Cohomology and the Algebraic Structure of Contextuality in Measurement Based Quantum Computation

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Okay, Roberts, Bartlett and Raussendorf recently introduced a new cohomological approach to contextuality in measurement based quantum computing. We give an abstract description of their obstruction and the algebraic structure it exploits, using the sheaf theoretic framework of Abramsky and Brandenburger. At this level of generality we contrast their approach to the Čech cohomology obstruction of Abramsky, Mansfield and Barbosa and give a direct proof that Čech cohomology is at least as powerful.

1 Introduction

Contextuality is a fundamental feature of quantum mechanics that has been shown to play a central role in certain models of quantum computing [11, 16]. For instance, a result by Raussendorf shows that a measurement based quantum computer with mod 2 linear side processing requires a strongly contextual resource to perform universal computation [16].

The sheaf theoretic framework of Abramsky and Brandenburger describes contextuality using the powerful language of sheaf theory [3]. One of the insights of this approach is that contextuality in a range of examples is characterised by the non-vanishing of a cohomological obstruction that is derived using Čech cohomology [4, 2, 7].

More recently Okay et al. described an obstruction for contextuality in measurement based quantum computation (MBQC) that is based on group cohomology [15]. While the Čech cohomology obstruction is well defined for any set of quantum measurements, their obstruction exploits the algebraic structure of the Pauli measurements used in MBQC. We give a more abstract account of this approach using the sheaf theoretic framework. We briefly state our results:

- Local (resp. global) value assignments in MBQC induce local (resp. global) trivialisations of a sequence

$$\mathbb{Z}_2 \longrightarrow X \longrightarrow X/\mathbb{Z}_2$$

  where X is a commutative partial monoid encoding the compositional structure of commuting measurements.
- Mermin's square and GHZ have natural interpretations in terms of this sequence.
- Okay et al.'s obstruction can be defined as an obstruction to a local trivialisation of a sequence of this form to extend globally.
- We give a direct proof that the vanishing of the Čech cohomology obstruction implies the vanishing of Okay et al.'s obstruction.

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This paper is organised as follows. In Section 2 we review the sheaf theoretic formulation of quantum contextuality, the Čech cohomology obstruction, and the issue of completeness and generalised all-versus-nothing arguments. In Section 3 we derive Okay et al.'s obstruction as a generalisation of the cohomological characterisation of trivial group extensions. Finally, in section 4, we apply this obstruction to contextuality and compare it to the Čech cohomology obstruction.

2 Preliminaries

Sheaf theoretic formulation of contextuality. In the sheaf theoretic approach to contextuality the type of an experiment is described by a *measurement scenario* $(X,\mathscr{M},O)$, where

- The set of *measurements* X is a discrete topological space.
- The measurement cover $\mathscr{M} \subset \mathscr{P}(X)$ is a cover of X and furthermore an *anti-chain* ($V \subset C \in \mathscr{M} \Rightarrow V \notin \mathscr{M}$). A subset $V \subset X$ is *compatible* if $V \subset C$ for some *context* $C \in \mathscr{M}$.
- O is the set of outcomes.

The *event sheaf* $\mathscr{E} : X^{\mathrm{op}} \to \mathbf{Set} :: V \mapsto O^V$ assigns to a set of measurements the set of joint outcomes, or *sections*, and restricts a section $s \in \mathscr{E}(V)$ to a section $s|_U \in \mathscr{E}(U)$ for $U \subset V$ with function restriction.

The data describing a particular experiment of type $(X,\mathscr{M},O)$ is specified by an *empirical model*. Contextuality is often defined in terms of probabilities [5]. We will instead be concerned with the stronger notion of *possibilistic* contextuality [12, 14]. A (possibilistic) *empirical model* $\mathscr{S} : (X,\mathscr{M},O)$ is a subpresheaf $\mathscr{S} \subset \mathscr{E} : X^{\mathrm{op}} \to \mathbf{Set}$ satisfying the conditions

1. $\mathscr{S}(C) \neq \emptyset$ for all $C \in \mathscr{M}$.
2. $\mathscr{S}$ is *flasque beneath the cover*: $U \subset V \subset C \in \mathscr{M} \implies \mathscr{S}(U \subset V) : \mathscr{S}(V) \to \mathscr{S}(U)$ is surjective.
3. Every *compatible family* induces a global section: A family $\{s_C \in \mathscr{S}(C)\}_{C\in\mathscr{M}}$ is *compatible* if $s_C|_{C\cap C'} = s_{C'}|_{C\cap C'}$ for all $C, C' \in \mathscr{M}$. We require that every compatible family is the family of restrictions of some global section.

We say that $\mathscr{S} : (X,\mathscr{M},O)$ is

- *logically contextual at* $s \in \mathscr{S}(C)$ if there is no $g \in \mathscr{S}(X)$ with $g|_C = s$.
- *non-contextual* if $\mathscr{S}$ is not logically contextual at any $s \in \mathscr{S}(C)$.
- *strongly contextual* if $\mathscr{S}$ is logically contextual at every $s \in \mathscr{S}(C)$, equivalently $\mathscr{S}(X) = \emptyset$.

Quantum contextuality. If X is a set of Hermitian measurements then we define a measurement scenario $(X,\mathscr{M},O)$, where each context $C \in \mathscr{M}$ is a maximal subset of mutually commuting measurements and O is the combined set of eigenvalues.

