On Upper Bounds on the Church-Rosser Theorem

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The Church-Rosser theorem in the type-free λ-calculus is well investigated both for β-equality and β-reduction. We provide a new proof of the theorem for β-equality with no use of parallel reductions, but simply with Takahashi's translation (Gross-Knuth strategy). Based on this, upper bounds for reduction sequences on the theorem are obtained as the fourth level of the Grzegorczyk hierarchy.

1 Introduction

1.1 Background

The Church-Rosser theorem [3] is one of the most fundamental properties on rewriting systems, which guarantees uniqueness of computation and consistency of a formal system. For instance, for proof trees and formulae of logic the unique normal forms of the corresponding terms and types in a Pure Type System (PTS) can be chosen as their denotations [21] via the Curry-Howard isomorphism.

The Church-Rosser theorem for β-reduction states that if $M \twoheadrightarrow N_1$ and $M \twoheadrightarrow N_2$ then we have $N_1 \twoheadrightarrow P$ and $N_2 \twoheadrightarrow P$ for some P. Here, we write $\twoheadrightarrow$ for the reflexive and transitive closure of one-step reduction $\rightarrow$. Two proof techniques of the theorem are well known; tracing the residuals of redexes along a sequence of reductions [3, 1, 8], and working with parallel reduction [4, 1, 8, 19] known as the method of Tait and Martin-Löf. Moreover, a simpler proof of the theorem is established only with Takahashi's translation [19] (the Gross-Knuth reduction strategy [1]), but with no use of parallel reduction [12, 5].

On the other hand, the Church-Rosser theorem for β-equality states that if $M =_\beta N$ then there exists P such that $M \twoheadrightarrow P$ and $N \twoheadrightarrow P$. Here, we write $M =_\beta N$ iff M is obtained from N by a finite series of reductions ($\twoheadrightarrow$) and reversed reductions ($\twoheadleftarrow$). As the Church-Rosser theorem for β-reduction has been well studied, to the best of our knowledge the Church-Rosser theorem for β-equality is always *secondary* proved as a corollary from the theorem for β-reduction [3, 4, 1, 8].

One of our motivations is to analyze quantitative properties in general of reduction systems. For instance, measures for developments are investigated by Hindley [7] and de Vrijer [18]. Statman [16] proved that deciding the $\beta\eta$-equality of typable λ-terms is not elementary recursive. Schwichtenberg [14] analysed the complexity of normalization in the simply typed lambda-calculus, and showed that the number of reduction steps necessary to reach the normal form is bounded by a function at the fourth level of the Grzegorczyk hierarchy $\mathcal{E}^4$ [6], i.e., a non-elementary recursive function. Later Beckmann [2] determined the exact bounds for the reduction length of a term in the simply typed λ-calculus. Xi [22] showed bounds for the number of reduction steps on the standardization theorem, and its application to normalization. In addition, Ketema and Simonsen [9] extensively studied valley sizes of confluence and the Church-Rosser property in term rewriting and λ-calculus as a function of given term sizes and reduction lengths. However, there are no known bounds for the Church-Rosser theorem for β-equality.

In this study, we are also interested in quantitative analysis of the witness of the Church-Rosser theorem: how to find common contractums with the least size and with the least number of reduction

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steps. For the theorem for β-equality ($M =_\beta N$ implies $M \twoheadrightarrow^{l_3} P$ and $N \twoheadrightarrow^{l_4} P$ for some P), we study functions that set bounds on the least size of a common contractum P, and the least number of reduction steps l_3 and l_4 required to arrive at a common contractum, involving the term sizes of M and N, and the length of $=_\beta$. For the theorem for β-reduction ($M \twoheadrightarrow^{l_1} N_1$ and $M \twoheadrightarrow^{l_2} N_2$ implies $N_1 \twoheadrightarrow^{l_3} P$ and $N_2 \twoheadrightarrow^{l_4} P$ for some P), we study functions that set bounds on the least size of a common contractum P, and the least number of reduction steps l_3 and l_4 required to arrive at a common contractum, involving the term size of M and the lengths of l_1 and l_2.

1.2 New results of this paper

In this paper, first we investigate *directly* the Church-Rosser theorem for β-equality *constructively* from the viewpoint of Takahashi translation [19]. Although the two statements are equivalent to each other, the theorem for β-reduction is a special case of that for β-equality. Our investigation shows that a common contractum of M and N such that $M =_\beta N$ is determined by (i) M and the number of occurrences of reduction ($\rightarrow$) appeared in $=_\beta$, and also by (ii) N and that of reversed reduction ($\leftarrow$). The main lemma plays a key role and reveals a new invariant involved in the equality $=_\beta$, independently of an exponential combination of reduction and reversed reduction. Next, in terms of iteration of translations, this characterization of the Church-Rosser theorem makes it possible to analyse how large common contractums are and how many reduction-steps are required to obtain them. From this, we obtain an upper bound function for the theorem in the fourth level of the Grzegorczyk hierarchy. In addition, the theorem for β-reduction is handled as a *special case* of the theorem for β-equality, where the key notion is contracting new redexes under development.

1.3 Outline of paper

This paper is organized as follows. Section 1 is devoted to background, related work, and new results of this paper. Section 2 gives preliminaries including basic definitions and notions. Following the main lemma, Section 3 provides a new proof of the Church-Rosser theorem for β-equality. Based on this, reduction length and term size for the theorem are analyzed in Section 4, and then we compare with related results. Section 5 concludes with remarks, related work, and further work.

2 Preliminaries

The set of λ-terms denoted by Λ is defined with a countable set of variables as follows.

Definition 1 (λ-terms)

$$M,N,P,Q \in \Lambda ::= x \mid (\lambda x.M) \mid (MN)$$

We write $M \equiv N$ for the syntactical identity under renaming of bound variables. We suppose that every bound variable is distinct from free variables. The set of free variables in M is denoted by $\mathrm{FV}(M)$.

If M is a subterm of N then we write $M \sqsubseteq N$ for this. In particular, we write $M \sqsubset N$ if M is a proper subterm of N. If $P \sqsubseteq M$ and $Q \sqsubseteq M$, and moreover there exist no terms N such that $N \sqsubseteq P$ and $N \sqsubseteq Q$, then we write $P \parallel Q$ for this, i.e., P and Q have non-overlapping parts of M.

