C}_{PU}^* . Let \mathcal{D} be a C^* -algebra and let $d: \mathcal{D} \rightarrow \mathcal{A}$ be a PU-map such that $f \circ d = g \circ d$. We must show that there is a unique PU-map $h: \mathcal{D} \rightarrow \mathcal{E}$ such that $d = e \circ h$. Note that d maps \mathcal{A} into \mathcal{E} . The map $h: \mathcal{D} \rightarrow \mathcal{E}$ is simply the restriction of $d: \mathcal{D} \rightarrow \mathcal{A}$ in the codomain. Hence e is the equaliser of f, g in \mathbf{C}_{PU}^* .

Note that in the argument above h is a PU-map since d is a PU-map. If d were a MIU-map, then h would be a MIU-map too. Hence e is the equaliser of f, g in the category $\mathbf{C}_{\text{MIU}}^*$ as well.

Hence $\mathbf{C}_{\text{MIU}}^*$ has all equalisers and $U: \mathbf{C}_{\text{MIU}}^* \rightarrow \mathbf{C}_{\text{PU}}^*$ preserves equalisers. Hence $\mathbf{C}_{\text{MIU}}^*$ has all small limits and $U: \mathbf{C}_{\text{MIU}}^* \rightarrow \mathbf{C}_{\text{PU}}^*$ preserves all small limits.

(Note that while we have seen that \mathbf{C}_{PU}^* has all small products, and it was easy to see that $\mathbf{C}_{\text{MIU}}^*$ has all equalisers, it is not clear whether \mathbf{C}_{PU}^* has all equalisers. Indeed, if $f, g: \mathcal{A} \rightarrow \mathcal{B}$ are PU-maps, then the set $\{a \in \mathcal{A} : f(a) = g(a)\}$ need not be a C^* -subalgebra of \mathcal{A} .)

(Condition (iii)). Let \mathcal{A} be a C^* -algebra. We must find a set I and for each $i \in I$ a PU-map $f_i: \mathcal{A} \rightarrow \mathcal{A}_i$ such that for every PU-map $f: \mathcal{A} \rightarrow \mathcal{B}$ there is a (not necessarily unique) $i \in I$ and $h: \mathcal{A}_i \rightarrow \mathcal{B}$ such that $f = h \circ f_i$.

Note that if $f: \mathcal{A} \rightarrow \mathcal{B}$ is a PU-map, then the range of the PU-map f need not be a C^* -subalgebra of \mathcal{B} . (If the range of PU-maps would have been C^* -algebras, then we could have taken I to be the set of all ideals of \mathcal{A} , and $f_J: \mathcal{A} \rightarrow \mathcal{A}/J$ to be the quotient map for any ideal J of \mathcal{A} .)

¹See Corollary 1 of [7].

Nevertheless, given a PU-map $f: \mathcal{A} \rightarrow \mathcal{B}$ there is a smallest C^* -subalgebra, say \mathcal{B}' , of \mathcal{B} that contains the range of f . **We claim that $\#\mathcal{B}' \leq \#(\mathcal{A}^{\mathbb{N}})$** where $\#\mathcal{B}'$ is the cardinality of \mathcal{B}' and $\#(\mathcal{A}^{\mathbb{N}})$ is the cardinality of $\mathcal{A}^{\mathbb{N}}$.²

If we can find proof for our claim, the rest is easy. Indeed, to begin note that the collection of all C^* -algebras is not a small set. However, given a set U , the collection of all C^* -algebras \mathcal{C} whose elements come from U (so $\mathcal{C} \subseteq U$) is a small set. Now, let $\kappa := \#(\mathcal{A}^{\mathbb{N}})$ be the cardinality of $\mathcal{A}^{\mathbb{N}}$ (so κ is itself a set) and take

$$I := \{ (\mathcal{C}, c) : \mathcal{C} \text{ is a } C^*\text{-algebra on a subset of } \kappa \text{ and } c: \mathcal{A} \rightarrow \mathcal{C} \text{ is a PU-map} \}.$$

Since the collection of C^* -algebras \mathcal{C} with $\mathcal{C} \subseteq \kappa$ is small, and since the collection of PU-maps from \mathcal{A} to \mathcal{C} is small for any C^* -algebra \mathcal{C} , it follows that I is small.

For each $i \in I$ with $i \equiv (\mathcal{C}, c)$ define $\mathcal{A}_i := \mathcal{C}$ and $f_i := c$.

Let $f: \mathcal{A} \rightarrow \mathcal{B}$ be a PU-map. We must find $i \in I$ and a MIU-map $h: \mathcal{A}_i \rightarrow \mathcal{B}$ such that $h \circ f_i = f$. Let \mathcal{B}' be the smallest C^* -subalgebra that contains the range of f . By our claim we have $\#\mathcal{B}' \leq \#(\mathcal{A}^{\mathbb{N}}) \equiv \kappa$. By renaming the elements of \mathcal{B}' we can find a C^* -algebra \mathcal{C} isomorphic to \mathcal{B}' whose elements come from κ . Let $\varphi: \mathcal{C} \rightarrow \mathcal{B}'$ be the isomorphism.

Note that $c := \varphi^{-1} \circ f: \mathcal{A} \rightarrow \mathcal{C}$ is a PU-map. So we have $i := (\mathcal{C}, c) \in I$. Further, the inclusion $e: \mathcal{B}' \rightarrow \mathcal{B}$ is a MIU-map, as is φ . So we have:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow[f]{\text{PU}} & \mathcal{B} \\ c \downarrow \text{PU} & & \text{MIU} \uparrow e \\ \mathcal{C} & \xrightarrow[\varphi]{\text{MIU}} & \mathcal{B}' \end{array}$$

Now, $h := e \circ \varphi: \mathcal{C} \rightarrow \mathcal{B}$ is a MIU-map with $f = h \circ f_i$. Hence Cond. (iii) holds.