A value assignment $s : V \to O$, for a compatible set $V = \{M_1, M_2, \cdots, M_n\} \subset X$, is *consistent with quantum mechanics* if there exists a state $|\psi\rangle$ such that the joint outcome specified by s is consistent with $|\psi\rangle$ according to the Born rule

$$\|P_1P_2\cdots P_n|\psi\rangle\|^2 \neq 0$$

where we require that $s(M_i)$ is an eigenvalue of M_i and P_i denotes the projector onto the corresponding eigenspace.

The *state independent*, and *state dependent* models $\mathscr{S}_X, \mathscr{S}_{X,\psi} : (X, \mathscr{M}, O)$ are defined at any $V \subset C \in \mathscr{M}$ *below the cover* as respectively

$$\mathscr{S}_X(V) := \{s : V \to O \mid s \text{ is consistent with quantum mechanics}\}$$

and

$$\mathscr{S}_{X,\psi}(V) := \{s : V \to O \mid s \text{ is consistent with } |\psi\rangle\}$$

and *above the cover* by the condition that every compatible family induces a global section. It can be shown that the requirement that $\mathscr{S}_{X,\psi}$ and $\mathscr{S}_X$ are flasque beneath the cover is equivalent to the no-signalling principle [3].

Definition 2.1. The Pauli n-group P_n is the matrix group of n-fold tensor products of the Pauli matrices

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_x := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

along with multiplicative factors $\pm 1, \pm i$. The elements of P_n with multiplicative factor ± 1 specify n-qubit measurements with outcomes in $\{1, -1\}$. As is customary, we identify the groups $\{1, -1\} \cong \mathbb{Z}_2$ and write $\sigma^i \in P_n$, where $\sigma \in \{\sigma_x, \sigma_y, \sigma_y\}$, for the n-fold tensor product that is σ at qubit i and I everywhere else.

Lemma 2.1. *Let $X \subset P_n$ be a set of measurements, $C \subset X$ a context and $s : C \to \mathbb{Z}_2$ a value assignment that is consistent with quantum mechanics.*

a) $s(M_1M_2) = s(M_1) \oplus s(M_2)$ for all $M_1, M_2 \in C$ such that $M_1M_2 \in X$.

b) If $I \in X$ then $I \in C$ and $s(I) = 0$. Similarly if $-I \in X$ then $-I \in C$ and $s(-I) = 1$.

Proof. b) is clear. For a) let $M_1, M_2 \in C$ and take any state $|\psi\rangle$ such that $M|\psi\rangle = s(M)|\psi\rangle$ for all $M \in C$.

$$s(M_1M_2)|\psi\rangle = M_1M_2|\psi\rangle = M_1(s(M_2)|\psi\rangle) = s(M_1)s(M_2)|\psi\rangle$$

Hence $s(M_1M_2) = s(M_1) \oplus s(M_2)$ with the identification $\{-1, 1\} \cong \mathbb{Z}_2$. ☐

Example 2.1 (Mermin's square)**.** Let $\mathscr{S}_X : (X, \mathscr{M}, \mathbb{Z}_2)$ be the state independent model induced by the set of measurements displayed in *Mermin's square*

$$\begin{array}{cccc}
\sigma_x^1 & \sigma_x^2 & \sigma_x^1\sigma_x^2 & I \\[1em]
\sigma_z^2 & \sigma_z^1 & \sigma_z^1\sigma_z^2 & I \\[1em]
\sigma_x^1\sigma_z^2 & \sigma_z^1\sigma_x^2 & \sigma_y^1\sigma_y^2 & I \\[1em]
I & I & -I &
\end{array}$$

Observe that the measurements displayed in any row or column M_1, M_2, M_3, M_4 defines a context and furthermore satisfies $M_1M_2M_3 = M_4$, where $M_4 = \pm I$. By Lemma 2.1 any local section $s \in \mathscr{S}(C)$ therefore satisfies one of the following equations

$$\sigma_x^1 \oplus \sigma_x^2 \oplus \sigma_x^1\sigma_x^2 = 0 \tag{1}$$
$$\sigma_z^1 \oplus \sigma_z^2 \oplus \sigma_z^1\sigma_z^2 = 0 \tag{2}$$
$$\sigma_x^1 \oplus \sigma_z^2 \oplus \sigma_x^1\sigma_z^2 = 0 \tag{3}$$
$$\sigma_z^1 \oplus \sigma_x^2 \oplus \sigma_z^1\sigma_x^2 = 0 \tag{4}$$
$$\sigma_x^1\sigma_z^2 \oplus \sigma_z^1\sigma_x^2 \oplus \sigma_y^1\sigma_y^2 = 0 \tag{5}$$
$$\sigma_x^1\sigma_x^2 \oplus \sigma_z^1\sigma_z^2 \oplus \sigma_y^1\sigma_y^2 = 1 \tag{6}$$

Any global section $g \in \mathscr{S}_X(C)$ therefore simultaneously satisfies all equations. However, these equations are mutually inconsistent. Summing together all of the equations gives $0 = 1$, because each measurement appears in exactly two equations. $\mathscr{S}_X$ is therefore strongly contextual.