Definition 2 (β-reduction) *One step β-reduction $\rightarrow$ is defined as follows, where $M[x := N]$ denotes a result of substituting N for every free occurrence of x in M.*

1. $(\lambda x.M)N \rightarrow M[x := N]$

2. *If $M \to N$ then $PM \to PN$, $MP \to MP$, and $\lambda x.M \to \lambda x.N$.*

A term of the form of $(\lambda x.P)Q \sqsubseteq M$ is called a redex of M. A redex is denoted by R or S, and we write $R : M \to N$ if N is obtained from M by contracting the redex $R \sqsubseteq M$. We write $\twoheadrightarrow$ for the reflexive and transitive closure of $\to$. If $R_1 : M_0 \to M_1, \ldots, R_n : M_{n-1} \to M_n$ $(n \geq 0)$, then for this we write $R_0 \ldots R_n : M_0 \twoheadrightarrow^n M_n$, and the *reduction sequence* is denoted by the list $[M_0, M_1, \ldots, M_n]$. For operating on a list, we suppose fundamental list functions, `append`, `reverse`, and `tail` (`cdr`).

Definition 3 (β-equality) *A term M is β-equal to N with reduction sequence ls, denoted by $M =_\beta N$ with ls is defined as follows:*

1. *If $M \twoheadrightarrow N$ with reduction sequence ls, then $M =_\beta N$ with ls.*
2. *If $M =_\beta N$ with ls, then $N =_\beta M$ with* $\texttt{reverse}(ls)$.
3. *If $M =_\beta P$ with ls_1 and $P =_\beta N$ with ls_2, then $M =_\beta N$ with* $\texttt{append}(ls_1, \texttt{tail}(ls_2))$.

Note that $M =_\beta N$ with reduction sequence ls iff there exist terms $M_0, \ldots, M_n (n \geq 0)$ in this order such that $ls = [M_0, \ldots, M_n]$, $M_0 \equiv M, M_n \equiv N$, and either $M_i \to M_{i+1}$ or $M_{i+1} \to M_i$ for each $0 \leq i \leq n-1$. In this case, we say that the *length* of $=_\beta$ is n, denoted by $=_\beta^n$. The arrow in $M_i \to M_{i+1}$ is called a *right arrow*, and the arrow in $M_{i+1} \to M_i$ is called a *left arrow*, denoted also by $M_i \leftarrow M_{i+1}$.

Definition 4 (Term size) *Define a function $|\ | : \Lambda \to \mathbf{N}$ as follows.*

1. $|x| = 1$
2. $|\lambda x.M| = 1 + |M|$
3. $|MN| = 1 + |M| + |N|$

Definition 5 (Takahashi's * and iteration) *The notion of Takahashi translation M^* [19], that is, the Gross-Knuth reduction strategy [1] is defined as follows.*

1. $x^* = x$
2. $((\lambda x.M)N)^* = M^*[x := N^*]$
3. $(MN)^* = M^* N^*$
4. $(\lambda x.M)^* = \lambda x.M^*$

The 3rd case above is available provided that M is not in the form of a λ-abstraction. We write an iteration of the translation [20] as follows.

1. $M^{0*} = M$
2. $M^{n*} = (M^{(n-1)*})^*$

We write $\sharp(x \in M)$ for the number of free occurrences of the variable x in M.

Lemma 1 $|M[x := N]| = |M| + \sharp(x \in M) \times (|N| - 1)$.

Proof. By straightforward induction on M. □

Definition 6 ($\mathsf{Redex}(M)$) *The set of all redex occurrences in a term M is denoted by* $\mathsf{Redex}(M)$. *The cardinality of the set* $\mathsf{Redex}(M)$ *is denoted by* $\sharp\mathsf{Redex}(M)$.

Lemma 2 ($\sharp\mathsf{Redex}(M)$) *We have* $\sharp\mathsf{Redex}(M) \leq \frac{1}{2}|M| - 1$ *for* $|M| \geq 4$.

Proof. Note that $\sharp\mathsf{Redex}(M) = 0$ for $|M| < 4$. By straightforward induction on M for $|M| \geq 4$. □

Lemma 3 (Substitution) *If* $M_1 \twoheadrightarrow^{l_1} N_1$ *and* $M_2 \twoheadrightarrow^{l_2} N_2$*, then* $M_1[x := M_2] \twoheadrightarrow^{l} N_1[x := N_2]$ *where* $l = l_1 + \sharp(x \in M_1) \times l_2$.

Proof. By induction on the derivation of $M_1 \twoheadrightarrow^{l_1} N_1$. The case of $l_1 = 0$ requires induction on $M_1 \equiv N_1$. We also need induction on the derivation of $M_1 \to N_1$, and we show here one of the interesting cases.

1. Case of $(\lambda y.M)N \twoheadrightarrow^1 M[y := N]$:

$$\begin{array}{rcl} (\lambda y.M[x := M_2])(N[x := M_2]) & \twoheadrightarrow^{m_1} & (\lambda y.M[x := N_2])(N[x := M_2]) \text{ by IH1} \\ & \twoheadrightarrow^{m_2} & (\lambda y.M[x := N_2])(N[x := N_2]) \text{ by IH2} \\ & \twoheadrightarrow^{1} & (M[x := N_2])[y := (N[x := N_2])] \end{array}$$

Here, IH1 is $\lambda y.M[x := M_2] \twoheadrightarrow^{m_1} \lambda y.M[x := N_2]$ with $m_1 = \sharp(x \in M) \times l_2$. IH2 is $N[x := M_2] \twoheadrightarrow^{m_2} N[x := N_2]$ with $m_2 = \sharp(x \in N) \times l_2$. Therefore,

$$\begin{array}{rcl} l & = & m_1 + m_2 + 1 \\ & = & 1 + \sharp(x \in M) \times l_2 + \sharp(x \in N) \times l_2 \\ & = & 1 + \sharp(x \in ((\lambda y.M)N)) \times l_2. \end{array}$$

□

Proposition 1 (Term size after n-step reduction) *If* $M \twoheadrightarrow^n N$ $(n \geq 1)$ *then*

$$|N| < 8\left(\frac{|M|}{8}\right)^{2^n}.$$

Proof. By induction on n.

1. Case of $n = 1$, where $M \to M_1$:

   The following inequality can be proved by induction on the derivation of $M \to M_1$:

$$|M_1| \leq \frac{|M|^2}{2^3} - 1$$

2. Case of $n = k + 1$, where $M \to M_1 \twoheadrightarrow^k M_{k+1}$:

$$\begin{array}{rcll} |M_{k+1}| & < & 8\left(\frac{|M_1|}{8}\right)^{2^k} & \text{from the induction hypothesis} \\ & < & 8\left(\left(\frac{|M|}{8}\right)^{2}\right)^{2^k} & \text{from } |M_1| < \frac{1}{8}|M|^2 \\ & = & 8\left(\frac{|M|}{8}\right)^{2^{(k+1)}} & \end{array}$$

□

Lemma 4 (Size of M^*) *We have* $|M^*| \leq 2^{|M|-1}$.

