Coordination Games on Directed Graphs

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We study natural strategic games on directed graphs, which capture the idea of coordination in the absence of globally common strategies. We show that these games do not need to have a pure Nash equilibrium and that the problem of determining their existence is NP-complete. The same holds for strong equilibria. We also exhibit some classes of games for which strong equilibria exist and prove that a strong equilibrium can then be found in linear time.

1 Introduction

In this paper we study a simple and natural class of strategic games. Assume a finite directed graph. Suppose that each node selects a colour from a private set of colours available for it. The payoff to a node is the number of (in)neighbours who chose the same colour.

These games are typical examples of coordination games. Recall that the idea behind *coordination* in strategic games is that players are rewarded for choosing common strategies. The games we study here are specific coordination games in the absence of globally common strategies.

Recently, we studied in [2], and more fully in [3], a very similar class of games in which the graphs were assumed to be undirected. However, the transition from undirected to directed graphs drastically changes the status of the games. For instance, for the case of directed graphs Nash equilibria do not need to exist, while they always exist when the graph is undirected. Consequently, in [2] and [3] we focused on the problem of existence of strong equilibria. We also argued there that such games are of relevance for the cluster analysis, the task of which is to partition in a meaningful way the nodes of a graph. The same applies here. Indeed, once the strategies are possible cluster names, a Nash equilibrium naturally corresponds to a 'satisfactory' clustering of the underlying graph.

The above two classes of games are also similar in that both are special cases of a number of well-studied types of games. One of them are ***polymatrix games*** introduced in [15]. In these games the payoff for each player is the sum of the payoffs from the individual two player games he plays with each other player separately. Another are ***graphical games*** introduced in [10]. In these games the payoff of each player depends only on the strategies of its neighbours in a given in advance graph structure over the set of players.

In addition both classes of games satisfy the ***positive population monotonicity*** (PPM) property introduced in [11] that states that the payoff of each player weakly increases if another player switches to his strategy. Coordination games on graphs are examples of games on networks, a vast research area recently surveyed in [9]. Other related references can be found in [3].

1.1 Plan of the paper and the results

In the next section we introduce preliminary definitions, following [3]. We define the coordination games on directed graphs in Section 3. In Section 4 we exhibit a number of cases when a strong equilibrium

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exists. Next, in Section 5 we study complexity of the problem of existence of Nash and strong equilibria and the problem of determining the complexity of finding a strong equilibrium in a natural case when it is known to exist. Finally, in Section 6 we discuss future directions.

The main results are as follows. If the underlying graph is a DAG, is complete or is such that every strongly connected component (SCC) is a simple cycle, then strong equilibria always exist and they can always be reached from any initial joint strategy by a sequence of coalitional improvement steps. The same is the case when only two colours are used.

In general Nash equilibria do not need to exist and the problem of determining their existence is NP-complete. The same is the case for strong equilibria. We also show that when every SCC is a simple cycle, then strong equilibrium can always be found in linear time.

2 Preliminaries

A ***strategic game*** $\mathcal{G} = (S_1, \ldots, S_n,\ p_1, \ldots, p_n)$ with $n > 1$ players, consists of a non-empty set S_i of ***strategies*** and a ***payoff function*** $p_i : S_1 \times \cdots \times S_n \to \mathbb{R}$, for each player i.

We denote $S_1 \times \cdots \times S_n$ by S, call each element $s \in S$ a ***joint strategy*** and abbreviate the sequence $(s_j)_{j \neq i}$ to s_{-i}. Occasionally we write (s_i, s_{-i}) instead of s. We call a strategy s_i of player i a ***best response*** to a joint strategy s_{-i} of his opponents if for all $s'_i \in S_i$, $p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i})$.

Fix a strategic game $\mathcal{G}$. We say that $\mathcal{G}$ satisfies the ***positive population monotonicity (PPM)*** if for all joint strategies s and players i, j, $p_i(s) \leq p_i(s_i, s_{-j})$. (Note that (s_i, s_{-j}) refers to the joint strategy in which player j chooses s_i.) So if more players (here just player j) choose player i's strategy and the remaining players do not change their strategies, then i's payoff weakly increases.

We call a non-empty subset $K := \{k_1, \ldots, k_m\}$ of the set of players $N := \{1, \ldots, n\}$ a ***coalition***. Given a joint strategy s we abbreviate the sequence $(s_{k_1}, \ldots, s_{k_m})$ of strategies to s_K and $S_{k_1} \times \cdots \times S_{k_m}$ to S_K. We also write (s_K, s_{-K}) instead of s. If there is a strategy x such that $s_i = x$ for all players $i \in K$, we also write (x_K, s_{-K}) for s.

Given two joint strategies s' and s and a coalition K, we say that s' is a ***deviation of the players in*** K from s if $K = \{i \in N \mid s_i \neq s'_i\}$. We denote this by $s \xrightarrow{K} s'$. If in addition $p_i(s') > p_i(s)$ holds for all $i \in K$, we say that the deviation s' from s is ***profitable***. Further, we say that the players in K ***can profitably deviate from*** s if there exists a profitable deviation of these players from s.

Next, we call a joint strategy s a ***k-equilibrium***, where $k \in \{1, \ldots, n\}$, if no coalition of at most k players can profitably deviate from s. Using this definition, a ***Nash equilibrium*** is a 1-equilibrium and a ***strong equilibrium***, see [5], is an n-equilibrium.