Example 2.2 (GHZ). Let $\mathscr{S}_{X,\mathrm{GHZ}} : (X, \mathscr{M}, \mathbb{Z}_2)$ be the state dependent model induced by $|\mathrm{GHZ}\rangle := (|000\rangle + |111\rangle)/\sqrt{2}$ and $X := \bigotimes_{i=1}^{3} \pm\{\sigma_x, \sigma_y, I\}$.

$|\mathrm{GHZ}\rangle$ is a $+1$-eigenstate of $\sigma_x^1\sigma_x^2\sigma_x^3$ while it is a -1-eigenstate of $\sigma_x^1\sigma_y^2\sigma_y^3$, $\sigma_y^1\sigma_x^2\sigma_y^3$, and $\sigma_y^1\sigma_y^2\sigma_x^3$. With the identification $\{-1, 1\} \cong \mathbb{Z}_2$ this means that any global section $g \in \mathscr{S}_{X,\mathrm{GHZ}}(X)$ satisfies the following four equations.

$$\sigma_x^1\sigma_x^2\sigma_x^3 = 0, \quad \sigma_x^1\sigma_y^2\sigma_y^3 = 1, \quad \sigma_y^1\sigma_x^2\sigma_y^3 = 1, \quad \sigma_y^1\sigma_y^2\sigma_x^3 = 1$$

By Lemma 2.1 a) g then also satisfies the equations

$$\sigma_x^1 \oplus \sigma_x^2 \oplus \sigma_x^3 = 0 \tag{7}$$
$$\sigma_x^1 \oplus \sigma_y^2 \oplus \sigma_y^3 = 1 \tag{8}$$
$$\sigma_y^1 \oplus \sigma_x^2 \oplus \sigma_y^3 = 1 \tag{9}$$
$$\sigma_y^1 \oplus \sigma_y^2 \oplus \sigma_x^3 = 1 \tag{10}$$

However, summing together equations (7)-(10) results in $0 = 1$. $\mathscr{S}_{X,\mathrm{GHZ}}$ is therefore strongly contextual.

Čech cohomology. Let $\mathscr{U}$ be an open cover of a topological space X and $\mathscr{F} : X^{\mathrm{op}} \to \mathbf{AbGrp}$ a presheaf of abelian groups. The *q-simplices* $\mathscr{N}^q(\mathscr{U})$ of the nerve of $\mathscr{U}$ is the set of all tuples $\sigma = (U_0, U_1, \cdots, U_q) \in \mathscr{U}^{q+1}$ with non-trivial overlap $|\sigma| := \bigcap_{i=0}^{q} U_i \neq \emptyset$. The q-cochains $C^q(\mathscr{U}, \mathscr{F}) := \bigoplus_{U \in \mathscr{N}^q(\mathscr{U})} \mathscr{F}(|U|)$ is the abelian group of all assignments ω of a coefficient $\omega(\sigma) \in \mathscr{F}(|\sigma|)$ to each simplex $q \in \mathscr{N}^q(\mathscr{U})$ such that $\omega(\sigma) \neq 0$ for at most finitely many σ. Using the notation $\partial_i\sigma$ to denote the q simplex obtained from a $q+1$ simplex σ by omitting the i'th element we define for each q a coboundary map $d^q : C^q(\mathscr{U}, \mathscr{F}) \to C^{q+1}(\mathscr{U}, \mathscr{F})$ at each q-cochain ω and $q+1$-simplex σ as

$$d^q(\omega)(\sigma) := \sum_{i=0}^{q} (-1)^i \mathrm{res}_{|\sigma|}^{|\partial_i\sigma|}(\omega(\partial_i\sigma))$$

where $\mathrm{res}_V^U := \mathscr{F}(U \subset V) : \mathscr{F}(V) \to \mathscr{F}(U)$. It can be verified that $d^{q+1} \circ d^q = 0$ and so

$$0 \xrightarrow{d^{-1}:=0} C^0(\mathscr{U}, \mathscr{F}) \xrightarrow{d^0} C^1(\mathscr{U}, \mathscr{F}) \xrightarrow{d^1} C^2(\mathscr{U}, \mathscr{F}) \xrightarrow{d^2} \cdots$$

is a cochain complex. The *q'th Čech cohomology group* $H^q(\mathscr{U}, \mathscr{F})$ is the quotient $Z^q(\mathscr{U}, \mathscr{F})/B^q(\mathscr{U}, \mathscr{F})$ of the *q-cocycles* $Z^q(\mathscr{U}, \mathscr{F}) := \ker d^q$ over the *q-coboundaries* $B^q(\mathscr{U}, \mathscr{F}) := \mathrm{im}\, d^{q-1}$.