Let us prove our claim. Let \mathcal{A} and \mathcal{B} be C^* -algebras and let $f: \mathcal{A} \rightarrow \mathcal{B}$ be a PU-map. Let \mathcal{B}' be the smallest C^* -subalgebra that contains the range of f .

We must show that $\#\mathcal{B}' \leq \#(\mathcal{A}^{\mathbb{N}})$.

Let us first take care of pathological case. Note that if \mathcal{A} is trivial, i.e. $\mathcal{A} = \{0\}$, then $\mathcal{B}' = \{0\}$, so $\#(\mathcal{A}^{\mathbb{N}}) = 1 = \#\mathcal{B}'$. Now, let us assume that \mathcal{A} is not trivial. Then we have an injection $\mathbb{C} \rightarrow \mathcal{A}$ given by $\lambda \mapsto \lambda \cdot 1$, and thus $\#\mathbb{C} \leq \#\mathcal{A}$.

The trick to prove $\#\mathcal{B}' \leq \#(\mathcal{A}^{\mathbb{N}})$ is to find a more explicit description of \mathcal{B}' . Let T be the set of terms formed using a unary operation $(-)^*$ (involution) and two binary operations, \cdot (multiplication) and $+$ (addition), starting from the elements of \mathcal{A} . Let $f_T: T \rightarrow \mathcal{B}'$ be the map (recursively) given by, for $a \in \mathcal{A}$, and $s, t \in T$,

$$\begin{aligned} f_T(a) &= f(a) \\ f_T(s^*) &= (f_T(s))^* \\ f_T(s \cdot t) &= f_T(s) \cdot f_T(t) \\ f_T(s + t) &= f_T(s) + f_T(t). \end{aligned}$$

²Although it has no bearing on the validity of the proof one might wonder if the simpler statement $\#\mathcal{B}' \leq \#\mathcal{A}$ holds as well. Indeed, if $\#\mathcal{A} = \#\mathbb{C}$ or $\#\mathcal{A} = \#(2^X)$ for some infinite set X , then we have $\#\mathcal{A} = \#(\mathcal{A}^{\mathbb{N}})$, and so $\#\mathcal{B}' \leq \#\mathcal{A}$. However, not every uncountable set is of the form 2^X for some infinite set X , and in fact, if $\#\mathcal{A} = \aleph_\omega$, then $\#(\mathcal{A}^{\mathbb{N}}) > \#\mathcal{A}$ by Corollary 3.9.6 of [2]

Note that the range of f_B , let us call it $\text{Ran}f_B$, is a $*$ -subalgebra of \mathcal{B}' . We will prove that $\#\text{Ran}f_B \leq \#\mathcal{A}$. Since f_B is a surjection of T onto $\text{Ran}f_B$ it suffices to prove that $\#T \leq \#\mathcal{A}$. In fact, we will show that $\#T = \#\mathcal{A}$.

First note that \mathcal{A} is infinite, and $\mathcal{A} \subseteq T$, so T is infinite as well. To prove that $\#T = \#\mathcal{A}$ we write the elements of T as words (with the use of brackets). Indeed, with $Q := \mathcal{A} \cup \{“.”, “+”, “*”, “()”, “()”\}$ there is an obvious injection from T into the set Q^* of words over Q . Since \mathcal{A} is infinite, and $Q \setminus \mathcal{A}$ is finite we have $\#Q = \#\mathcal{A}$ by Hilbert’s hotel. Recall that $Q^* = \bigcup_{n=0}^{\infty} Q^n$. Since Q is infinite, we also have $\#(\mathbb{N} \times Q) = \#Q$ and even $\#(Q \times Q) = \#Q$ (see Theorem 3.7.7 of [2]), so $\#Q = \#(Q^n)$ for all $n > 0$. It follows that

$$\begin{aligned} \#(Q^*) &= \#(\bigcup_{n=0}^{\infty} Q^n) \\ &= \#(1 + \bigcup_{n=1}^{\infty} Q^n) \\ &= \#(1 + \mathbb{N} \times Q) \\ &= \#Q. \end{aligned}$$

Since there is an injection from T to Q^* we have $\#\mathcal{A} \leq \#T \leq \#(Q^*) = \#Q = \#\mathcal{A}$ and so $\#T = \#\mathcal{A}$. Hence $\#\text{Ran}f_B \leq \#\mathcal{A}$.

Since $\text{Ran}f_B$ is a $*$ -algebra that contains $\text{Ran}f$, the closure $\overline{\text{Ran}f_B}$ of $\text{Ran}f_B$ with respect to the norm on \mathcal{B}' is a C^* -algebra that contains $\text{Ran}f$. As \mathcal{B}' is the smallest C^* -subalgebra that contains $\text{Ran}f$, we see that $\mathcal{B}' = \overline{\text{Ran}f_B}$.

Let S be the set of all Cauchy sequences in $\text{Ran}f_B$. As every point in \mathcal{B}' is the limit of a Cauchy sequence in $\text{Ran}f_B$, we get $\#\mathcal{B}' \leq \#S$. Thus:

$$\begin{aligned} \#\mathcal{B}' &\leq \#S \\ &\leq \#(\text{Ran}f_B)^{\mathbb{N}} && \text{as } S \subseteq (\text{Ran}f_B)^{\mathbb{N}} \\ &\leq \#(\mathcal{A}^{\mathbb{N}}) && \text{as } \#\text{Ran}f_B \leq \#\mathcal{A}. \end{aligned}$$

Thus we have proven our claim.