Given a joint strategy s, the ***social welfare*** of s is defined as,

$$SW(s) = \sum_{i \in N} p_i(s).$$

A ***coalitional improvement path*** (*c-improvement path*), is a maximal sequence $\rho = (s^1, s^2, \ldots)$ of joint strategies such that for every $k > 1$ there is a coalition K such that s^k is a profitable deviation of the players in K from s^{k-1}. If ρ is finite then by $last(\rho)$ we denote the last element of the sequence. Clearly, if a c-improvement path is finite, its last element is a strong equilibrium. We say that $\mathcal{G}$ has the ***finite c-improvement property*** (***c-FIP***) if every c-improvement path is finite. Further, we say that the function $P : S \to A$, where A is a set, is a ***generalized ordinal c-potential***, also called ***generalized strong potential***, see [7, 8], for $\mathcal{G}$ if for some strict partial ordering $(P(S), \succ)$ the fact that s' is a profitable deviation of the players in some coalition from s implies that $P(s') \succ P(s)$.

If a finite game admits a generalized ordinal c-potential then it has the c-FIP. The converse also holds, see, e.g., [3]. We say that $\mathscr{G}$ is ***c-weakly acyclic*** if for every joint strategy there exists a finite c-improvement path that starts at it. Note that games that have the c-FIP or are c-weakly acyclic game have a strong equilibrium.

We call a c-improvement path an ***improvement path*** if each deviating coalition consists of one player. The notions of a game having the ***FIP*** or being ***weakly acyclic***, see [16, 13], are then defined by referring to improvement paths instead of c-improvement paths.

3 Coordination games on directed graphs

We now introduce the class of games we are interested in. Fix a finite set of colours M and a weighted directed graph (G,w) without self loops in which each edge e has a non-negative weight w_e associated with. We say that a node j is a ***neighbour*** of the node i if there is an edge $j \to i$ in G. Let N_i denote the set of all neighbours of node i in the graph G. By a ***colour assignment*** we mean a function that assigns to each node of G a finite non-empty set of colours. For technical reasons we also introduce the concept of a ***bonus***, which is a function β that to each node i and a colour c assigns a natural number $\beta(i,c)$. (We allow zero as a natural number.)

Given a weighted graph (G,w), a colour assignment A and a bonus function β we define a strategic game $\mathscr{G}(G,w,A,\beta)$ as follows:

- the players are the nodes,
- the set of strategies of player (node) i is the set of colours $A(i)$; we refer to the strategies as ***colours*** and to joint strategies as ***colourings***,
- each payoff function is defined by

$$p_i(s) = \sum_{j \in N_i,\, s_i = s_j} w_{j \to i} + \beta(i, s_i).$$

So each node simultaneously chooses a colour and the payoff to the node is the sum of the weights of the edges from its neighbours that chose its colour augmented by the bonus to the node from choosing the colour. We call these games ***coordination games on directed graphs***, from now on just ***coordination games***. Because weights are non-negative each coordination game satisfies the PPM.

In the paper we mostly consider the case when all weights are 1 and all bonuses are 0. Then each payoff function is simply defined by

$$p_i(s) := |\{j \in N_i \mid s_i = s_j\}|.$$

Example 1 Consider the directed graph and the colour assignment depicted in Figure 1.

Take the joint strategy s that consists of the underlined strategies. Then the payoffs are as follows:

- 0 for the nodes 1, 7, 8 and 9,
- 1 for the nodes 2, 4, 5, 6,
- 2 for the node 3.

Note that the above joint strategy is not a Nash equilibrium. For example, node 1 can profitably deviate to colour a. □

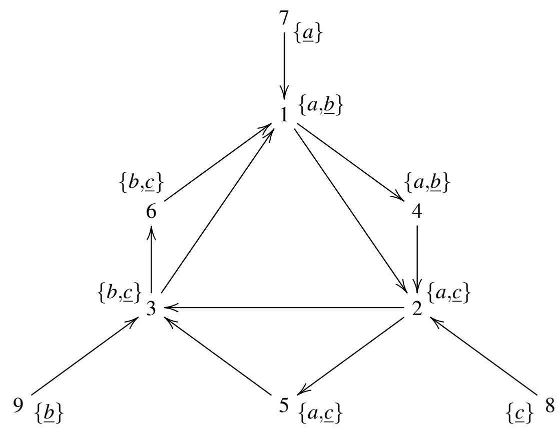

Figure 1: A directed graph with a colour assignment.

In what follows we study the problem of existence of Nash equilibria or strong equilibria in coordination games.

Finally, given a directed graph G and a set of nodes K, we denote by $G[K]$ the subgraph of G induced by K.

4 Strong equilibria

In this section we focus on the existence of strong equilibria. To start with, we have the following positive result.

Theorem 2 *Every coordination game whose underlying graph is a DAG has the c-FIP and a fortiori a strong equilibrium. Further, each Nash equilibrium is a strong equilibrium.*

Proof. Given a DAG $G := (V, E)$, where $V = \{1, \ldots, n\}$, we fix a permutation π of $1, \ldots, n$ such that for all $i, j \in V$

$$\text{if } i < j, \text{ then } (\pi(j) \to \pi(i)) \notin E. \tag{1}$$

So if $i < j$, then the payoff of the node $\pi(i)$ does not depend on the strategy selected by the node $\pi(j)$.

Then given a coordination game whose underlying directed graph is the DAG G we associate with each joint strategy s the sequence $p_{\pi(1)}(s), \ldots, p_{\pi(n)}(s)$ that we abbreviate to $p_\pi(s)$. We now claim that $p_\pi : S \to \mathbb{R}^n$ is a generalized ordinal c-potential when we take for the partial ordering $\succ$ on $p_\pi(S)$ the lexicographic ordering $>_{lex}$ on the sequences of reals.

Suppose that some coalition K profitably deviates from the joint strategy s to s'. Choose the smallest j such that $\pi(j) \in K$. Then $p_{\pi(j)}(s') > p_{\pi(j)}(s)$ and by (1) $p_{\pi(i)}(s') = p_{\pi(i)}(s)$ for $i < j$. This implies that $p_\pi(s') >_{lex} p_\pi(s)$, as desired. Hence the game has the c-FIP.