The cohomological obstruction. Suppose now that $\mathscr{S} : (X, \mathscr{M}, O)$ is an empirical model and $s_0 \in \mathscr{S}(C_0)$ is a local section. The cohomological obstruction to s_0 lifting to a global section is defined in terms of the presheaf $\mathscr{F} := F_{\mathbb{Z}} \circ \mathscr{S} : X^{\mathrm{op}} \to \mathbf{AbGrp}$ of formal linear combinations of local sections, and two auxiliary presheafs

$$\mathscr{F}_{\tilde{C}_0} :: U \mapsto \ker \mathscr{F}(U \cap C_0 \subset U) \quad \mathscr{F}|_{C_0} :: U \mapsto \mathscr{F}(C_0 \cap U)$$

At any $U \subset X$ these presheafs are related to $\mathscr{F}$ by a sequence

$$0 \longrightarrow \mathscr{F}_{\tilde{C}_0}(U) \hookrightarrow \mathscr{F}(U) \xrightarrow{\mathrm{res}^{U}_{U\cap C_0}} \mathscr{F}|_{C_0}(U) \longrightarrow 0$$

which in fact is exact, because $\mathscr{F}$ is flasque beneath the cover. When lifted to the level of cochain complexes it therefore gives rise to a short exact sequence

$$0 \longrightarrow C^*(\mathscr{M},\mathscr{F}_{\tilde{C}_0}) \longrightarrow C^*(\mathscr{M},\mathscr{F}) \longrightarrow C^*(\mathscr{M},\mathscr{F}|_{C_0}) \longrightarrow 0$$

Using standard techniques from homological algebra this short exact sequence of cochain complexes induces a *long exact sequence* of cohomology groups

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathscr{M},\mathscr{F}_{\tilde{C}_0}) \arrow[r] & H^0(\mathscr{M},\mathscr{F}) \arrow[r] & H^0(\mathscr{M},\mathscr{F}|_{C_0}) \arrow[dll, "\gamma"'] & \\ & H^1(\mathscr{M},\mathscr{F}_{\tilde{C}_0}) \arrow[r] & H^1(\mathscr{M},\mathscr{F}) \arrow[r] & H^1(\mathscr{M},\mathscr{S}|_{C_0}) \arrow[r] & \cdots \end{tikzcd}$$

where γ is the connecting homomorphism [21]. Using the identification $\mathscr{F}(C_0) \cong H^0(\mathscr{M},\mathscr{F}|_{C_0})$ we define the *obstruction for* s_0 to extend to a global section to be $\gamma(1\cdot s_0) \in H^1(\mathscr{M},\mathscr{F}_{\tilde{C}_0})$.

Lemma 2.2 ([4])**.** *If the cover $\mathscr{M}$ is* connected[1] *then $\gamma(1\cdot s_0) = 0$ if and only if $1\cdot s_0$ extends to a compatible family of $F_{\mathbb{Z}}\mathscr{S}$.*

The obstruction is clearly *sound*. If $g \in \mathscr{S}(X)$ then $1\cdot g|_{C_0}$ extends to the compatible family $\{1\cdot g|_C\}_{C\in\mathscr{M}}$. However, in general it is not *complete*. If $\gamma(1\cdot s) = 0$, then $1\cdot s$ extends to a compatible family of $F_{\mathbb{Z}}\mathscr{S}$, but this family might not correspond to any global section of $\mathscr{S}$. Such a false positive occurs for example in the case of Hardy's paradox [2, 7, 8].

Generalised AvN models. Examples 2.1 and 2.2 illustrate a type of contextuality proof that Mermin called 'all versus nothing' [13]. These proofs can be understood as exhibiting an inconsistent set of equations over $\mathbb{Z}_2$ that is locally satisfied by the model. The Čech cohomology obstruction is complete for the *generalised AvN models*, the class of models that locally satisfies a system of inconsistent equations over any ring R [2]. Let R be a ring and suppose that $\mathscr{S} : (X,\mathscr{M},R)$ is an empirical model. An *R-linear equation* ϕ at a context $C \in \mathscr{M}$ is a formal sum

$$\sum_{x\in C} \mathbf{r}(x)x = a$$

where $\mathbf{r} : C \to R$ and $a \in R$. A local section $s : C \to R$ *satisfies* ϕ, written $s \models \phi$, if $\sum_{x\in C} \mathbf{r}(x)\cdot s(x) = a$, where $\cdot$ denotes multiplication in R. The *R-linear theory of $\mathscr{S}$* is the set of all R-linear equations that are consistent with $\mathscr{S}$.

$$\mathrm{Th}_R(\mathscr{S}) := \bigcup_{C\in\mathscr{M}} \{\phi \text{ is an } R\text{-linear equation at } C \mid s \models \phi, \forall s \in \mathscr{S}(C)\}$$

Definition 2.2. $\mathscr{S}$ is AvN_R if its R-linear theory is *inconsistent*. i.e. there is no $s : X \to R$ such that $s|_C \models \phi$, for every context $C \in \mathscr{M}$ and formula $\phi \in \mathrm{Th}_R(\mathscr{S})$ at C.

Theorem 2.1 ([2])**.** *If $\mathscr{S}$ is AvN_R then $\gamma(1\cdot s) \neq 0$ for all $C \in \mathscr{M}$ and $s \in \mathscr{S}(C)$.*

[1] i.e. All pairs $C, C' \in \mathscr{M}$ are connected by a sequence $C_0 = C, C_1, C_2, \cdots, C_{n-1}, C_n = C'$ with $C_i \cap C_{i+1} \neq \emptyset$. This assumption is harmless because non-connected components are completely independent in terms of contextuality. Incidentally all of the scenarios we will consider are connected.

3 The group cohomology obstruction

If G is a commutative group and $H \leq G$ is a subgroup then it is not always the case that $G \cong H \times G/H$. More generally, if $H \leq K \leq G$ then a *local trivialisation* $\phi : K \cong H \times K/H$ might not arise as a restriction of any *global trivialisation* $\phi' : G \cong H \times G/H$. In group cohomology the local trivialisations that can be extended globally are characterised by a vanishing cohomological obstruction [6, 20]. The obstruction of Okay et al. can be understood as a natural generalisation of this idea to the case where G is a commutative partial monoid.