Proof. By straightforward induction on M. □

Definition 7 (Residuals [3, 8]) *Let* $\mathscr{R} \subseteq \mathsf{Redex}(M)$*. Let* $R \in \mathscr{R}$*, and* $R : M \to N$*. Then the set of residuals of* $\mathscr{R}$ *in* N *with respect to* R*, denoted by* $\mathsf{Res}(\mathscr{R}/R : M \to N)$ *is defined by the smallest set satisfying the following conditions:*

1. *Case of $S \in \mathscr{R}$ and $S \parallel R$:*
   Then we have $S \in \mathsf{Res}(\mathscr{R}/R : M \to N)$.
2. *Case of $S \in \mathscr{R}$ and $S \equiv R$:*
   Then we have $S \notin \mathsf{Res}(\mathscr{R}/R : M \to N)$.
3. *Case of $S \in \mathscr{R}$ and $S \equiv (\lambda x.M_1)N_1$ and $R \sqsubset M_1$ for some $M_1, N_1 \sqsubset M$:*
   Then we have $S' \in \mathsf{Res}(\mathscr{R}/R : M \to N)$ such that $R : S \to S'$ for $S' \sqsubset N$.
4. *Case of $S \in \mathscr{R}$ and $S \equiv (\lambda x.M_1)N_1$ and $R \sqsubset N_1$ for some $M_1, N_1 \sqsubset M$:*
   Then we have $S' \in \mathsf{Res}(\mathscr{R}/R : M \to N)$ such that $R : S \to S'$ for $S' \sqsubset N$.
5. *Case of $S \in \mathscr{R}$ and $R \equiv (\lambda x.M_1)N_1$ and $S \sqsubset M_1$ for some $M_1, N_1 \sqsubset M$:*
   Then we have $S[x := N_1] \in \mathsf{Res}(\mathscr{R}/R : M \to N)$ such that $S[x := N_1] \sqsubset M_1[x := N_1]$ where $R : (\lambda x.M_1)N_1 \to M_1[x := N_1]$.
6. *Case of $S \in \mathscr{R}$ and $R \equiv (\lambda x.M_1)N_1$ and $S \sqsubset N_1$ for some $M_1, N_1 \sqsubset M$:*
   Then we have $S \in \mathsf{Res}(\mathscr{R}/R : M \to N)$ for every occurrence S such that $S \sqsubset M_1[x := N_1]$ where $R : (\lambda x.M_1)N_1 \to M_1[x := N_1]$.

Definition 8 (Complete development [1]) *Let $\mathscr{R} \subseteq \mathsf{Redex}(M)$. A reduction path $R_0R_1 \ldots : M \equiv M_0 \to M_1 \to \cdots$ is a development of $\langle M, \mathscr{R}\rangle$ if and only if each redex $R_i \sqsubseteq M_i$ is in the set $\mathscr{R}_i$ $(i \geq 0)$ such that $\mathscr{R}_0 = \mathscr{R}$ and $\mathscr{R}_i = \mathsf{Res}(\mathscr{R}_{i-1}/R_{i-1} : M_{i-1} \to M_i)$. If $\mathscr{R}_k = \emptyset$ for some k, then the development is called complete.*

Definition 9 (Minimal complete development [8]) *Let $\mathscr{R} \subseteq \mathsf{Redex}(M)$. A redex occurrence $R \in \mathscr{R}$ is called minimal if there is no $S \in \mathscr{R}$ such that $S \sqsubset R$ (i.e., R properly contains no other $S \in \mathscr{R}$).*

Let $\mathscr{R} = \{R_0, \ldots, R_{n-1}\}$. Let $\mathscr{R}_0 = \mathscr{R}$ and $\mathscr{R}_i = \mathsf{Res}(\mathscr{R}_{i-1}/R_{i-1})$. A reduction path $M \twoheadrightarrow^n N$ is a minimal complete development of $\mathscr{R}$ if and only if we contract any minimal $R_i \in \mathscr{R}_i$ at each reduction step. This development is also called an inside-out development that yields shortest complete developments [10, 15].

We write $M \Rightarrow N$ if N is obtained from M by a minimal complete development of a subset $\{R_1, \ldots, R_n\}$ of $\mathsf{Redex}(M)$. In this case, we write $R_1 \ldots R_n : M \Rightarrow^n N$.

Note that we can repeat this development at most n-times with respect to $\mathscr{R} = \{R_0, \cdots, R_{n-1}\}$ until no residuals of $\mathscr{R}$ are left, since we never have the fifth or sixth case in Definition 7, and then we have $R \notin \mathsf{Res}(\mathscr{R}/R)$.

Definition 10 (Reduction of new redexes) *Let $R : M \to N$. If there exists a redex occurrence $S \in \mathsf{Redex}(N)$ but $S \notin \mathsf{Res}(\mathsf{Redex}(M)/R : M \to N)$, then we say that the reduction $R : M \to N$ creates a new redex $S \sqsubseteq N$, and N contains a created redex after contracting R.*

Let σ be a reduction path $R_0R_1 \ldots : M \equiv M_0 \to M_1 \to \cdots$. We define the set of new redex occurrences denoted by $\mathsf{NewRed}(M_{i+1})$ $(i \geq 0)$ as follows:

$$\mathsf{NewRed}(M_{i+1}) \quad = \quad \{R \in \mathsf{Redex}(M_{i+1}) \mid R \notin \mathsf{Res}(\mathsf{Redex}(M_i)/R_i)\}.$$

A redex occurrence $R_j \sqsubseteq M_j$ $(1 \leq j)$ in σ is called new if $R_j \in \mathsf{NewRed}(M_i)$ for some $i \leq j$. The reduction path σ contains k reductions of new redexes if σ contracts k of the new redexes.