To prove the second claim, take a Nash equilibrium s and suppose it is not a strong equilibrium. Then some coalition K can profitably deviate from s to s'. Choose the smallest j such that $\pi(j) \in K$. Then $p_{\pi(j)}(s') > p_{\pi(j)}(s)$ and by (1) the payoff of $\pi(j)$ does not depend on the strategies selected by the other members of the coalition K. Hence $p_{\pi(j)}(s') = p_{\pi(j)}(s'_{\pi(j)}, s_{-\pi(j)})$, which contradicts the assumption that s is a Nash equilibrium. $\square$

The next result deals with a class of coordination games introduced in [3]. Given the set of colours M, we say that a directed graph G is ***colour complete*** (with respect to a colour assignment A) if for every colour $x \in M$ each component of $G[V_x]$ is complete, where $V_x = \{i \in V \mid x \in A_i\}$. In particular, every complete graph is colour complete.

Theorem 3 *Every coordination game on a colour complete directed graph has the c-FIP and a fortiori a strong equilibrium.*

Proof. In [3] it is proved that every uniform game has the c-FIP, where we call a coordination game on a directed graph G ***uniform*** if for every joint strategy s and for every edge $i \to j \in E$ it holds: if $s_i = s_j$ then $p_i(s) = p_j(s)$. (In [3] only undirected graphs are considered, but the proof remains valid without any change.) Clearly every coordination game on a colour complete directed graph is uniform. □

It is difficult to come up with other classes of directed graphs for which the coordination game has the c-FIP. Indeed, consider the following example.

Example 4 Consider a coordination game on a simple cycle $1 \to 2 \to \ldots \to n \to 1$, where $n \geq 3$ and such that the nodes share at least two colours, say a and b. Take the initial colouring $(a, b, \ldots, b)$. Then both $(a, \underline{b}, b, \ldots, b), (a, a, b, \ldots, b)$ and $(\underline{a}, a, b, \ldots, b), (b, a, b, \ldots, b)$ are profitable deviations. (To increase readability we underlined the strategies that were modified.) After these two steps we obtain a colouring that is a rotation of the first one. Iterating we obtain an infinite improvement path.

Hence the coordination game does not have the FIP and a fortiori the c-FIP. □

However, a weaker result holds, which, for reasons that will soon become clear, we prove for a larger class of games.

Theorem 5 *Every coordination game with bonuses on a simple cycle is c-weakly acyclic, so a fortiori has a strong equilibrium.*

To prove it, we first establish a weaker claim.

Lemma 6 *Every coordination game with bonuses on a simple cycle is weakly acyclic.*

Proof. To fix the notation, suppose that the considered graph is $1 \to 2 \to \ldots \to n \to 1$. Below for $i \in \{2, \ldots, n\}$, $i \ominus 1 = i - 1$, and $1 \ominus 1 = n$.

Let $MA(i)$ be the set of available colours to player i with the maximal bonus, i.e., $MA(i) = \{c \in A(i) \mid \beta(i, c) = \max_{d \in A(i)} \beta(i, d)\}$. Let $BR(i, s_{-i}) = \{c \in MA(i) \mid \text{colour } c \text{ is a best response of player } i \text{ to } s_{-i}\}$ be the set of best responses among the colours with the highest bonus only. The set $BR(i, s_{-i})$ is never empty because of the game structure and the fact that bonuses are natural numbers. Indeed, if $s_{i \ominus 1} \in MA(i)$, then $BR(i, s_{-i}) = \{s_{i \ominus 1}\}$ and otherwise $BR(i, s_{-i})$ is a non-empty subset of $MA(i)$.

Below we stipulate that whenever a player i updates in a joint strategy s his strategy to a best response to s_{-i}, he always selects a strategy from $BR(i, s_{-i})$.

Consider an initial joint strategy s. We construct a finite improvement path that starts with s as follows.

Phase 1. We proceed around the cycle and consider the players 1, 2, ..., $n-1$ in that order. For each player in turn, if his current strategy is not a best response, we update it to a best response respecting the above proviso. When this phase ends the current strategy of each of the players 1, 2, ..., $n-1$ is a best response.

If at this moment the current strategy of player n is also a best response, the current joint strategy s' is a Nash equilibrium and the path is constructed. Otherwise we move to the next phase.

Phase 2. We repeat the same process as in Phase 1, but starting with s' and player n and proceeding at most n steps. From now on at each step at least $n-1$ players have a best response strategy. So if at a certain moment the current strategy of the considered player is a best response, the current joint strategy is a Nash equilibrium and the path is constructed. Otherwise, after n steps, we move to the final phase.
Phase 3. We repeat the same process as in Phase 2. Now in the initial joint strategy each player i has a strategy from $MA(i)$. Because of the definition of $BR(i, s_{-i})$ each player can improve his payoff only if he switches to the strategy selected by his predecessor. So after at most n steps this phase terminates and we obtain a Nash equilibrium. □

By Lemma 6 every coordination game on a simple cycle has a Nash equilibrium. However, not every Nash equilibrium is then a strong equilibrium.

Example 7 Consider the directed graph depicted in Figure 2, together with the sets of colours associated with the nodes.

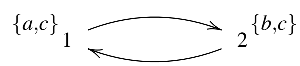

Figure 2: Nash equilibria versus strong equilibria

Clearly (a,b) is a Nash equilibrium. However, it is not a strong equilibrium since the coalition $\{1,2\}$ can profitably deviate to (c,c), which is a strong equilibrium. □

On the other hand, the following observation holds.

Lemma 8 *Consider a coordination game with bonuses on a simple cycle with n nodes. Then every Nash equilibrium is a k-equilibrium for all $k \in \{1,\ldots,n-1\}$.*

Proof. Take a Nash equilibrium s. It suffices to prove that it is an $(n-1)$-equilibrium. Suppose otherwise. Then for some coalition K of size $\leq n-1$ and a joint strategy s', $s \xrightarrow{K} s'$ is a profitable deviation.