Definition 3.1. Let A be a commutative group, X a commutative partial monoid, and $i : A \to X$ an injective homomorphism. Consider the sequence $A \xrightarrow{i} X \xrightarrow{\pi} X/A$, where $\pi : X \to X/A$ is the canonical quotient of the group action $l_A : A \times X \to X :: (a,x) \mapsto i(a) + x$. [2]

- A *left splitting* is a homomorphism $s : X \to A$ such that $s \circ i = \mathrm{id}_A$.
- A *right splitting* is a homomorphism $h : X/A \to X$ such that $\pi \circ h = \mathrm{id}_{X/A}$.
- A *trivialisation* is a homomorphism $\phi : X \to A \times X/A$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "\mathrm{in}_1"'] & X \arrow[r, "\pi"] \arrow[d, "\phi"] & X/A \\ & A \times X/A \arrow[ur, "\mathrm{proj}_2"'] & \end{tikzcd}$$

  Where $\times$ denotes the cartesian product, and in_1, proj_2 refers to the associated inclusion and projection maps respectively.

In this section we will show that under the assumption that l_A is free the group cohomology obstruction can be generalised to an obstruction for a *local* trivialisation $\phi : C \to A \times C/A$, where $i(A) \subset C \subset X$ is a submonoid, to extend globally. We first translate the problem into one about right splittings.

The splitting lemma. It follows from a general fact about the cartesian product [1] that the maps

$$\phi \mapsto \mathrm{proj}_1 \circ \phi, \quad s \mapsto \langle s, \pi \rangle$$

where $\langle s, \pi \rangle := x \mapsto (s(x), \pi(x))$, defines a bijective correspondence between left splittings and trivialisations. Because this correspondence is compatible with restrictions, the problem of extending a trivialisation is equivalent to the problem of extending a left splitting. When l_A is free something similar is true about right splittings.

Lemma 3.1. *Any trivialisation ϕ is in fact an isomorphism*

Proof. Write $\phi_1 := \mathrm{proj}_1 \circ \phi$ and define $\phi^{-1}(a, [x]) := x + i(a - \phi_1(x))$. This is well defined independently of the representative x because ϕ_1 is a splitting. Using $\phi = \langle \phi_1, \pi \rangle$ and the properties of left splittings it is straightforward to verify that ϕ^{-1} is both a left and right inverse to ϕ. □

Lemma 3.2 (Splitting lemma)**.** *Suppose that l_A is free, $i(A) \subset C \subset X$ is a submonoid, and that $\phi : C \to C \times C/A$ is a trivialisation. The following conditions are then equivalent.*

1. *There exists a left splitting $s : X \to A$ such that $s|_C = \mathrm{proj}_1 \circ \phi$.*
2. *There exists a right splitting $h : X/A \to X$ such that $h|_{C/A} = \phi^{-1} \circ in_2$.*
3. *There exists a trivialisation $\phi' : X \to A \times X/A$ such that $\phi'|_C = \phi$.*

[2] Observe that $i(a) + x$ is always defined, even when X is partial, because $0 + x = i(-a) + (i(a) + x)$

Proof. 1. $\Leftrightarrow$ 3. $\phi \mapsto \mathrm{proj}_1 \circ \phi$ is a bijection between trivialisations and left splittings and furthermore compatible with restrictions: $s'|_C = s \iff \phi'|_C = \phi$ whenever $\phi \mapsto s$, and $\phi' \mapsto s'$.
2 $\Leftrightarrow$ 3. We show that the map $\phi \mapsto \phi^{-1} \circ \mathrm{in}_2$ from trivialisations to right splittings, has a left inverse. For any right splitting $h : X/A \to X$ let $\Phi(h) := \langle s, \pi \rangle$ where $s : X \to A$ is defined by the equation

$$h(\pi(x)) = x - i(s(x))$$

which has a unique solution because l_A is free. To see that s in fact is a splitting note first that $h(\pi(x+y)) = h(\pi(x)) + h(\pi(y))$ and hence

$$x + y - i(s(x+y)) = x - i(s(x)) + y - i(s(y))$$

because s is unique we therefore have $s(x+y) = s(x) + s(y)$. For $s \circ i = \mathrm{id}_A$ we have $h(\pi(i(a))) = h(\pi(0))$, therefore by uniqueness we have $s(i(a)) = a$. Finally,

$$\begin{aligned}(\phi^{-1} \circ \mathrm{in}_2)(\pi(x)) &= \phi^{-1}(0, \pi(x)) \\ &= x + i(0 - \phi_1(x)) \\ &= x - i(\phi_1(x))\end{aligned}$$

hence by uniqueness $\Phi(\phi^{-1} \circ \mathrm{in}_2) = \phi$, and so Φ is a left inverse of $\phi \mapsto \phi^{-1} \circ \mathrm{in}_2$. Because the map is defined pointwise it is clear that it is compatible with restrictions. $\square$

An obstruction to global trivialisations.