Hence Conditions (i)–(iii) hold and $U: \mathbf{C}_{\text{MIU}}^* \longrightarrow \mathbf{C}_{\text{PU}}^*$ has a left adjoint. \square

We have seen that $U: \mathbf{C}_{\text{MIU}}^* \longrightarrow \mathbf{C}_{\text{PU}}^*$ has a left adjoint $F: \mathbf{C}_{\text{PU}}^* \longrightarrow \mathbf{C}_{\text{MIU}}^*$. This adjunction gives a comonad FU on $\mathbf{C}_{\text{MIU}}^*$, which in turns gives us two categories: the Eilenberg–Moore category $\mathcal{E}\mathcal{M}(FU)$ of FU -coalgebras and the Kleisli category $\mathcal{K}\ell(FU)$. We claim that \mathbf{C}_{PU}^* is isomorphic to $\mathcal{K}\ell(FU)$ since $\mathbf{C}_{\text{MIU}}^*$ is a subcategory of \mathbf{C}_{PU}^* with the same objects.

This is a special case of a more general phenomenon which we discuss in the next section (in terms of monads instead of comonads), see Theorem 9.

2 Kleislian Adjunctions

Beck’s Theorem (see [6], VI.7) gives a criterion for when an adjunction $F \dashv U$ “is” an adjunction between \mathbf{C} and $\mathcal{E}\mathcal{M}(UF)$. We give a similar (but easier) criterion for when an adjunction “is” an adjunction between \mathbf{C} and $\mathcal{K}\ell(UF)$. The criterion is not new; e.g., it is mentioned in [5] (paragraph 8.6) without proof or reference, and it can be seen as a consequence of Exercise VI.5.2 of [6] (if one realises that an equivalence which is bijective on objects is an isomorphism). Proofs can be found in the appendix.

Notation 6. Let $F: \mathbf{C} \rightarrow \mathbf{D}$ be a functor with right adjoint U . Denote the unit of the adjunction by $\eta: \text{id}_{\mathbf{D}} \rightarrow UF$, and the counit by $\varepsilon: FU \rightarrow \text{id}_{\mathbf{C}}$.

Recall that UF is a monad with unit η and as multiplication, for C from \mathbf{C} ,

$$\mu_C := U\varepsilon_{FC}: UFUFC \rightarrow UFC.$$

Let $\mathcal{K}l(UF)$ be the Kleisli category of the monad UF . So $\mathcal{K}l(UF)$ has the same objects as \mathbf{C} , and the morphisms in $\mathcal{K}l(UF)$ from C_1 to C_2 are the morphism in \mathbf{C} from C_1 to UFC_2 . Given C from \mathbf{C} the identity in $\mathcal{K}l(UF)$ on C is η_C . If C_1, C_2, C_3 , $f: C_1 \rightarrow C_2$, $g: C_2 \rightarrow C_3$ from \mathbf{C} are given, g after f in $\mathcal{K}l(UF)$ is

$$g \odot f := \mu_{C_3} \circ UFg \circ f.$$

Let $V: \mathbf{C} \rightarrow \mathcal{K}l(UF)$ be given by, for $f: C_1 \rightarrow C_2$ from \mathbf{C} ,

$$Vf := \eta_{C_2} \circ f: C_1 \rightarrow UFC_2.$$

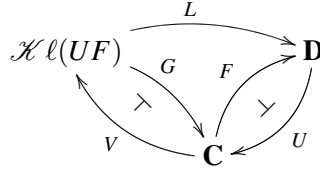
Let $G: \mathcal{K}l(UF) \rightarrow \mathbf{C}$ be given by, for $f: C_1 \rightarrow UFC_2$ from \mathbf{C} ,

$$Gf := \mu_{C_2} \circ UFf: UFC_1 \rightarrow UFC_2.$$

The following is Exercise VI.5.1 of [6].

Lemma 7. Let $F: \mathbf{C} \rightarrow \mathbf{D}$ be a functor with a right adjoint U .

Then there is a unique functor $L: \mathcal{K}l(UF) \rightarrow \mathbf{D}$ (called the comparison functor) such that $U \circ L = G$ and $L \circ V = F$ (see Notation 6).



Definition 8. Let \mathbf{C} and \mathbf{D} be categories.

- (i) A functor $F: \mathbf{C} \rightarrow \mathbf{D}$ is called *Kleislian* when it has a right adjoint $U: \mathbf{D} \rightarrow \mathbf{C}$, and the functor $L: \mathcal{K}l(UF) \rightarrow \mathbf{D}$ from Lemma 7 is an isomorphism.
- (ii) We say that \mathbf{D} is *Kleislian over \mathbf{C}* when there is a Kleislian functor $F: \mathbf{C} \rightarrow \mathbf{D}$.

Theorem 9. Let $F: \mathbf{C} \rightarrow \mathbf{D}$ be a functor with a right adjoint U .

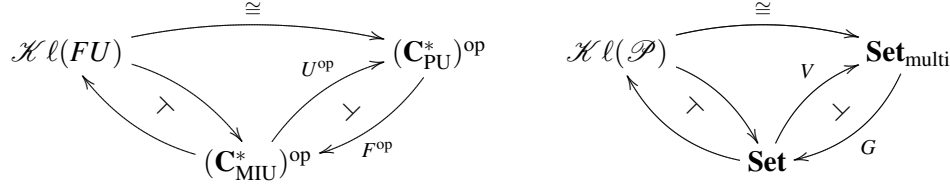
The following are equivalent.