Assume $k \ominus 1 = k-1$ if $k > 1$ and $1 \ominus 1 = n$. Take some $i \in K$ such that $i \ominus 1 \notin K$. We have $p_i(s') > p_i(s)$. Also $p_i(s'_i, s_{-i}) = p_i(s')$, since $s_{i\ominus 1} = s'_{i\ominus 1}$. So $p_i(s'_i, s_{-i}) > p_i(s)$, which contradicts the fact that s is a Nash equilibrium. □

Proof of Theorem 5. Take a joint strategy s. By Lemma 6 a finite improvement path starts at s and ends in a Nash equilibrium s'. By Lemma 8 s' is an $(n-1)$-equilibrium. If s' is not a strong equilibrium, then a profitable deviation $s' \xrightarrow{N} s''$ exists, where, recall, N is the set of all players. Because of the game structure the social welfare along each c-improvement path weakly increases, while in the last step the social welfare strictly increases. So $SW(s'') > SW(s)$.

If s'' is not a strong equilibrium, we repeat the above procedure starting with s''. Since each time the social welfare strictly increases, eventually this process stops and we obtain a finite c-improvement path. □

Using Theorem 5, we now show that every coordination game in which all strongly connected components are simple cycles is c-weakly acyclic. We first introduce some notations and make use of the following well-known decomposition result.

Theorem 9 ([6], page 92) *Every directed graph is a directed acyclic graph of its strongly connected components.*

Given a graph $G = (V,E)$, let $D = (V_D, E_D)$ be the corresponding directed acyclic graph (DAG) obtained by the above decomposition theorem and let $g : 2^V \to V_D$ be the function that maps each strongly connected component (SCC) in G to a node in D. Let $g^{-1}(v) = X \subseteq V$ where $g(X) = v$. Note that for each $i \in V$, there is a unique $v \in V_D$ such that $i \in g^{-1}(v)$, we denote this node by v_i. Let $|V_D| = m$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$ be a topological ordering of V_D (this is well-defined since D is a DAG). We define a labelling function $l_D : V_D \to \{1, \ldots, m\}$ as follows: for all $v \in V_D$, $l_D(v) = j$ iff $\theta_j = v$. We can extend l_D to a function $l : V \to \{1, \ldots, m\}$ in the natural way: for all $i \in V$, $l(i) = l_D(v)$ if $i \in g^{-1}(v)$.

Note that for each node $v \in V_D$, either $g^{-1}(v) = \{i\}$ for some $i \in V$ or $g^{-1}(v) = X \subseteq V$ with $|X| \geq 2$ and the subgraph of G induced by the set of nodes X forms an SCC. Also, note that every $v \in V_D$ and a joint strategy s in $\mathscr{G}$, defines a coordination game with bonuses $\mathscr{G}_v$ on the graph $G(v,s) = (V', E')$ which is the subgraph induced by the set of nodes $V' = g^{-1}(v)$. For $i \in V'$ and $a \in A(i)$ we put $\beta(i,a) := |\{j \in N_i \setminus V' \mid s_j = a\}|$.

Theorem 10 *Every coordination game on a directed graph G in which all strongly connected components of G are simple cycles is c-weakly acyclic and a fortiori has a strong equilibrium.*

Proof. Consider a coordination game $\mathscr{G}$ on a graph $G = (V,E)$ where all SCCs are simple cycles. Let $D = (V_D, E_D)$ be the corresponding DAG with $|V_D| = m$. Since all SCCs in G are simple cycles, it follows that for all $v \in V_D$, either $g^{-1}(v) = \{i\}$ or $g^{-1}(v) = X \subseteq V$ such that the induced graph on X forms a simple cycle in G.

Let $v \in V_D$ such that the induced graph on $g^{-1}(v)$ forms a simple cycle in G. For a joint strategy t in $\mathscr{G}$, consider the resulting game $\mathscr{G}_v$ on the graph (V', E'). Let $s^0 = t_{V'}$ (the restriction of the joint strategy t to nodes in V') and let $\rho : s^0, s^1, \ldots, s^k$ be a finite c-improvement path in $\mathscr{G}_v$ which is guaranteed to exist by Theorem 5. Define $CPath(\mathscr{G}_v, t)$ as follows:

$$CPath(\mathscr{G}_v, t) = \begin{cases} \varepsilon & \text{if } t_{V'} \text{ is a strong equilibrium in } \mathscr{G}_v, \\ \lambda_t(s^1), \ldots, \lambda_t(s^k) & \text{otherwise,} \end{cases}$$

where for all $h \in \{1, \ldots, k\}$, $\lambda_t(s^h)$ is the joint strategy in $\mathscr{G}$ defined as: for all $i \in V$, $(\lambda_t(s^h))_i = s^h_i$ if $i \in V'$ and $(\lambda_t(s^h))_i = t_i$ if $i \notin V'$.

For a joint strategy t in $\mathscr{G}$ and $v \in V_D$, if the underlying graph of the coordination game $\mathscr{G}_v$ with bonuses consists of exactly one node, then the game is trivially c-weakly acyclic. $CPath(\mathscr{G}_v, t)$ is then defined analogously.

Let t^0 be an arbitrary joint strategy in $\mathscr{G}$. We define a sequence of joint strategies as follows:

- $\rho_0 = t^0$,
- for $h \in \{0, 1, \ldots, m-1\}$, let $\rho_{h+1} = \rho_h \cdot CPath(\mathscr{G}_v, t^h)$ where $l_D(v) = h+1$ and $t^h = last(\rho_h)$.