Definition 3.2. We define the *relative cohomology groups* $H^*(M,N;G)$ of commutative partial monoids $N \subset M$ with coefficients in an abelian group G. For each $n \geq 0$ let M_n, and similarly N_n, be defined by $M_0 = \{()\}$ and for $n > 0$

$$M_n := \{(m_1, m_2, \cdots, m_n) \in M^n \mid m_1 + m_2 + \cdots + m_n \text{ is defined}\}$$

The *relative n-cochains* $C^n(M,N;G) := \{f : M_n \to G \mid f|_{N_n} = 0\}$ is the abelian group of functions from M_n to G that vanish on N_n, and the coboundary maps

$$0 = C^0(M,N;G) \xrightarrow{d^0} C^1(M,N;G) \xrightarrow{d^1} C^2(M,N;G) \xrightarrow{d^2} \cdots$$

are given by

$$\begin{aligned} d^n(f)(m_0, m_1, \cdots, m_n) :=& f(m_1, \cdots, m_n) \\ &+ \sum_{i=1}^{n} (-1)^i f(m_0, \cdots, m_{i-1}, m_i + m_{i+1}, m_{i+2}, \cdots, m_n) \\ &+ (-1)^n f(m_0, \cdots, m_{n-1}) \end{aligned}$$

$H^n(M,N;G) := Z^n(M,N;G)/B^n(M,N;G)$ is defined as the quotient of the *relative n-cocycles* $Z^n(M,N;G) := \ker d^n$ over the *relative n-coboundaries* $B^n(M,N;G) := \mathrm{im}\, d^{n-1}$. Note that $f|_{N_n} = 0 \implies d^n(f)|_{N_{n+1}} = 0$. It can also be shown that $d^{n+1} \circ d^n = 0$. However, for our purpose it is sufficient to check this for the maps

$$d^2(f)(m_1, m_2, m_3) = f(m_2, m_3) - f(m_1 + m_2, m_3) + f(m_1, m_2 + m_3) - f(m_1, m_2) \tag{11}$$

$$d^1(f)(m_1, m_2) = f(m_2) - f(m_1 + m_2) + f(m_1) \tag{12}$$

$$d^0 = 0 \tag{13}$$

which is easily done.

Suppose now that l_A is free and that $\phi : C \to A \times C/A$ is a trivialisation for some submonoid $i(A) \subset C \subset X$. By the splitting lemma we can equivalently consider the splitting $R(\phi) := \phi^{-1} \circ \mathrm{in}_2 : C/A \to C$.

Definition 3.3. Let $\eta : X/A \to X$ be any choice of representatives that coincides with $R(\phi)$ on C/A. The *cohomological obstruction* to ϕ is the cohomology class $[\beta] \in H^2(X/A, C/A; A)$ of β, where $\beta \in Z^2(X/A, C/A; A)$ is uniquely defined by

$$\eta(q_1 + q_2) = \eta(q_1) + \eta(q_2) + i(\beta(q_1, q_2)) \tag{14}$$

for all $q_1, q_2 \in X/A$ with $q_1 + q_2$ defined.

Lemma 3.3. *The obstruction is well defined and independent of the choice of representatives.*

Proof. First note that β is unique because l_A is free, and a relative cochain because $\eta|_{C/A} = R(\phi)$ is a homomorphism. Next, to show that β is a cocycle we use (14) and associativity to expand $\eta(q_0 + q_1 + q_2)$ as both

$$\eta(q_1) + \eta(q_2) + \eta(q_3) + i(\beta(q_1, q_2 + q_3) + \beta(q_2, q_3))$$

and

$$\eta(q_1) + \eta(q_2) + \eta(q_3) + i(\beta(q_1 + q_2, q_3) + \beta(q_1, q_2))$$

Because these terms are equal and l_A is free

$$\beta(q_1, q_2 + q_3) + \beta(q_2, q_3) = \beta(q_1 + q_2, q_3) + \beta(q_1, q_2)$$

Comparing this to (11) gives $d^2(\beta) = 0$, as required. Finally, to see that $[\beta]$ is independent of the choice of representatives suppose that we instead chose $\eta' := \eta + i \circ \gamma$, for some $\gamma \in C^1(X/A, C/A; A)$, and similarly defined β'. Expanding (14) in the case of η' in terms of η and $i \circ \gamma$ gives

$$\eta(q_1 + q_2) = \eta(q_1) + \eta(q_2) + i(\beta'(q_1, q_2) + \gamma(q_1) - \gamma(q_1 + q_2) + \gamma(q_2))$$

By uniqueness we therefore have $\beta = \beta' + d^1(\gamma)$ and hence $[\beta] = [\beta']$. □

Theorem 3.1. *The following conditions are equivalent.*

1. *There exists a trivialisation $\phi' : X \to A \times X/A$ such that $\phi'|_C = \phi$.*
2. $[\beta] = 0$

Proof. 1. $\iff$ There exists a right splitting $h : X/A \to X$ such that $h|_{C/A} = R(\phi)$.
$\iff$ There exists $\gamma \in C^1(X/A, C/A; A)$ such that $\eta + i \circ \gamma$ is a homomorphism.
$\iff$ There exists $\gamma \in C^1(X/A, C/A; A)$ such that for all $q_1, q_2 \in M/A$ with $q_1 + q_2$ defined

$$\eta(q_1 + q_2) = \eta(q_1) + \eta(q_2) + i(\gamma(q_1) - \gamma(q_1 + q_2) + \gamma(q_2))$$

$\iff$ There exists $\gamma \in C^1(X/A, C/A; A)$ such that $\beta = d(\gamma)$
$\iff$ $[\beta] = 0 \in H^2(X/A, C/A; A)$. □

4 The group cohomological approach to contextuality

Suppose that $X \subset \bigotimes_{i=1}^{n} \pm\{\sigma_x, \sigma_y, \sigma_z, I\}$ is a set of Pauli measurements satisfying the two conditions

1. $\{I, -I\} \subset X$.
2. $M_1, M_2 \in X$ and $M_1 M_2 = M_2 M_1 \implies M_1 M_2 \in X$.