- (i) F is Kleislian (see Definition 8).
- (ii) F is bijective on objects (i.e. for every object D from \mathbf{D} there is a unique object C from \mathbf{C} such that $FC = D$).

Corollary 10. The embedding $U^{\text{op}}: (\mathbf{C}_{\text{MIU}}^*)^{\text{op}} \rightarrow (\mathbf{C}_{\text{PU}}^*)^{\text{op}}$ is Kleislian (see Def. 8).

Proof. By Theorem 9 we must show that U^{op} has a left adjoint and is bijective on objects. Since the embedding $U: \mathbf{C}_{\text{MIU}}^* \rightarrow \mathbf{C}_{\text{PU}}^*$ has a left adjoint $F: \mathbf{C}_{\text{PU}}^* \rightarrow \mathbf{C}_{\text{MIU}}^*$ it follows that $F^{\text{op}}: (\mathbf{C}_{\text{PU}}^*)^{\text{op}} \rightarrow (\mathbf{C}_{\text{MIU}}^*)^{\text{op}}$ is the right adjoint of U^{op} . Thus U^{op} has a left adjoint. Further, as $\mathbf{C}_{\text{MIU}}^*$ and \mathbf{C}_{PU}^* have the same objects, U is bijective on objects, and so is U^{op} . Hence U^{op} is Kleislian. \square

In summary, the embedding $U: \mathbf{C}_{\text{MIU}}^* \longrightarrow \mathbf{C}_{\text{PU}}^*$ has a left adjoint F (and so $F^{\text{op}}: (\mathbf{C}_{\text{MIU}}^*)^{\text{op}} \rightarrow (\mathbf{C}_{\text{PU}}^*)^{\text{op}}$ is right adjoint to U^{op}), and the unique functor from the Kleisli category $\mathcal{K}l(FU)$ of the monad FU on $(\mathbf{C}_{\text{MIU}}^*)^{\text{op}}$ to $(\mathbf{C}_{\text{PU}}^*)^{\text{op}}$ that makes the two triangles in the diagram below on the left commute is an isomorphism.



For the category $\mathbf{Set}_{\text{multi}}$ of multimaps between sets used in the introduction to describe the semantics of non-deterministic programs the situation is the same, see the diagram above to the right.

(The functor V is the obvious embedding. The right adjoint G of V sends a multimap f from X to Y to the function $Gf: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ that assigns to a subset $A \in \mathcal{P}(X)$ the image of A under f . Note that $GV = \mathcal{P}$.)

3 Discussion

3.1 Variations

Example 11 (Subunital maps). Let $\mathbf{C}_{\text{PsU}}^*$ be the category of C^* -algebras and the positive linear maps f between them that are *subunital*, i.e. $f(1) \leq 1$. The morphisms of $\mathbf{C}_{\text{PsU}}^*$ are called *PsU-maps*.

It is not hard to see that the products in $\mathbf{C}_{\text{PsU}}^*$ are the same as in $\mathbf{C}_{\text{MIU}}^*$, and that the equaliser in $\mathbf{C}_{\text{MIU}}^*$ of a pair f, g of MIU-maps is the equaliser of f, g in $\mathbf{C}_{\text{PsU}}^*$ as well. Thus the embedding $U: \mathbf{C}_{\text{MIU}}^* \longrightarrow \mathbf{C}_{\text{PsU}}^*$ preserves limits. Using the same argument as in Theorem 5 but with “PU-map” replaced by “PsU-map” one can show that U satisfies the Solution Set Condition. Hence U has a left adjoint by Freyd’s Adjoint Function Theorem, say $F: \mathbf{C}_{\text{PsU}}^* \longrightarrow \mathbf{C}_{\text{MIU}}^*$.

Since $\mathbf{C}_{\text{PsU}}^*$ has the same objects as $\mathbf{C}_{\text{MIU}}^*$ (namely the C^* -algebras) the functor $U^{\text{op}}: (\mathbf{C}_{\text{MIU}}^*)^{\text{op}} \longrightarrow (\mathbf{C}_{\text{PsU}}^*)^{\text{op}}$ is bijective on objects and thus Kleislian (by Th. 9).

Hence $(\mathbf{C}_{\text{PsU}}^*)^{\text{op}}$ is Kleislian over $(\mathbf{C}_{\text{MIU}}^*)^{\text{op}}$.

Example 12 (Bounded linear maps). Let \mathbf{C}_{p}^* be the category of positive bounded linear maps between C^* -algebras. We will show that $(\mathbf{C}_{\text{p}}^*)^{\text{op}}$ is *not* Kleislian over $(\mathbf{C}_{\text{MIU}}^*)^{\text{op}}$. Indeed, if it were then $(\mathbf{C}_{\text{p}}^*)^{\text{op}}$ would be cocomplete, but it is not: there is no ω -fold product of \mathbb{C} in \mathbf{C}_{p}^* . To see this, suppose that there is a ω -fold product \mathcal{P} in \mathbf{C}_{p}^* with projections $\pi_i: \mathcal{P} \rightarrow \mathbb{C}$ for $i \in \omega$. Since π_i is a bounded linear map for $i \in \omega$, it has finite operator norm, say $\|\pi_i\|$. By symmetry, $\|\pi_i\| = \|\pi_j\|$ for all $i, j \in \omega$. Write $K := \|\pi_0\| = \|\pi_1\| = \|\pi_2\| = \dots$. Define $f_i: \mathbb{C} \rightarrow \mathbb{C}$ by $f_i(z) = iz$ for all $z \in \mathbb{C}$ and $i \in \omega$. Then f_i is a positive bounded linear map for each $i \in \omega$. Since \mathcal{P} is the ω -fold product of \mathbb{C} , there is a (unique positive) bounded linear map $f: \mathbb{C} \rightarrow \mathcal{P}$ such that $\pi_i \circ f = f_i$ for all $i \in \omega$. For each $N \in \omega$ we have

$$N = \|f_N(1)\| \leq \|f_N\| = \|\pi_N \circ f\| \leq \|\pi_N\| \|f\| = K \|f\|.$$

Thus $K\|f\|$ is greater than any number, which is absurd.