Let $\rho = \rho_m$. From the definition of ρ_m and $CPath$, it follows that ρ is finite sequence of joint strategies in $\mathscr{G}$. By induction on the length of ρ, we can claim that for every subsequent pair of joint strategies t^k and t^{k+1} in ρ, there is a coalition $K \subseteq V$ for which t^{k+1} is a profitable deviation from t^k. To complete the proof, it suffices to argue that ρ is maximal, or equivalently, that $last(\rho)$ is a strong equilibrium.

Suppose $last(\rho)$ is not a strong equilibrium. Then there exists $K \subseteq V$ and a joint strategy s such that there is a profitable deviation of players in K from $last(\rho)$ to s. Let d be the least element of the set $\{l(i) \mid i \in K\}$ and $X = K \cap \{i \in V \mid l(i) = d\}$. By definition of a profitable deviation, we have that for all $i \in X$, $p_i(s) > p_i(last(\rho))$. Note that for all $i \in X$ and for all $j \in N_i \setminus g^{-1}(v_i)$, we have $l(j) < d$. Therefore, $(N_i \setminus g^{-1}(v_i)) \cap K = \emptyset$. Also note that for all $j \in g^{-1}(v_i)$, $(last(\rho_d))_j = (last(\rho))_j$. But this implies that the coalition X has a profitable deviation from the joint strategy $(last(\rho_d))_X$ to s_X in the game $\mathscr{G}_{v_i}$. This contradicts the fact that $last(\rho_d)$ is a strong equilibrium in the game $\mathscr{G}_{v_i}$. □

We conclude this section by considering another class of coordination games. Example 4 shows that even when only two colours are used, the coordination game does not need to have the c-FIP. On the other hand, a weaker property does hold.

Theorem 11 *Every coordination game in which only two colours are used is c-weakly acyclic and a fortiori has a strong equilibrium.*

Proof. We prove the result for a more general class of games, namely the ones that satisfy the PPM. Call the colours blue and red, that we abbreviate to b and r. When a node selected blue we refer to it as a blue node, and the same for the red colour.

Take a joint strategy s^1. Consider a maximal sequence ξ of profitable deviations of the coalitions starting in s in which the nodes can only switch to blue. At each step the number of blue nodes increases, so ξ is finite. Let $s^1,\ldots,s^k$, where $k \geq 1$, be the successive joint strategies of ξ.

If s^k is a strong equilibrium, then ξ is the desired finite improvement path. Otherwise consider a maximal sequence χ of profitable deviations of the coalitions starting in s^k in which the nodes can only switch to red. χ is finite. Let $s^k, s^{k+1},\ldots,s^{k+l}$, where $l \geq 1$, be the successive joint strategies of χ.

We claim that s^{k+l} is a strong equilibrium. Suppose otherwise. Then for some joint strategy s', $s^{k+l} \xrightarrow{K} s'$ is a profitable deviation of some coalition K. Let L be the set of nodes from K that switched in this deviation to blue. By the definition of s^{k+l} the set L is non-empty.

Given a set of nodes M and a joint strategy s we denote by $(M:b, s_{-M})$ the joint strategy obtained from s by letting the nodes in M to select blue, and similarly for the red colour. Also it should be clear what joint strategy we denote by $(M:b, P\setminus M:r, s_{-P})$, where $M \subseteq P$.

We claim that $s^{k+l} \xrightarrow{L} (L:b, s^{k+l}_{-L})$ is a profitable deviation of the players in L. Indeed, we have for all $i \in L$

$$p_i(L:b, s^{k+l}_{-L}) > p_i(s^{k+l}), \tag{2}$$

since by the PPM $p_i(L:b, s^{k+l}_{-L}) \geq p_i(s')$ and by the assumption $p_i(s') > p_i(s^{k+l})$.

Let M be the set of nodes from L that are red in s^k. Suppose that M is non-empty. We show that then

$$p_M(M:r, L\setminus M:b, s^k_{-L}) < p_M(M:b, L\setminus M:b, s^k_{-L}). \tag{3}$$

Indeed, we have for all $i \in M$

$$\begin{array}{rl} & p_i(M:r, L\setminus M:b, s^k_{-L}) \leq p_i(M:r, L\setminus M:b, s^{k+l}_{-L}) \\ \leq & p_i(M:r, L\setminus M:r, s^{k+l}_{-L}) < p_i(M:b, L\setminus M:b, s^{k+l}_{-L}) \\ \leq & p_i(M:b, L\setminus M:b, s^k_{-L}), \end{array}$$

where the weak inequalities are due to the PPM and the strict inequality holds by the definition of L.

But $s^k = (M:r, L\setminus M:b, s^k_{-L})$, so (3) contradicts the definition of s^k. So M is empty, i.e., all nodes from L are blue in s^k. We now have for all $i \in L$

$$\begin{array}{rl} & p_i(L:r, s^k_{-L}) \leq p_i(L:r, s^{k+l}_{-L}) = p_i(s^{k+l}) \\ < & p_i(L:b, s^{k+l}_{-L}) \leq p_i(L:b, s^k_{-L}), \end{array}$$

where again the weak inequalities are due to the PPM and the strict inequality holds by (2).

But $(L:r, s^k_{-L}) = s^k$, so we proved that $s^k \xrightarrow{L} (L:b, s^k_{-L})$ is a profitable deviation. This yields a contradiction with the definition of s^k. □

When the underlying graph is symmetric and the set of strategies for every node is the same, the existence of strong equilibrium for coordination games with two colours follows from Proposition 2.2 in [12]. Theorem 11 shows a stronger result, that in the general case, these games are c-weakly acyclic. The following example shows that when three colours are used, Nash equilibria, so a fortiori strong equilibria do not need to exist.