3 New proof of the Church-Rosser theorem for β-equality

Proposition 2 (Complete development) *We have $M \twoheadrightarrow^l M^*$ where $l \le \frac{1}{2}|M| - 1$ for $|M| \ge 4$.*

Proof. By induction on the structure of M. Otherwise by the minimal complete development [8] with respect to $\mathsf{Redex}(M)$, where $l \le \sharp\mathsf{Redex}(M) \le \frac{1}{2}|M| - 1$ for $|M| \le 4$ by Lemma 2. □

Definition 11 (Iteration of exponentials $\mathbf{2}^m_n$, $\mathsf{F}(m,n)$) *Let m and n be natural numbers.*

1. *(1) $\mathbf{2}^m_0 = m$; (2) $\mathbf{2}^m_{n+1} = 2^{\mathbf{2}^m_n}$.*
2. *(1) $\mathsf{F}(m,0) = m$; (2) $\mathsf{F}(m,n+1) = 2^{\mathsf{F}(m,n)-1}$.*

Proposition 3 (Length to M^{n*}) *If $M \twoheadrightarrow M^* \twoheadrightarrow \cdots \twoheadrightarrow M^{n*}$, then the reduction length l with $M \twoheadrightarrow^l M^{n*}$ is bounded by $\mathsf{Len}(|M|,n)$, such that*

$$\mathsf{Len}(|M|,n) = \begin{cases} 0, & \text{for } n = 0 \\ \dfrac{1}{2}\displaystyle\sum_{k=0}^{n-1} \mathsf{F}(|M|,k) - n, & \text{for } n \ge 1 \end{cases}$$

and then we have $\mathsf{Len}(|M|,n) < \mathbf{2}^{|M|}_{n-1}$ for $n \ge 1$.

Proof. From Lemma 4, we have $|M^*| \le 2^{|M|-1}$, and hence $|M^{k*}| \le \mathsf{F}(|M|,k) < \mathbf{2}^{|M|}_k$ for $k \ge 1$. Let $M \twoheadrightarrow^{l_1} M^* \twoheadrightarrow^{l_2} \cdots \twoheadrightarrow^{l_n} M^{n*}$. Then from Proposition 2, each l_k is bounded by $\mathsf{F}(|M|,k-1)$:

$$l_k \;\le\; \frac{1}{2}|M^{(k-1)*}| - 1 \;\le\; \frac{1}{2}\mathsf{F}(|M|,k-1) - 1$$

Therefore, l is bounded by $\mathsf{Len}(|M|,n)$ that is smaller than $\mathbf{2}^{|M|}_{n-1}$ for $n \ge 1$.

$$l \;\le\; \sum_{k=1}^{n} l_k \;\le\; \frac{1}{2}\sum_{k=0}^{n-1}\mathsf{F}(|M|,k) - n \;=\; \mathsf{Len}(|M|,n) \;<\; \frac{1}{2}\sum_{k=0}^{n-1}\mathbf{2}^{|M|}_k - n \;<\; \mathbf{2}^{|M|}_{n-1} - n$$ □

Lemma 5 ((Weak) Cofinal property) *If $M \to N$ then $N \twoheadrightarrow^l M^*$ where $l \le \frac{1}{2}|N| - 1$ for $|N| \ge 4$.*

Proof. By induction on the derivation of $M \to N$. □

Lemma 6 $M^*[x := N^*] \twoheadrightarrow^l (M[x := N])^*$ *with $l \le |M^*| - 1$.*

Proof. By induction on the structure of M. We show one case M of M_1M_2.

1. Case $M_1 \equiv \lambda y.M_3$ for some M_3:

$$\begin{array}{rcll} ((\lambda y.M_3)M_2)^*[x := N^*] & = & M_3^*[x := N^*][y := M_2^*[x := N^*]] & \\ & \twoheadrightarrow^{m_1} & M_3^*[x := N^*][y := (M_2[x := N])^*] & \text{by IH1} \\ & \twoheadrightarrow^{m_2} & (M_3[x := N])^*[y := (M_2[x := N])^*] & \text{by IH2} \end{array}$$

Here, IH1 is $M_2^*[x := N^*] \twoheadrightarrow^{n_1} (M_2[x := N])^*$ with $n_1 \le |M_2^*| - 1$, and then we have $m_1 = \sharp(y \in (M_3^*[x := N^*])) \times n_1$ from Lemma 3.
IH2 is $M_3^*[x := N^*] \twoheadrightarrow^{m_2} (M_3[x := N])^*$ with $m_2 \le |M_3^*| - 1$. Hence,

$$\begin{array}{rcl} l & = & m_1 + m_2 \\ & \le & \sharp(y \in (M_3^*[x := N^*])) \times (|M_2^*| - 1) + |M_3^*| - 1 \\ & = & \sharp(y \in M_3^*) \times (|M_2^*| - 1) + |M_3^*| - 1 \ \text{ since } y \notin \mathrm{FV}(N^*) \\ & = & |M_3^*[y := M_2^*]| - 1. \end{array}$$

2. Case $M_1 \not\equiv \lambda y.M_3$:
   (a) Case $(M_1[x := N]) \equiv (\lambda z.P)$ for some P:

$$\begin{array}{rcl} (M_1^*[x := N^*])(M_2^*[x := N^*]) & \twoheadrightarrow^m & (M_1[x := N])^*(M_2[x := N])^* \text{ by IH} \\ & = & (\lambda z.P^*)(M_2[x := N])^* \\ & \twoheadrightarrow^1 & P^*[z := (M_2[x := N])^*] \\ & = & ((M_1 M_2)[x := N])^* \end{array}$$

   Now, IH are $M_1^*[x := N^*] \twoheadrightarrow^{n_1} (M_1[x := N])^*$ with $n_1 \leq |M_1^*| - 1$, and $M_2^*[x := N^*] \twoheadrightarrow^{n_2} (M_2[x := N])^*$ with $n_2 \leq |M_2^*| - 1$. Hence,

$$\begin{array}{rcl} l & = & m + 1 \\ & \leq & |M_1^*| - 1 + |M_2^*| - 1 + 1 \\ & < & |M_1^* M_2^*| - 1. \end{array}$$

   (b) Case $(M_1[x := N]) \not\equiv (\lambda z.P)$:
   This case is handled similarly to the above case, and then

$$\begin{array}{rcl} l & \leq & m \\ & = & |M_1^*| - 1 + |M_2^*| - 1 \\ & < & |M_1^* M_2^*| - 1. \end{array}$$

□

Proposition 4 (Monotonicity) *If $M \to N$ then $M^* \twoheadrightarrow^l N^*$ with $l \leq |M^*| - 1$.*

Proof. By induction on the derivation of $M \to N$. We show some of the interesting cases.

1. Case of $(\lambda x.M)N \to M[x := N]$:

$$\begin{array}{rcl} ((\lambda x.M)N)^* & = & M^*[x := N^*] \\ & \twoheadrightarrow^m & (M[x := N])^* \end{array}$$

   From Lemma 6, we have $m \leq |M^*[x := N^*]| - 1 = |((\lambda x.M)N)^*| - 1$.