Example 13 (Completely positive maps). For clarity’s sake we recall what it means for a linear map f between C^* -algebras to be completely positive (see [8]). For this we need some notation. Given a C^* -algebra \mathcal{A} , and $n \in \mathbb{N}$ let $M_n(\mathcal{A})$ denote the set of $n \times n$ -matrices with entries from \mathcal{A} . We leave it to the

reader to check that $M_n(\mathcal{A})$ is a $*$ -algebra with the obvious operations. In fact, it turns out that $M_n(\mathcal{A})$ is a C^* -algebra, but some care must be taken to define the norm on $M_n(\mathcal{A})$ as we will see below. Now, a linear map $f: \mathcal{A} \rightarrow \mathcal{B}$ is called *completely positive* when $M_n f$ is positive for each $n \in \mathbb{N}$, where $M_n f: M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ is the map obtained by applying f to each entry of a matrix in $M_n(\mathcal{A})$. Of course, “ $M_n f$ is positive” only makes sense once we know that $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$ are C^* -algebras.

Let \mathcal{A} be a C^* -algebra. We will put a C^* -norm on $M_n(\mathcal{A})$. Let \mathcal{H} be a Hilbert space and let $\pi: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, be an isometric MIU-map. We get a norm $\| - \|_\pi$ on $M_n(\mathcal{A})$ given by for $A \in M_n(\mathcal{A})$,

$$\|A\|_\pi = \|\xi((M_n\pi)(A))\|,$$

where $\xi((M_n\pi)(A)): \mathcal{H}^{\oplus n} \rightarrow \mathcal{H}^{\oplus n}$ is the bounded linear map represented by the matrix $(M_n\pi)(A)$, and $\|\xi((M_n\pi)(A))\|$ is the operator norm of $\xi((M_n\pi)(A))$ in $\mathcal{B}(\mathcal{H}^{\oplus n})$.

It is easy to see that $\| - \|_\pi$ satisfies the C^* -identity, $\|A^*A\|_\pi = \|A\|_\pi^2$ for all $A \in M_n(\mathcal{A})$. It is less obvious that $M_n(\mathcal{A})$ is complete with respect to $\| - \|_\pi$. To see this, first note that $\|A_{ij}\| \leq \|A\|_\pi$ for all i, j . So given a Cauchy sequence A_1, A_2, \dots in $M_n(\mathcal{A})$ we can form the entrywise limit A , that is, $A_{ij} = \lim_{m \rightarrow \infty} A_{ij}$. We leave it to the reader to check that A_{ij} is the limit of A_1, A_2, \dots , and thus $M_n(\mathcal{A})$ is complete with respect to $\| - \|_\pi$. Hence $M_n(\mathcal{A})$ is a C^* -algebra with norm $\| - \|_\pi$.

The C^* -norm $\| - \|_\pi$ does not depend on π . Indeed, let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces and let $\pi_1: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_1)$ and $\pi_2: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_2)$ be isometric MIU-maps; we will show that $\| - \|_{\pi_1} = \| - \|_{\pi_2}$. Recall that the norm $\| - \|_{\pi_i}$ induces an order \leq_{π_i} on $M_n(\mathcal{A})$ given by $0 \leq_{\pi_i} A$ iff $\|A - \|A\|_{\pi_i}\|_{\pi_i} \leq \|A\|_{\pi_i}$ where $A \in M_n(\mathcal{A})$. Since $\|A\|_{\pi_i}^2 = \inf\{\lambda \in [0, \infty): A^*A \leq_{\pi_i} \lambda\}$ for all $A \in M_n(\mathcal{A})$, to prove $\| - \|_{\pi_1} = \| - \|_{\pi_2}$ it suffices to show that the orders \leq_{π_1} and \leq_{π_2} coincide. But this is easy when one recalls that $A \in M_n(\mathcal{A})$ is positive iff A is of the form B^*B for some $B \in M_n(\mathcal{A})$.

The completely positive linear maps that preserve the unit are called *CPU-maps*. Let $\mathbf{C}_{\text{CPU}}^*$ be the category of CPU-maps between C^* -algebras. Since $M_n(f)$ is a MIU-map when f is a MIU-map and a MIU-map is positive, we see that any MIU-map is completely positive. Thus $\mathbf{C}_{\text{MIU}}^*$ is a subcategory of $\mathbf{C}_{\text{CPU}}^*$. We claim that $(\mathbf{C}_{\text{CPU}}^*)^{\text{op}}$ is Kleislian over $(\mathbf{C}_{\text{MIU}}^*)^{\text{op}}$.