Example 12 Consider the directed graph depicted in Figure 1 of Example 1, together with the sets of colours associated with the nodes. We argue that the coordination game associated with this graph does not have a Nash equilibrium. Note that for nodes 7, 8 and 9 the only option is to select the unique strategy in its strategy set. The best response for nodes 4, 5 and 6 is to always select the same strategy as nodes 1, 2 and 3 respectively. Therefore, to show that the game does not have a Nash equilibrium, it suffices to consider the strategies of nodes 1, 2 and 3. We denote this by the triple (s_1, s_2, s_3). Below we list all such joint strategies and we underline a strategy that is not a best response to the choice of other players: $(\underline{a},a,b)$, $(a,a,\underline{c})$, $(a,c,\underline{b})$, $(a,\underline{c},c)$, $(b,\underline{a},b)$, $(\underline{b},a,c)$, $(b,c,\underline{b})$ and $(\underline{b},c,c)$. □

Call now a graph a ***coloured DAG*** (with respect to a colour assignment A) if for each available colour x the components of the subgraph induced by the nodes having colour x are DAGs. In view of Theorem 3 it is tempting to try to generalize Theorem 2 to coloured DAGs. However, the directed graph depicted in Figure 1 is a coloured DAG and, as explained in the above example, the coordination game on this graph has no Nash equilibrium.

5 Complexity issues

Next, we study the complexity of the existence problems and of the problem of finding strong equilibria.

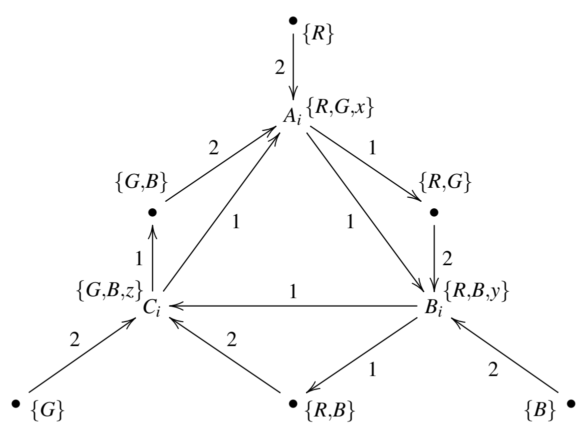

Figure 3: Gadget D_i with three parameters $x, y, z \in \{\top, \bot\}$ and three distinguished nodes A_i, B_i, C_i.

Theorem 13 *The Nash equilibrium existence problem in coordination games is NP-complete.*

Proof. The problem is in NP, since we can simply guess a colour assignment and checking whether it is a Nash equilibrium can be done in polynomial time.

To prove NP-hardness we provide a reduction from the 3-SAT problem, which is NP-complete. Notice that an edge with a natural number weight w can be simulated by adding w extra players to the game. More precisely, an edge $(i \to j)$ with the weight w can be simulated by the extra set of players

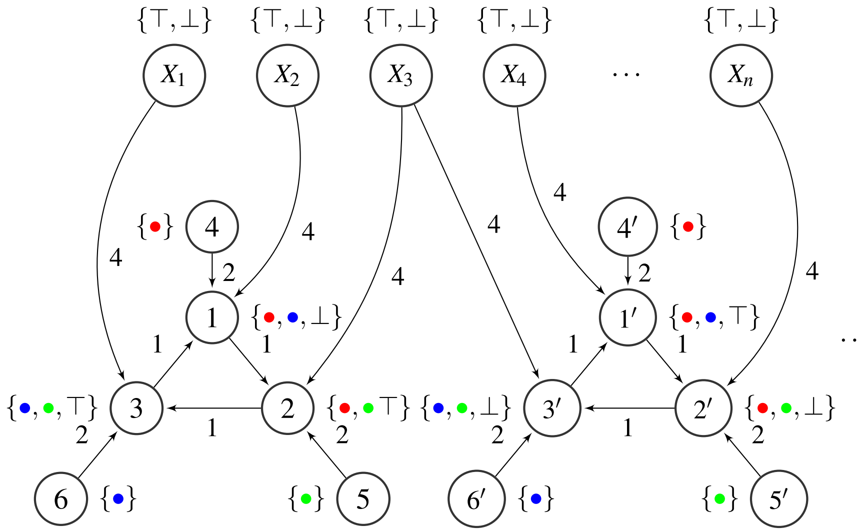

Figure 4: The game $\mathscr{G}_\phi$ corresponding to the formula $\phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee x_4 \vee \neg x_n)$, where in each gadget the nodes of indegree 1 are omitted.

$\{i_1, \ldots, i_w\}$ and the following $2 \cdot w$ unweighted edges: $\{(i \to i_1), (i \to i_2), \ldots, (i \to i_w), (i_1 \to j), (i_2 \to j), \ldots, (i_w \to j)\}$. Given a colour assignment in the game with the weighted edges, we then assign to each of the nodes $i_1, \ldots, i_w$ the colour set of the node i.

Therefore we will assume that edges can have such weights assigned to them, because this simplifies our construction. Assume we are given a 3-SAT formula

$$\phi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$$

with k clauses and n propositional variables $x_1, \ldots, x_n$, where each a_i, b_i, c_i is a literal equal to x_j or $\neg x_j$ for some j. We will construct a coordination game $\mathscr{G}_\phi$ of size $\mathscr{O}(k)$ such that $\mathscr{G}_\phi$ has a Nash equilibrium iff ϕ is satisfiable.

First, for every propositional variable x_i we have a corresponding node X_i in $\mathscr{G}_\phi$ with two possible colours $\top$ and $\bot$. Intuitively, for a given truth assignment, if x_i is true then $\top$ should be chosen for X_i and otherwise $\bot$ should be chosen. In our construction we make use of the following gadget, denoted by $D_i(x,y,z)$, with three parameters $x,y,z \in \{\top, \bot\}$ and i used just for labelling purposes, and presented in Figure 3. This gadget behaves similarly to the game without Nash equilibrium analyzed in Example 12.