2. Case of $PM \to PN$ from $M \to N$:
   (a) Case of $P \equiv \lambda x.P_1$ for some P_1:

$$\begin{array}{rcl} ((\lambda x.P_1)M)^* & = & P_1^*[x := M^*] \\ & \twoheadrightarrow^m & P_1^*[x := N^*] \text{ by IH} \\ & = & ((\lambda x.P_1)N)^* \end{array}$$

   Here, IH is $M^* \twoheadrightarrow^n N^*$ with $n \leq |M^*| - 1$, and $m = \sharp(x \in P_1^*) \times n$ from Lemma 3. Hence,

$$\begin{array}{rcl} l & = & m \\ & \leq & \sharp(x \in P_1^*) \times (|M^*| - 1) \\ & \leq & |P_1^*| + \sharp(x \in P_1^*) \times (|M^*| - 1) - 1 \\ & = & |P_1^*[x := M^*]| - 1. \end{array}$$

(b) Case of $P \not\equiv \lambda x.P_1$: Similarly handled. □

Lemma 7 (Main lemma) *Let $M =_\beta^k N$ with length $k = l + r$, where r is the number of occurrences of right arrow $\rightarrow$ in $=_\beta^k$, and l is that of left arrow $\leftarrow$ in $=_\beta^k$. Then we have both $M^{r*} \twoheadleftarrow N$ and $M \twoheadrightarrow N^{l*}$.*

Proof. By induction on the length of $=_\beta^k$.

(1) Case of $k = 1$ is handled by Lemma 5.

(2-1) Case of $(k+1)$, where $M =_\beta^k M_k \rightarrow M_{k+1}$:

From the induction hypothesis, we have $M_k \twoheadrightarrow M^{r*}$ and $M \twoheadrightarrow M_k^{l*}$ where $l + r = k$.

From $M_k \rightarrow M_{k+1}$, Lemma 5 gives $M_{k+1} \twoheadrightarrow M_k^*$, and then $M_k^* \twoheadrightarrow M^{(r+1)*}$ from the induction hypothesis $M_k \twoheadrightarrow M^{r*}$ and Proposition 4. Hence, we have $M_{k+1} \twoheadrightarrow M^{(r+1)*}$. On the other hand, we have $M_k^{l*} \twoheadrightarrow M_{k+1}^{l*}$ from $M_k \rightarrow M_{k+1}$ and the repeated application of Proposition 4. Then the induction hypothesis $M \twoheadrightarrow M_k^{l*}$ derives $M \twoheadrightarrow M_{k+1}^{l*}$, where $l + (r+1) = k+1$.

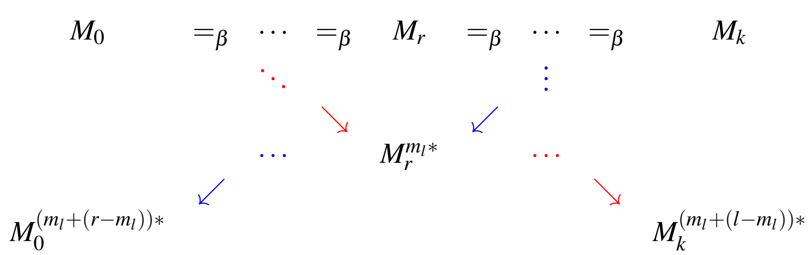
(2-2) Case of $(k+1)$, where $M =_\beta^k M_k \leftarrow M_{k+1}$:

From the induction hypothesis, we have $M_k \twoheadrightarrow M^{r*}$ and $M \twoheadrightarrow M_k^{l*}$ where $l + r = k$, and hence $M_{k+1} \twoheadrightarrow M^{r*}$. From $M_{k+1} \rightarrow M_k$ and Lemma 5, we have $M_k \twoheadrightarrow M_{k+1}^*$, and then $M_k^{l*} \twoheadrightarrow M_{k+1}^{(l+1)*}$. Hence, $M \twoheadrightarrow M_{k+1}^{(l+1)*}$ from the induction hypothesis $M \twoheadrightarrow M_k^{l*}$, where $(l+1) + r = k+1$. □

Given $M_0 =_\beta^k M_k$ with reduction sequence $[M_0, \ldots, M_k]$, then for natural numbers i and j with $0 \le i \le j \le k$, we write $\sharp r[i,j]$ for the number of occurrences of right arrow $\rightarrow$ which appears in $M_i =_\beta^{(j-i)} M_j$, and $\sharp l[i,j]$ for that of left arrow $\leftarrow$ in $M_i =_\beta^{(j-i)} M_j$. In particular, we have $\sharp l[0,k] + \sharp r[0,k] = k$.

Corollary 1 (Main lemma refined) *Let $M_0 =_\beta^k M_k$ with reduction sequence $[M_0, M_1, \ldots, M_k]$. Let $r = \sharp r[0,k]$ and $l = \sharp l[0,k]$. Then we have $M_0 \twoheadrightarrow M_r^{m_l*}$ and $M_r^{m_l*} \twoheadleftarrow M_k$, where $m_l = \sharp l[0,r] \le \min\{l, r\}$.*

Proof. From the main lemma, we have two reduction paths such that $M_0 \twoheadrightarrow M_k^{l*}$ and $M_0^{r*} \twoheadleftarrow M_k$, where the paths have a crossed point that is the term M_r^{n*} for some $n \le k$ as follows:



Let m_l be $\sharp l[0,r]$, then $\sharp l[r,k] = (l - m_l)$ and $\sharp r[r,k] = m_l$. Hence, from the main lemma, we have $M_0 \twoheadrightarrow M_r^{m_l*} \twoheadleftarrow M_k$ where $m_l \le \min\{l, r\}$. Moreover, we have $M_r \twoheadrightarrow M_k^{(l-m_l)*}$ by the main lemma again, and then $M_r^{m_l*} \twoheadrightarrow M_k^{((l-m_l)+m_l)*}$ from the repeated application of Proposition 4. Therefore, we indeed have $M_0 \twoheadrightarrow M_r^{m_l*} \twoheadrightarrow M_k^{l*}$. Similarly, we have $M_0^{r*} \twoheadleftarrow M_r^{m_l*} \twoheadleftarrow M_k$ as well. □

Example 1 *We demonstrate a simple example of $M_0 =_\beta^4 M_4$ with length 4, and list 2^4 patterns of the reduction graph consisting of the sequence $[M_0, M_1, M_2, M_3, M_4]$. The sixteen patterns can be classified into 5 groups, in which M_0 and M_4 have a pair of the same common reducts $\langle M_0^{r*}, M_4^{l*} \rangle$ where $r + l = 4$:*

1. *Common reducts $\langle M_0^{4*}, M_4^{0*} \rangle$ and a crossed point $M_4^{m_l*} \equiv M_4^{0*}$:*

   (1) $M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4$.