Let us show that U preserves limits. To show that U preserves equalisers, let $f, g: \mathcal{A} \rightarrow \mathcal{B}$ be MIU-maps. Then $\mathcal{E} := \{x \in \mathcal{A} : f(x) = g(x)\}$ is a C^* -subalgebra of \mathcal{A} and the embedding $e: \mathcal{E} \rightarrow \mathcal{A}$ is an isometric MIU-map. Then e is the equaliser of f, g in $\mathbf{C}_{\text{MIU}}^*$; we will show that e is the equaliser of f, g in $\mathbf{C}_{\text{CPU}}^*$. Let \mathcal{C} be a C^* -algebra, and let $c: \mathcal{C} \rightarrow \mathcal{A}$ be a CPU-map such that $f \circ c = g \circ c$. Let $d: \mathcal{C} \rightarrow \mathcal{E}$ be the restriction of c . It turns out we must prove that d is completely positive. Let $n \in \mathbb{N}$ be given. We must show that $M_n d: M_n \mathcal{C} \rightarrow M_n \mathcal{E}$ is positive. Note that $M_n e$ is an injective MIU-map and thus an isometry. So in order to prove that $M_n d$ is positive it suffices to show that $M_n e \circ M_n d = M_n(e \circ d) = M_n c$ is positive, which it is since c is completely positive. Thus e is the equaliser of f, g in $\mathbf{C}_{\text{CPU}}^*$. Hence U preserves equalisers.

To show that U preserves products, let I be a set and for each $i \in I$ let \mathcal{A}_i be a C^* -algebra. We will show that $\bigoplus_{i \in I} \mathcal{A}_i$ is the product of the \mathcal{A}_i in $\mathbf{C}_{\text{CPU}}^*$. Let \mathcal{C} be a C^* -algebra, and for each $i \in I$, let $f_i: \mathcal{C} \rightarrow \mathcal{A}_i$ be a CPU-map. As before, let $f: \mathcal{C} \rightarrow \bigoplus_{i \in I} \mathcal{A}_i$ be the map given by $f(x)(i) = f_i(x)$ for all $i \in I$ and $x \in \mathcal{C}$. Leaving the details to the reader it turns out that it suffices to show that f is completely positive. Let $n \in \mathbb{N}$ be given. We must prove that $M_n f: M_n(\mathcal{C}) \rightarrow M_n(\bigoplus_{i \in I} \mathcal{A}_i)$ is positive. Let $\varphi: M_n(\bigoplus_{i \in I} \mathcal{A}_i) \rightarrow \bigoplus_{i \in I} M_n(\mathcal{A}_i)$ be the unique MIU-map such that $\pi_i \circ \varphi = M_n \pi_i$ for all $i \in I$. Then φ is a MIU-isomorphism and thus to prove that $M_n f$ is positive, it suffices to show that $\varphi \circ M_n f$ is positive. Let $i \in I$ be given. We must prove that $\pi_i \circ \varphi \circ M_n f$ is positive. But we have $\pi_i \circ \varphi \circ M_n f = M_n \pi_i \circ M_n f = M_n(\pi_i \circ f) = M_n f_i$, which is positive since f is completely positive. Thus $\bigoplus_{i \in I} \mathcal{A}_i$ is the product of the \mathcal{A}_i in $\mathbf{C}_{\text{CPU}}^*$ and hence U preserves limits.

With the same argument as in Theorem 9 the functor U satisfies the Solution Set Condition and thus U has a left adjoint. It follows that $U^{\text{op}}: (\mathbf{C}_{\text{MIU}}^*)^{\text{op}} \rightarrow (\mathbf{C}_{\text{CPU}}^*)^{\text{op}}$ is Kleislian.

Example 14 (W^* -algebras). Let $\mathbf{W}_{\text{NMIU}}^*$ be the category of von Neumann algebras (also called W^* -algebras) and the MIU-maps between them that are normal, i.e., preserve suprema of upwards directed sets of self-adjoint elements. Let $\mathbf{W}_{\text{NPU}}^*$ be the category of von Neumann and normal PU-maps. Note that $\mathbf{W}_{\text{NMIU}}^*$ is a subcategory of $\mathbf{W}_{\text{NPU}}^*$. We will prove that $(\mathbf{W}_{\text{NPU}}^*)^{\text{op}}$ is Kleislian over $(\mathbf{W}_{\text{NMIU}}^*)^{\text{op}}$.

It suffices to show that U has a left adjoint. Again we follow the lines of the proof of Theorem 5. Products and equalisers in $\mathbf{W}_{\text{NMIU}}^*$ are the same as in $\mathbf{C}_{\text{MIU}}^*$. It is not hard to see that the embedding $U: \mathbf{W}_{\text{NMIU}}^* \rightarrow \mathbf{W}_{\text{NPU}}^*$ preserves limits. To see that U satisfies the Solution Set Condition we use the same method as before: given a von Neumann algebra \mathcal{A} , find a suitable cardinal κ such that the following is a solution set.

$$I := \{ (\mathcal{C}, c): \mathcal{C} \text{ is a von Neumann algebra on a subset of } \kappa \\ \text{and } c: \mathcal{A} \rightarrow \mathcal{C} \text{ is a normal PU-map} \},$$

Only this time we take $\kappa = \#(\wp(\wp(\mathcal{A})))$ instead of $\kappa = \#(\mathcal{A}^{\mathbb{N}})$. We leave the details to the reader, but it follows from the fact that given a subset X of a von Neumann algebra \mathcal{B} the smallest von Neumann subalgebra \mathcal{B}' that contains X has cardinality at most $\#(\wp(\wp(X)))$. Indeed, if \mathcal{H} is a Hilbert space such that $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ (perhaps after renaming the elements of \mathcal{B}), then \mathcal{B}' is the closure (in the weak operator topology on $\mathcal{B}(\mathcal{H})$) of the smallest $*$ -subalgebra containing X . Thus any element of \mathcal{B}' is the limit of a filter — a special type of net, see paragraph 12 of [9] — of $*$ -algebra terms over X , of which there are no more than $\#(\wp(\wp(X)))$.