What is important is that for all possible parameters values, the gadget $D_i(x,y,z)$ does not have a Nash equilibrium. Indeed, each of the nodes A_i, B_i, or C_i can always secure a payoff 2, so selecting $\top$ or $\bot$ is never a best response and hence in no Nash equilibrium a node chooses $\top$ or $\bot$. The rest of the reasoning is as in Example 12.

For any literal, l, let

$$\mathrm{pos}(l) := \begin{cases} \top & \text{if } l \text{ is a positive literal} \\ \bot & \text{otherwise} \end{cases}$$

For every clause $(a_i \vee b_i \vee c_i)$ in ϕ we add to the game graph $\mathscr{G}_\phi$ the $D_i(\mathrm{pos}(a_i), \mathrm{pos}(b_i), \mathrm{pos}(c_i))$ instance of the gadget. Finally, for every literal a_i, b_i, or c_i in ϕ, which is equal to x_j or $\neg x_j$ for some j, we add an edge from X_j to A_i, B_i, or C_i, respectively, with weight 4. We depict an example game $\mathscr{G}_\phi$ in Figure 4.

We claim that $\mathscr{G}_\phi$ has a Nash equilibrium iff ϕ is satisfiable.

($\Rightarrow$) Assume there is a Nash equilibrium s in the game $\mathscr{G}_\phi$. We claim that the truth assignment $\nu : \{x_1,\ldots,x_n\} \to \{\top,\bot\}$ that assigns to each x_j the colour selected by the node X_j in s makes ϕ true. Fix $i \in \{1,\ldots,k\}$. We need to show that ν makes one of the literals a_i, b_i, c_i of the clause $(a_i \vee b_i \vee c_i)$ true.

From the above observation about the gadgets it follows that at least one of the nodes A_i, B_i, C_i selected in s the same colour as its neighbour X_j. Without loss of generality suppose it is A_i. The only colour these two nodes, A_i and X_j, have in common is $\mathrm{pos}(a_i)$. So X_j selected in s $\mathrm{pos}(a_i)$, which by the definition of ν equals $\nu(x_j)$. Moreover, by construction x_j is the variable of the literal a_i. But $\nu(x_j) = \mathrm{pos}(a_i)$ implies that ν makes a_i true.

($\Leftarrow$) Assume ϕ is satisfiable. Take a truth assignment $\nu : \{x_1,\ldots,x_n\} \to \{\top,\bot\}$ that makes ϕ true. For all j, we assign to the node X_j the colour $\nu(x_j)$. We claim that this assignment can be extended to a Nash equilibrium in $\mathscr{G}_\phi$.

Fix $i \in \{1,\ldots,k\}$ and consider the $D_i(\mathrm{pos}(a_i),\mathrm{pos}(b_i),\mathrm{pos}(c_i))$ instance of the gadget. The truth assignment ν makes the clause $(a_i \vee b_i \vee c_i)$ true. Suppose without loss of generality that ν makes a_i true. We claim that then it is always a unique best response for the node A_i to select the colour $\mathrm{pos}(a_i)$.

Indeed, let j be such that $a_i = x_j$ or $a_i = \neg x_j$. Notice that the fact that ν makes a_i true implies that $\nu(x_j) = \mathrm{pos}(a_i)$. So when node A_i selects $\mathrm{pos}(a_i)$, the colour assigned to X_j, its payoff is 4.

This partial assignment of colours can be completed to a Nash equilibrium. Indeed, remove from the directed graph of $\mathscr{G}_\phi$ all X_j nodes and the nodes that secured the payoff 4, together with the edges that use any of these nodes. The resulting graph has no cycles, so by Theorem 2 the corresponding coordination game has a Nash equilibrium. Combining both assignments of colours we obtain a Nash equilibrium in $\mathscr{G}_\phi$. □

Corollary 14 *The strong equilibrium existence problem in coordination games is NP-complete.*

Proof. It suffices to note that in the above proof the ($\Rightarrow$) implication holds for a strong equilibrium, as well, while in the proof of the ($\Leftarrow$) implication by virtue of Theorem 2 actually a strong equilibrium is constructed. □

An interesting application of Theorem 13 is in the context of polymatrix games. These are finite strategic form games in which the influence of a pure strategy selected by any player on the payoff of any other player is always the same, regardless of what strategies other players select. Formally, for all pairs of players i and j there exists a partial payoff function a^{ij} such that for any joint strategy $s = (s_1,\ldots,s_n)$, the payoff of player i is given by $p_i(s) := \sum_{j\neq i} a^{ij}(s_i,s_j)$. In [14] we proved that deciding whether a polymatrix game has a Nash equilibrium is NP-complete. We can strengthen this result as follows.

Theorem 15 *Deciding whether a polymatrix game with 0/1 partial payoffs has a Nash equilibrium is NP-complete.*

Proof. We can efficiently translate any coordination game $\mathscr{G}(G,M,w,A,\beta)$ into a polymatrix game $\mathscr{P}$ with only 0/1 partial payoff as follows. The number of players in $\mathscr{P}$ will be equal to the number of nodes in G and the set of strategies for each player will be M. We define $a^{ij}(s_i,s_j) := 1$ if $s_i = s_j$ and $j \in N_i$, and $a^{ij}(s_i,s_j) := 0$ otherwise.

Notice that any joint strategy $s = (s_1,\ldots,s_n)$ in $\mathscr{G}$ is also a joint strategy in $\mathscr{M}$ with exactly the same payoff, because $p_i^{\mathscr{P}}(s) = \sum_{j\neq i} a^{ij}(s_i,s_j) = |\{j \in N_i \mid s_i = s_j\}| = p_i^{\mathscr{G}}(s)$. It follows that Nash equilibria in $\mathscr{G}$ and $\mathscr{P}$ coincide. In particular, there exists Nash equilibrium in $\mathscr{G}$ if and only if there exists one in $\mathscr{P}$, but the former problem was shown to be NP-hard in Theorem 13, so the latter is also NP-hard. On the other hand, for any polymatrix game we can guess a joint strategy and check whether it is a Nash equilibrium in polynomial time, which shows this decision problem is in fact NP-complete. □

Next, we determine the complexity of finding a strong equilibrium. We begin with the following auxiliary result.