In this case matrix multiplication gives each context $C \in \mathscr{M}$ the structure of a commutative monoid and the embedding $i : \mathbb{Z}_2 \to C :: k \mapsto (-1)^k I$ induces a sequence

$$\mathbb{Z}_2 \xrightarrow{\ i\ } C \xrightarrow{\ \pi\ } C/\mathbb{Z}_2$$

By Lemma 2.1 every $s : C \to \mathbb{Z}_2$ that is consistent with quantum mechanics is a left splitting. It follows that both the state independent and state dependent models $\mathscr{S}_X, \mathscr{S}_{X,\psi} : (X, \mathscr{M}, \mathbb{Z}_2)$ are instances of the following definition.

Definition 4.1. In this section we will assume that we are working with an empirical model $\mathscr{S} : (X, \mathscr{M}, A)$ with the additional structure:

1. The set of outcomes is a commutative group $(A, +_A, 0_A)$.
2. Each $C \in \mathscr{M}$ is a commutative monoid $(C, +_C, 0_C)$ and the monoid structures on different contexts are compatible. For all $C, C' \in \mathscr{M}$:
   (a) $0_C = 0_{C'}$.
   (b) $x, y \in C \cap C' \implies x +_C y = x +_{C'} y$.
3. We are given an embedding $i : A \to \bigcap_{C \in \mathscr{M}} C$ such that for each $C \in \mathscr{M}$ the action $A \times C \to C ::$ $(a, x) \mapsto i(a) +_C x$ is free.
4. Every local section $s \in \mathscr{S}(C)$ is a left splitting of the sequence $A \xrightarrow{\ i\ } C \xrightarrow{\ \pi\ } C/A$.

Group cohomology The monoid structures on different contexts are compatible and therefore "glue together" to define a commutative *partial* monoid $(X, +, 0)$ whose maximal submonoids correspond to the contexts. We consider the sequence

$$A \xrightarrow{\ i\ } X \xrightarrow{\ \pi\ } X/A \tag{15}$$

induced by $i : A \to \bigcap_{C \in \mathscr{M}} C$ and note that the action $l_A : A \times X \to X :: (a, x) \mapsto i(a) + x$ is free. Suppose now that $C \in \mathscr{M}$ is a particular context and $s \in \mathscr{S}(C)$ a local section. Because s is a splitting it induces a *local* trivialisation

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "\mathrm{in}_1"'] & C \arrow[r, "\pi"] \arrow[d, "\langle s,\pi\rangle"] & C/A \\ & A \times C/A \arrow[ur, "\mathrm{proj}_2"'] & \end{tikzcd}$$

of sequence (15).

Definition 4.2. $[\beta_s] \in H^2(X/A, C/A; A)$ is the cohomological obstruction to the existence of a trivialisation of sequence (15) that extends $\langle s, \pi \rangle : C \to A \times C/A$.

The obstruction is clearly *sound*. A global section $g \in \mathscr{S}(X)$ is a splitting because it's restriction to every context is a splitting. If furthermore $g|_C = s$ then $\langle s, \pi \rangle$ extends to $\langle g, \pi \rangle : X \to A \times X/A$, by Theorem 3.1 we therefore have $[\beta_s] = 0$.

Proofs of contextuality. Although the obstruction is sound, it is not in general *complete*. False positives can arise in the form of global extensions $\langle g, \pi \rangle$ of $\langle s, \pi \rangle$ that correspond to splittings $g \notin \mathscr{S}(X)$ that are not allowed by $\mathscr{S}$. For this purpose Okay et al. introduced 'topological' versions of Mermin's square and GHZ. These proofs can be understood as showing that there are no false positives in the form of right splittings. We note however, that the original proofs almost exactly spells out that there are no false positives in the form of left splittings.

Example 4.1 (Mermin's square)**.** Let $X \subset P_2$ be any set of Pauli measurements that is closed under products of commuting measurements and contains the measurements displayed in Mermin's square. We consider the state independent model $\mathscr{S}_X : (X, \mathscr{M}, \mathbb{Z}_2)$ which in this case satisfies Definition 4.1.

Observe that equations (1)-(6) induced by Mermin's square all can be rearranged to be on the form

$$M_1 \oplus M_2 = M_1 M_2$$

for $M_1, M_2 \in X$ with $M_1 M_2 = M_2 M_1$. That the equations are mutually inconsistent therefore literally says that there is no global left splitting. We therefore have $[\beta_s] \neq 0$ for every local section s of $\mathscr{S}$.