By a similar reasoning one sees that the opposite $(\mathbf{W}_{\text{NCPsU}}^*)^{\text{op}}$ of the category of normal completely positive subunital linear maps between von Neumann algebras is Kleislian over $(\mathbf{W}_{\text{NMIU}}^*)^{\text{op}}$. The existence of the adjoint to the inclusion $\mathbf{W}_{\text{NMIU}}^* \rightarrow \mathbf{W}_{\text{NCPsU}}^*$ is key in our construction of a model of Selinger and Valiron’s quantum lambda calculus by von Neumann algebras, see [1].

3.2 Concrete description

In this note we have shown that the embedding $U: \mathbf{C}_{\text{MIU}}^* \rightarrow \mathbf{C}_{\text{PU}}^*$ has a left adjoint F , but we miss a concrete description of $F\mathcal{A}$ for all but the simplest C^* -algebras \mathcal{A} . What constitutes a “concrete description” is perhaps a matter of taste or occasion, but let us pose that it should at least enable us to describe the Eilenberg–Moore category $\mathcal{E}\mathcal{M}(FU)$ of the comonad FU . More concretely, it should settle the following problem.

Problem 15. *Writing **BOUS** for the category of positive linear maps that preserve the unit between Banach order unit spaces, determine whether $\mathcal{E}\mathcal{M}(FU) \cong \mathbf{BOUS}$.*

(An order unit space is an ordered vector space V over \mathbb{R} with an element 1 , the order unit, such that for all $v \in V$ there is $\lambda \in [0, \infty)$ such that $-\lambda \cdot 1 \leq v \leq \lambda \cdot 1$. The smallest such λ is denoted by $\|v\|$. See [4] for more details. If $v \mapsto \|v\|$ gives a complete norm, V is called a Banach order unit space.)

3.3 MIU versus PU

A second “problem” is to give a physical description (if there is any) of what it means for a quantum program’s semantics to be a MIU-map (and not just a PU-map). A step in this direction might be to define for a C^* -algebra \mathcal{A} , a PU-map $\varphi: \mathcal{A} \rightarrow \mathbb{C}$, and $a, b \in \mathcal{A}$ the quantity

$$\text{Cov}_{\varphi}(a, b) := \varphi(a^*b) - \varphi(a)^*\varphi(b)$$

and interpret it as the covariance between the observables a and b in state \wp of the quantum system \mathcal{A} . Let $T: \mathcal{A} \rightarrow \mathcal{B}$ be a PU-map between C^* -algebras (so perhaps T is the semantics of a quantum program). Then it is not hard to verify that T is a MIU-map if and only if T preserves covariance, that is,

$$\text{Cov}_{\wp}(Ta, Tb) = \text{Cov}_{\wp \circ T}(a, b) \quad \text{for all } a, b \in \mathcal{A}.$$

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A Additional Proofs

Proof of Lemma 7. Define $LC := FC$ for all objects C of $\mathcal{K}\ell(UF)$ and

$$Lf := \varepsilon_{FC_2} \circ Ff$$

for $f: C_1 \rightarrow UFC_2$ from \mathbf{C} . We claim this gives a functor $L: \mathcal{K}\ell(UF) \rightarrow \mathbf{D}$.

(*L preserves the identity*) Let C be an object of $\mathcal{K}\ell(UF)$, that is, an object of \mathbf{C} . Then the identity on C in $\mathcal{K}\ell(UF)$ is η_C . We have $L(\eta_C) = \varepsilon_{FC} \circ F\eta_C = \text{id}_{FC}$.

(*L preserves composition*) Let $f: C_1 \rightarrow UFC_2$ and $g: C_2 \rightarrow UFC_3$ from \mathbf{C} be given. We must prove that $L(g \odot f) = Lg \circ Lf$. We have:

$$\begin{aligned}
L(g \odot f) &= L(\mu_{C_3} \circ UFg \circ f) && \text{by def. of } g \odot f \\
&= \varepsilon_{FC_3} \circ F\mu_{C_3} \circ FUFg \circ Ff && \text{by def. of } L \\
&= \varepsilon_{FC_3} \circ FU\varepsilon_{FC_3} \circ FUFg \circ Ff && \text{by def. of } \mu_{C_3} \\
&= \varepsilon_{FC_3} \circ Fg \circ \varepsilon_{FC_2} \circ Ff && \text{by nat. of } \eta \\
&= Lg \circ Lf && \text{by def. of } L
\end{aligned}$$

Hence L is a functor from $\mathcal{K}\ell(UF)$ to \mathbf{D} .

Let us prove that $U \circ L = G$. For $f: C_1 \rightarrow UFC_2$ from \mathbf{C} we have

$$\begin{aligned}
ULf &= U(\varepsilon_{FC_2} \circ Ff) && \text{by def. of } L \\
&= U\varepsilon_{FC_2} \circ UFf && \\
&= \mu_{C_2} \circ UFf && \text{by def. of } \mu_{C_2} \\
&= Gf && \text{by def. of } Gf.
\end{aligned}$$

Let us prove that $L \circ V = F$. For $f: C_1 \rightarrow C_2$ from \mathbf{C} be given, we have

$$\begin{aligned}
LVf &= L(\eta_{C_2} \circ f) && \text{by def. of } V \\
&= \varepsilon_{FC_2} \circ F\eta_{C_2} \circ Ff && \text{by def. of } L \\
&= Ff && \text{by counit–unit eq.}
\end{aligned}$$

We have proven that there is a functor $L: \mathcal{K}\ell(UF) \rightarrow \mathbf{D}$ such that $U \circ L = G$ and $L \circ V = F$. We must still prove that it is as such unique.