2. *Common reducts* $\langle M_0^{3*}, M_4^{*}\rangle$ *and crossed points* $M_3^{m_l*}$ *of two kinds:*
   (1) $M_0 \leftarrow M_1 \to M_2 \to M_3 \to M_4$; (2) $M_0 \to M_1 \leftarrow M_2 \to M_3 \to M_4$ *with* $M_3^{m_l*} \equiv M_3^{*}$;
   (3) $M_0 \to M_1 \to M_2 \leftarrow M_3 \to M_4$; (4) $M_0 \to M_1 \to M_2 \to M_3 \leftarrow M_4$ *with* $M_3^{m_l*} \equiv M_3^{0*}$.
3. $\langle M_0^{2*}, M_4^{2*}\rangle$ *and crossed points* $M_2^{m_l*}$ *of three kinds:*
   (1) $M_0 \leftarrow M_1 \to M_2 \leftarrow M_3 \to M_4$; (2) $M_0 \leftarrow M_1 \leftarrow M_2 \to M_3 \to M_4$ *with* $M_2^{m_l*} \equiv M_2^{2*}$;
   (3) $M_0 \leftarrow M_1 \to M_2 \to M_3 \leftarrow M_4$; (4) $M_0 \to M_1 \leftarrow M_2 \to M_3 \leftarrow M_4$ *with* $M_2^{m_l*} \equiv M_2^{*}$;
   (5) $M_0 \to M_1 \leftarrow M_2 \leftarrow M_3 \to M_4$; (6) $M_0 \to M_1 \to M_2 \leftarrow M_3 \leftarrow M_4$ *with* $M_2^{m_l*} \equiv M_2^{0*}$.
4. $\langle M_0^{*}, M_4^{3*}\rangle$ *and crossed points* $M_1^{m_l*}$ *of two kinds:*
   (1) $M_0 \leftarrow M_1 \to M_2 \leftarrow M_3 \leftarrow M_4$; (2) $M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \to M_4$ *with* $M_1^{m_l*} \equiv M_1^{*}$;
   (3) $M_0 \leftarrow M_1 \leftarrow M_2 \to M_3 \leftarrow M_4$; (4) $M_0 \to M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow M_4$ *with* $M_1^{m_l*} \equiv M_1^{0*}$.
5. $\langle M_0^{0*}, M_4^{4*}\rangle$ *and a crossed point* $M_0^{m_l*} \equiv M_0^{0*}$*:*
   (1) $M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow M_4$.

Observe that a crossed point $M_r^{m_l*}$ *in Corollary 1 gives a "good" common contractum such that the number* m_l*, i.e., iteration of the translation* $*$ *is minimum, see also the trivial cases above; Case 1, Case 2 (4), Case 3 (6), Case 4 (4), and Case 5. Consider two reduction paths: (i) a reduction path from* $M_r^{m_l*}$ *to* M_0^{r*}*, and (ii) a reduction path from* $M_r^{m_l*}$ *to* M_k^{l*}*, see the picture in the proof of Corollary 1. In general, the reduction paths (i) and (ii) form the boundary line between common contractums and non-common ones. Let* B *be a term in the boundary (i) or (ii). Then any term* M *such that* $B \twoheadrightarrow M$ *is a common contractum of* M_0 *and* M_k*. In this sense, the term* $M_r^{m_l*}$ *where* $0 \le m_l \le \min\{l,r\}$ *can be considered as an optimum common reduct of* M_0 *and* M_k *in terms of Takahashi translation. Moreover, the refined lemma gives a divide and conquer method such that* $M_0 =_\beta^k M_k$ *is divided into* $M_0 =_\beta^r M_r$ *and* $M_r =_\beta^l M_k$*, where the base case is a valley such that* $M_0 \twoheadrightarrow M_r \twoheadleftarrow M_k$ *with minimal* M_r *and* $m_l = 0$*, as shown by the trivial cases above.*

The results of Lemma 7 and Corollary 1 can be unified as follows. The main theorem shows that every term in the reduction sequence ls of $M_0 =_\beta^k M_k$ generates a common contractum: For every term M in ls, there exists a natural number $n \le \max\{l,r\}$ such that M^{n*} is a common contractum of M_0 and M_k. Moreover, there exist a term N in ls and a natural number $m \le \min\{l,r\}$ such that N^{m*} is a common contractum of all the terms in ls.

Theorem 1 (Main theorem for β-equality) *Let* $M_0 =_\beta^k M_k$ *with reduction sequence* $[M_0,\dots,M_k]$*. Let* $l = \sharp l[0,k]$ *and* $r = \sharp r[0,k]$*. Then there exist the following common reducts:*

1. *We have* $M_0 \twoheadrightarrow M_{r-i}^{\sharp r[r-i,k]*}$ *and* $M_{r-i}^{\sharp r[r-i,k]*} \twoheadleftarrow M_k$ *for each* $i = 0,\dots,r$*. We also have* $M_0 \twoheadrightarrow M_{r+j}^{\sharp l[0,r+j]*}$ *and* $M_{r+j}^{\sharp l[0,r+j]*} \twoheadleftarrow M_k$ *for each* $j = 0,\dots,l$*.*
2. *For every term* M *in the reduction sequence, we have* $M \twoheadrightarrow M_r^{m_l*}$ *where* $m_l = \sharp l[0,r]$*.*

Proof. Both 1 and 2 are proved similarly from Lemma 7, Corollary 1, and monotonicity. We show the case 2 here. Let M_i be a term in the reduction sequence of $M_0 =_\beta^k M_k$ where $0 \le i \le r$. Take $a = \sharp r[0,i]$, then $M_a^{\sharp l[0,a]}$ is a crossed point of $M_0 \twoheadrightarrow M_i^{\sharp l[0,i]*}$ and $M_i \twoheadrightarrow M_0^{\sharp r[0,i]*}$. From $M_i \twoheadrightarrow M_r^{\sharp l[i,r]*}$ and monotonicity, we have $M_i^{\sharp l[0,i]*} \twoheadrightarrow M_r^{m_l*}$ where $m_l = \sharp l[0,i] + \sharp l[i,r]$. Hence, we have $M_i \twoheadrightarrow M_a^{\sharp l[0,a]*} \twoheadrightarrow M_i^{\sharp l[0,i]*} \twoheadrightarrow M_r^{m_l*}$. The case of $r \le i \le k$ is also verified similarly. □

Note that the case of $i = r$ and $j = l$ implies the main lemma, since $\sharp r[0,k] = r$ and $\sharp l[0,r+l] = \sharp l[0,k] = l$. Note also that the case of $i = 0 = j$ implies the refinement, since $\sharp l[0,r] = m_l = \sharp r[r,k]$.