Lemma 4.1. *Suppose that $X \subset P_n$ is a set of Pauli measurements that contains the identity and is closed under commuting products. For any state $|\psi\rangle$ the set of measurements*

$$X_\psi := \{M \in X \mid M|\psi\rangle = \pm|\psi\rangle\}$$

whose outcome is uniquely determined by $|\psi\rangle$ is a submonoid of X.

Proof. Because the Pauli measurements $\sigma_x, \sigma_y, \sigma_z$ pairwise *anti-commute*

$$\sigma_x \sigma_z = -\sigma_z \sigma_x, \quad \sigma_x \sigma_y = -\sigma_y \sigma_x, \quad \sigma_y \sigma_z = -\sigma_z \sigma_x$$

all $M_1, M_2 \in X$ either commute, or anti-commute. The condition that $M_1, M_2 \in X_\psi$ forces the former. Furthermore note that X_ψ contains I and is closed under products. $\square$

Example 4.2 (GHZ)**.** Let $X := \bigotimes_{i=1}^{3} \pm\{\sigma_x, \sigma_y, \sigma_z, I\}$. First note that the state dependent model $\mathscr{S}_{X,\mathrm{GHZ}} : (X, \mathscr{M}, \mathbb{Z}_2)$ is an instance of Definition 4.1 because X is closed under commuting products and contains $\pm I$. Next, consider the set X_{GHZ} of measurements whose outcome is uniquely determined by $|\mathrm{GHZ}\rangle$ and observe that equations (7-10) in Example 2.2 are all of the form

$$M_1 \oplus M_2 \oplus M_3 = s_{\mathrm{GHZ}}(M_1 M_2 M_3)$$

where $M_1, M_2, M_3 \in X$ are compatible, $M_1 M_2 M_3 \in X_{\mathrm{GHZ}}$, and $s_{\mathrm{GHZ}}(M_1 M_2 M_3)$ is the unique outcome that is consistent with $|\mathrm{GHZ}\rangle$. That the equations are mutually inconsistent therefore ensures that there is no global splitting $g : X \to \mathbb{Z}_2$ whose restriction to X_{GHZ} is s_{GHZ}. It follows that if $C \in \mathscr{M}$ is any context that contains X_{GHZ} then $[\beta_s] \neq 0$ for every $s \in \mathscr{S}_{X,\mathrm{GHZ}}(C)$. Note that such a context exists because the maximal submonoids of X are the contexts and by Lemma 4.1 X_{GHZ} is a monoid.

Comparison with Čech cohomology. The Čech cohomology obstruction is defined for all empirical models, but this generality comes at a price. It is not a complete characterisation of contextuality. It is therefore natural to ask if there are any examples of contextuality that is detected by group cohomology, but not Čech cohomology. Because Mermin's square and GHZ are examples of all versus nothing arguments we know that Čech cohomology detects contextuality in both cases. We now show more generally that if the group cohomology obstruction is non-trivial, then the Čech cohomology obstruction is also non-trivial.

Theorem 4.1. *Let $s_0 \in \mathscr{S}(C_0)$ be any local section. Then*

$$\gamma(1 \cdot s_0) = 0 \implies [\beta_{s_0}] = 0$$

Proof. First note that it follows from $i(A) \subset \bigcap_{C \in \mathscr{M}} C$ that the cover $\mathscr{M}$ is connected. Therefore, if $\gamma(1 \cdot s_0) = 0$ then there is some compatible family $\{r_C \in F_{\mathbb{Z}}\mathscr{S}(C)\}_{C \in \mathscr{M}}$ such that $r_{C_0} = 1 \cdot s_0$. Observe now that any such family in fact is a compatible family of formal affine combinations: For any $C \in \mathscr{M}$

$$\sum_{s \in \mathscr{S}(C)} r_C(s) \cdot s|_{C \cap C_0} = r_C|_{C \cap C_0} = r_{C_0}|_{C \cap C_0} = 1 \cdot s_0|_{C \cap C_0}$$

hence $\sum_{s \in \mathscr{S}(C)} r_C(s) = 1$. We now use the unique module action[3] of $\mathbb{Z}$ on A to collapse this compatible family to a function $g : X \to A$.

$$g(x) := \sum_{s \in \mathscr{S}(C)} r_C(s) \cdot s(x), \quad \text{where } C \in \mathscr{M} \text{ is any context with } x \in C$$

Because the set of splittings is closed under affine combinations this function is in fact a splitting which furthermore extends s_0. We therefore have $[\beta_{s_0}] = 0$. □

5 Conclusion

We have considered two different applications of cohomological techniques to contextuality in MBQC. While the Čech cohomology obstruction is defined for any set of quantum measurements, the group cohomology obstruction relies on the specific algebraic structure of the Pauli measurements. We have given an abstract account of this approach using the sheaf theoretic framework. At this level of generality we observe that although both approaches rely on structural assumptions to be complete, there is a direct way in which the Čech cohomology obstruction subsumes the group cohomology obstruction.

Our presentation of the group cohomology approach deviates from Okay et al.'s in that we have defined a single obstruction that applies to both state independent and state dependent contextuality. We have shown that this obstruction detects contextuality in the state independent case of Mermin's square and the state dependent case of GHZ.

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[3] i.e. $0 \cdot a = 0$, and for $n \geq 1$: $n \cdot a := a + a + \cdots + a$ (n times) and $-n \cdot a = -(n \cdot a)$.

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