Let $L': \mathcal{K}\ell(UF) \rightarrow \mathbf{D}$ be a functor such that $U \circ L' = G$ and $L' \circ V = F$. We must show that $L = L'$. Let us first prove that L' and L agree on objects. Let C be an object of $\mathcal{K}\ell(UF)$, i.e., C is an object of \mathbf{C} . Since $L' \circ V = F$ and $VC = C$ we have $L'C = L'VC = FC = LC$. Now, let $f: C_1 \rightarrow UFC_2$ from \mathbf{C} be given (so f is a morphism in $\mathcal{K}\ell(UF)$ from C_1 to C_2). We must show that $L'f = LU \equiv \varepsilon_{FC_2} \circ Ff$. Note that since F is the left adjoint of U there is a unique morphism $\bar{f}: FC_1 \rightarrow FC_2$ in \mathbf{D} such that $U\bar{f} \circ \eta_{C_1} = f$. To prove that $L'f = Lf$, we show that both Lf and $L'f$ have this property. We have

$$\begin{aligned}
UL'f \circ \eta_{C_1} &= Gf \circ \eta_{C_1} && \text{as } U \circ L' = G \text{ by assumpt.} \\
&= \mu_{C_2} \circ UFf \circ \eta_{C_1} && \text{by def. of } G \\
&= \mu_{C_2} \circ \eta_{UFC_2} \circ f && \text{by nat. of } \eta \\
&= f && \text{as } UF \text{ is a monad.}
\end{aligned}$$

By a similar argument we get $ULf \circ \eta_{C_1} = f$. Hence $Lf = L'f$. □

Proof of Theorem 9. We use the symbols from Notation 6.

(i) \implies (ii) Suppose that L is an isomorphism. We must prove that F is bijective on objects. Note that $F = L \circ V$, so it suffices to show that both L and V are bijective on objects. Clearly, L is bijective on objects as L is an isomorphism, and $V: \mathbf{C} \rightarrow \mathcal{K}\ell(UF)$ is bijective on objects since the objects of $\mathcal{K}\ell(UF)$ are those of \mathbf{C} and $VC = C$ for all C from \mathbf{C} .

(ii) \implies (i) Suppose that (ii) holds. We prove that L is an isomorphism by giving its inverse. Let D be an object from \mathbf{D} . Note that since F is bijective on objects there is a unique object C from \mathbf{C} such that $FD = C$. Define $KC := D$.

Let $g: D_1 \rightarrow D_2$ from \mathbf{D} be given. Note that by definition of K we have:

$$KD_1 \xrightarrow{\eta_{KD_1}} UFKD_1 \equiv UD_1 \xrightarrow{Ug} UD_2 \equiv UFKD_2$$

Now, define $Kg: KD_1 \rightarrow UFKD_2$ in \mathbf{D} by $Kg := Ug \circ \eta_{KD_1}$.

We claim that this gives a functor $K: \mathbf{D} \rightarrow \mathcal{K}\ell(UF)$.

(K preserves the identity) For an object D of \mathbf{D} we have

$$Kid_D = Uid_D \circ \eta_{KD} = \eta_{KD},$$

and η_{KD} is the identity on KD in $\mathcal{K}\ell(UF)$.

(K preserves composition) Let $f: D_1 \rightarrow D_2$ and $g: D_2 \rightarrow D_3$ from \mathbf{D} be given. We must prove that $K(g \circ f) = K(g) \circ K(f)$. We have

$$\begin{aligned} K(g) \circ K(f) &= \mu_{KD_3} \circ UFKg \circ Kf && \text{by def. of } \circ \\ &= \mu_{KD_3} \circ UFUg \circ UF\eta_{KD_2} \circ Uf \circ \eta_{KD_1} && \text{by def. of } K \\ &= U\varepsilon_{D_3} \circ UFUg \circ UF\eta_{KD_2} \circ Uf \circ \eta_{KD_1} && \text{by def. of } \mu \\ &= Ug \circ U\varepsilon_{D_2} \circ UF\eta_{KD_2} \circ Uf \circ \eta_{KD_1} && \text{by nat. of } \varepsilon \\ &= Ug \circ Uf \circ \eta_{KD_1} && \text{by counit-unit eq.} \\ &= K(g \circ f) && \text{by def of } K. \end{aligned}$$

Hence K is a functor from \mathbf{D} to $\mathcal{K}\ell(UF)$. We will show that K is the inverse of L . For this we must prove that $K \circ L = \text{id}_{\mathbf{D}}$ and $L \circ K = \text{id}_{\mathcal{K}\ell(UF)}$.

For a morphism $g: D_1 \rightarrow D_2$ from \mathbf{D} , we have

$$\begin{aligned} LKg &= L(Ug \circ \eta_{KD_1}) && \text{by def. of } K \\ &= \varepsilon_{FKD_2} \circ FUG \circ F\eta_{KD_1} && \text{by def. of } L \\ &= g \circ \varepsilon_{FKD_1} \circ F\eta_{KD_1} && \text{by nat. of } \varepsilon \\ &= g && \text{by counit-unit eq.} \end{aligned}$$

For a morphism $f: C_1 \rightarrow UFC_2$ in \mathbf{C} we have

$$\begin{aligned} KLf &= K(\varepsilon_{FC_2} \circ Ff) && \text{by def. of } L \\ KLfdd &= U\varepsilon_{FC_2} \circ UFf \circ \eta_{KFC_1} && \text{by def. of } K \\ &= U\varepsilon_{FC_2} \circ \eta_{UFC_2} \circ f && \text{by nat. of } \eta \\ &= f && \text{by counit-unit eq.} \end{aligned}$$

Hence K is the inverse of L , so L is an isomorphism. \square