Corollary 2 (Church-Rosser theorem for β-reduction) *Let $P_n \leftarrow \cdots \leftarrow P_1 \leftarrow M \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_m$ $(1 \leq n \leq m)$. Then we have $P_n \twoheadrightarrow Q_m^{n*}$ and $Q_m \twoheadrightarrow Q_m^{n*}$. We also have $P_n \twoheadrightarrow Q_{(m-n)}^{n*}$ and $Q_m \twoheadrightarrow Q_{(m-n)}^{n*}$.*

Proof. From the main lemma and the refinement where $Q_0 \equiv M$. □

Theorem 2 (Improved Church-Rosser theorem for β-reduction) *Let $P_n \leftarrow \cdots \leftarrow P_1 \leftarrow M \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_m$ $(1 \leq n \leq m)$. If $P_n \leftarrow \cdots \leftarrow P_1 \leftarrow M$ contains a-times reductions of new redexes $(0 \leq a \leq n-1)$, and $M \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_m$ contains b-times reductions of new redexes $(0 \leq b \leq m-1)$, then we have both $P_n \twoheadrightarrow Q_m^{(a+1)*}$ and $Q_m \twoheadrightarrow P_n^{(b+1)*}$.*

Proof. We show the claim that if a reduction path σ of $R_0R_1 \ldots R_n : M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_{n+1}$ contains a-times reductions of new redexes $(1 \leq a \leq n-1)$ then $M_{n+1} \twoheadrightarrow M^{(a+1)*}$, from which the theorem is derived by repeated application of Proposition 4.

We prove the claim by induction on a.

1. Case of $a = 0$:

   We have $R_0R_1 \ldots R_n : M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_{n+1}$, where none of R_i $(0 \leq i \leq n)$ is a new redex. The reduction path is a development of M with respect to a subset of $\mathsf{Redex}(M)$. Then we have $M_j \twoheadrightarrow M^*$ $(0 \leq j \leq n+1)$, since all developments of $\mathsf{Redex}(M)$ are finite [7, 1] and end with some N such that $N \twoheadrightarrow M^*$.

2. Case of $a = k+1$:

   We have $R_0R_1 \ldots R_{n-1}R_nR_{n+1} \ldots R_m : M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n \rightarrow M_{n+1} \rightarrow \cdots \rightarrow M_{m+1}$ $(m \geq 0)$, where $R_0R_1 \ldots R_{n-1} : M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n$ contains k reductions of new redexes $(0 \leq k \leq n-1)$. Moreover, the redex R_n is a new redex, and $R_{n+1} \ldots R_m : M_{n+1} \rightarrow \cdots \rightarrow M_{m+1}$ contains no new redexes. Then the reduction path $R_nR_{n+1} \ldots R_m : M_n \rightarrow M_{n+1} \rightarrow \cdots \rightarrow M_{m+1}$ is a development of M_n with respect to a subset of $\mathsf{Redex}(M_n)$, and hence $M_{m+1} \twoheadrightarrow M_n^*$. On the other hand, from the induction hypothesis applied to the reduction path $R_0R_1 \ldots R_{n-1} : M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n$ with k reductions of new redexes, we have $M_n \twoheadrightarrow M^{(k+1)*}$. Therefore, we have $M_{m+1} \twoheadrightarrow M^{(k+2)*}$ by repeated application of Proposition 4. □

4 Quantitative analysis and comparison with related results

4.1 Measure functions

For quantitative analysis, we list important measure functions, TermSize, Mon, and Rev.

Definition 12 (TermSize) *We define* $\mathsf{TermSize}(M =_\beta N)$ *by induction on the derivation.*

1. *If $M \twoheadrightarrow^r N$ then* $\mathsf{TermSize}(M =_\beta N) = 8(\frac{|M|}{8})^{2^r}$.
2. *If $M =_\beta N$ is derived from $N =_\beta M$, then define* $\mathsf{TermSize}(M =_\beta N)$ *by* $\mathsf{TermSize}(N =_\beta M)$.
3. *If $M =_\beta N$ is derived from $M =_\beta P$ and $P =_\beta N$, then define* $\mathsf{TermSize}(M =_\beta N)$ *as follows:* $\max\{\mathsf{TermSize}(M =_\beta P), \mathsf{TermSize}(P =_\beta N)\}$.

Proposition 5 (TermSize) *Let $M_0 =_\beta^k M_k$ with reduction sequence ls. Then $|M| \leq \mathsf{TermSize}(M_0 =_\beta^k M_k)$ for each term M in ls, and $\mathsf{TermSize}(M_0 =_\beta^k M_k) \leq |N|^{2^k}$ for some term N in ls.*

Proof. By induction on the derivation of $=_\beta$ together with Definition 12 and Proposition 1. □

Definition 13 (Monotonicity)

$$\mathsf{Mon}(|M|,m,n) = \begin{cases} 2^{|M|^{2^m}}, & \textit{for } n=1 \\ 2^{2^{[2^{\mathsf{Mon}(|M|,m,n-1)}\times \mathbf{2}^{|M|}_{(n-2)}]}}, & \textit{for } n>1 \end{cases}$$

Proposition 6 (Monotonicity) *If $M \twoheadrightarrow^m N$, then $M^{n*} \twoheadrightarrow^l N^{n*}$ with $l \leq \mathsf{Mon}(|M|,m,n)$.*

Proof. By induction on n.