Theorem 16 *A strong equilibrium of a coordination game with bonuses on a simple cycle can be found in linear time.*

Proof. Let n be the number of players in the game and C the number of possible colours. We assume adjacency list representation for the game graph, binary representation of the bonuses and that the list of colours available to player i is given as a list of length $|A(i)|$ of elements of size $\log C$. Formally, the size of the input for player i only is $\Theta(|A(i)|\log C + \sum_{c\in A(i)} \log(\beta(i,c)+1))$; the sum of these over $i = 1,\dots,n$ gives the total size of the input.

First note that for any colour assignment, the best response of the i-th player can be found in time linear in the size of her part of the input just by checking all possible colours in $A(i)$. Second, each phase of the algorithm in Lemma 6 looks for the best response (with a preference given to colours with a higher bonus) of each player at most once, which will require time linear in the size of the whole input. The algorithm requires at most three such phases before a Nash equilibrium is found, so it runs in linear time.

Note that thanks to Lemma 8 we know that any NE in such a game structure is already a $(n-1)$-equilibrium, so the only way this joint strategy is not a strong equilibrium is when all n players can strictly improve their payoff. However, in any Nash equilibrium a player has to have her payoff at at most one below the maximum possible one, because that is the minimum payoff for picking a colour with the highest bonus. Moreover, player's payoff can only be a natural number.

Therefore, the only possibility when a NE is not a strong equilibrium is when there is a joint strategy which gives all the players their maximum possible payoff, i.e. each player is assigned a colour with the highest possible bonus as well as gets an extra $+1$ to her payoff for colour agreement with her only neighbour. The latter implies that all the players need to pick the same colour in such a joint strategy.

To check whether such a joint strategy exists we do the following. Let $p = \mathrm{argmin}_i|A(i)|$ be the player with the least number of colours to choose from. We pick the set of her colours with the maximal bonus and intersect it with the set of colours with the maximal bonus for every other player. An intersection of two sets represented as lists of length a and b of elements of size K can be done in $\Theta(aK+bK)$ time, so the total running time will be $\Theta(n|A(p)|\log C + \sum_{i=1}^{n}|A(i)|\log C) = \Theta(\sum_{i=1}^{n}|A(i)|\log C)$, because $|A(p)| \leq |A(i)|$ for all i, which is linear. If the final set is empty then any NE is a strong equilibrium and otherwise we know how to construct one. □

Corollary 17 *A strong equilibrium of a coordination game on a graph in which all strongly connected components are simple cycles can be computed in linear time.*

6 Conclusions

We presented here a study of a simple class of coordination games on directed graphs. We focused on the existence of Nash and strong equilibria. We also studied the complexity of checking for the existence of

Nash and strong equilibria, as well as the complexity of computing a strong equilibrium in certain cases where it is guaranteed to exist.

A number of open problems remain. We showed that in general Nash equilibria and strong equilibria are not guaranteed to exist. However, if the underlying graph is a DAG, is colour complete or is such that every SCC is a simple cycle, then strong equilibria always exist. It would be interesting to identify other classes of graphs for which Nash or strong equilibria exist.

The proof of Lemma 6 shows that in the case of a simple cycle, starting from any initial joint strategy a Nash equilibrium can be found by an improvement path of length at most $3n$. Also, each step of such a path can be constructed in linear time. Additionally, the proof of Theorem 5 shows that a strong equilibrium can be found by an improvement path of length at most $3n+1$, possibly augmented by a single profitable deviation of all players. It would be interesting to extend this analysis of bounds on the lengths of improvement paths to other cases when a Nash or a strong equilibrium is known to exist.

In the future we plan to study the inefficiency of equilibria in coordination games on directed graphs. Also, we plan to study coordination games on finite directed weighted graphs. While we already defined here these games, we used weights solely as a means to simplify the argument in the proof of Theorem 13. It should be noted that Lemma 6 does not hold for finite directed weighted graphs and, as a consequence, Theorems 5, 10, and 16 do not hold either. A counterexample to Lemma 6 can be constructed by modifying the game in Figure 1 as follows. Nodes 4, 5, 6 are removed and replaced by assigning weight 2 to all the edges in the cycle. Nodes 7, 8, 9 are also removed and replaced by a +1 bonus to the colour of the node removed. It is easy to see that the behaviour of this new game will mimic the game in Figure 1. On the other hand, Theorem 2 and its proof is still valid for finite directed weighted graphs as well is Theorem 13, because checking whether a colour assignment is a Nash equilibrium can still be done in polynomial time for them.

As an example of coordination games on weighted directed graphs consider the problem of a choice of the trade treaties between various countries. Assume a directed weighted graph in which the nodes are the countries and the weight on an edge $i \to j$ corresponds to the percentage of the overall import of country j from country i. Suppose additionally that each country should choose a specific trade treaty, that the options for the countries differ (for instance because of its geographic location) and that each treaty offers the same tax-free advantages. Then once the countries choose the treaties, the payoff to each country is the aggregate percentage of its import that is tax-free.

The case of weighted directed graphs can be seen as a minor modification of the ***social network games*** with obligatory product selection that we introduced and analyzed in [4]. These are games associated with a threshold model of a social network introduced in [1] which is based on weighted graphs with thresholds. The difference consists of using thresholds equal to 0. However, setting the thresholds to 0 essentially changes the nature of the games and crucially affects the validity of several arguments.

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