1. Case of $n=1$:

   If $M \twoheadrightarrow^m M_m$, then $M^* \twoheadrightarrow^l M_m^*$ with $l \leq 2^{|M|^{2^m}}$. Indeed, from Proposition 1, we have $|M_m| < |M|^{2^m}$. If $M_0 \to M_1$ then we have $M_0^* \twoheadrightarrow^{l_1} M_1^*$ with $l_1 < 2^{|M_0|}$ from Proposition 4 and Lemma 4. Hence, from $M_0 \to M_1 \to \cdots \to M_m$, we have $M_0^* \twoheadrightarrow^{l_1} M_1^* \twoheadrightarrow^{l_2} \cdots \twoheadrightarrow^{l_m} M_m^*$ where

$$l \;=\; \sum_{i=1}^{m} l_i \;<\; \sum_{i=0}^{m-1} 2^{|M_i|} \;<\; \sum_{i=0}^{m-1} 2^{|M_0|^{2^i}} \;<\; 2^{|M_0|^{2^m}}.$$

2. Case of $n \geq 1$:

   From the induction hypothesis, we have $M^{n*} \twoheadrightarrow^l N^{n*}$ with $l < \mathsf{Mon}(|M|,m,n)$. Therefore, we have $M^{(n+1)*} \twoheadrightarrow^{l'} N^{(n+1)*}$ with

$$l' \;<\; 2^{|M^{n*}|^{2^l}} \;<\; 2^{|M^{n*}|^{2^{\mathsf{Mon}(|M|,m,n)}}}, \quad \text{where } |M^{n*}| < \mathbf{2}_n^{|M|}. \qquad \square$$

Lemma 8 (Cofinal property) *If $M \twoheadrightarrow^n N$ $(n \geq 1)$, then $N \twoheadrightarrow^l M^{n*}$ with $l < \mathsf{Rev}(|M|,n)$ as follows:*

$$\mathsf{Rev}(|M|,n) = \begin{cases} \frac{1}{2}|M|^2, & \textit{for } n=1 \\ \frac{1}{2}|M|^{2^n} + 2^{|M|^{2^{[n-1+\mathsf{Rev}(|M|,n-1)]}}}, & \textit{for } n>1 \end{cases}$$

Proof. The case $\mathsf{Rev}(|M|,1)$ is by Lemma 5. For $n>1$, $\mathsf{Rev}(|M|,n)$ follows $\mathsf{Mon}(|M|,n,1)$ from Proposition 6 and $|N| < |M|^{2^n}$ from Proposition 1. $\square$

4.2 Quantitative analysis of Church-Rosser for β-reduction

We show two bound functions $f(l,|M|,r) = \langle m,n\rangle$ such that for the peak $N_1 \twoheadleftarrow^l M \twoheadrightarrow^r N_2$, the valley size of $N_1 \twoheadrightarrow^a P \twoheadleftarrow^b N_2$ for some P is bounded by $a \leq m$ and $b \leq n$. The first function $\mathsf{CR\text{-}red}(l,M,r) = \langle m, N_1^{r*}, n\rangle$ provides a common reduct N_1^{r*}, following the proof of the main lemma with Mon. The second one $\mathsf{V\text{-}size}(l,M,r) = \langle m, M^{r*}, n\rangle$ gives a common reduct M^{r*} simply using Rev provided that $l \leq r$.

Definition 14 (CR-red) *1.* $\mathsf{CR\text{-}red}(l,M,1) = \langle \frac{1}{2}|M|^{2^l}, N_1^*, \frac{1}{2}|M|^2 + 2^{|M|^{2^l}}\rangle$

2. $\mathsf{CR\text{-}red}(l,M,r) =$

$$\texttt{let } \langle m, N_1^{(r-1)*}, n\rangle \texttt{ be } \mathsf{CR\text{-}red}(l,M,r-1) \texttt{ in } \langle \mathbf{2}^{|M|^{2^l}}_{(r-1)}, N_1^{r*}, \tfrac{1}{2}|M|^{2^r} + 2^{|M|^{2^{[r-1+n]}}}\rangle \textit{ for } r>1$$

Proposition 7 (CR-red) *If $N_1 \twoheadleftarrow^l M \twoheadrightarrow^r N_2$, then we have $\mathsf{CR\text{-}red}(l,M,r) = \langle m, N_1^{r*}, n\rangle$ such that $N_1 \twoheadrightarrow^a N_1^{r*} \twoheadleftarrow^b N_2$ with $a \leq m$ and $b \leq n$.*

Proof. By induction on r.

1. Case $r = 1$:
   We have $M^* \twoheadleftarrow^a N_2$ with $a \le \frac{1}{2}|N_2| \le \frac{1}{2}|M|^2$. Then $N_1^* \twoheadleftarrow^b M^*$ with $b \le \mathsf{Mon}(|M|, l, 1) = 2^{|M|^{2^l}}$. On the other hand, we have a common contractum N_1^* such that $N_1 \twoheadrightarrow^c N_1^*$ with $c \le \frac{1}{2}|N_1| \le \frac{1}{2}|M|^{2^l}$.

2. Case of $r > 1$:
   From the induction hypothesis, we have $\langle m, N_1^{(r-1)}, n\rangle = \mathsf{CR\text{-}red}(l, M, r-1)$ such that $M \twoheadrightarrow^{(r-1)} N_3 \to N_2$ and $N_1^{(r-1)*} \twoheadleftarrow^b N_3$ with $b \le n$ for some N_3. Then we have $N_3^* \twoheadleftarrow^c N_2$ with $c \le \frac{1}{2}|N_2| \le \frac{1}{2}|M|^{2^r}$, and hence $N_1^{r*} \twoheadleftarrow^d N_3^*$ where

$$d \;\le\; \mathsf{Mon}(|N_3|, n, 1) \;\le\; \mathsf{Mon}(|M|^{2^{(r-1)}}, n, 1) \;=\; 2^{(|M|^{2^{(r-1)}})^{2^n}} \;=\; 2^{|M|^{2^{[r+n-1]}}}.$$

   Therefore, we have a common reduct N_1^{r*} such that $N_1 \twoheadrightarrow^e N_1^{r*}$ with $e \le \mathsf{Len}(|N_1|, r) \le \mathbf{2}_{(r-1)}^{|M|^{2^l}}$. □

Definition 15 (V-size) $\mathsf{V\text{-}size}(l, M, r) = \langle \mathsf{Rev}(|M|, l) + \mathbf{2}_{r-1}^{|M|}, M^{r*}, \mathsf{Rev}(M, r)\rangle$ *for* $1 \le l \le r$.

Proposition 8 (V-size) *If* $N_1 \twoheadleftarrow^l M \twoheadrightarrow^r N_2$ *with* $l \le r$*, then we have* $\mathsf{V\text{-}size}(l, M, r) = \langle m, M^{r*}, n\rangle$ *such that* $N_1 \twoheadrightarrow^a M^{r*} \twoheadleftarrow^b N_2$ *with* $a \le m$ *and* $b \le n$.

Proof. Suppose that $l \le r$. We have $N_1 \twoheadrightarrow^a M^{l*}$ with $a \le \mathsf{Rev}(|M|, l)$ and $M^{r*} \twoheadleftarrow^b N_2$ with $b \le \mathsf{Rev}(|M|, r)$, respectively. From $l \le r$, we have $M^{l*} \twoheadrightarrow^c M^{r*}$ where

$$c \;\le\; \mathsf{Len}(|M^{l*}|, r-l) \;\le\; \mathbf{2}_{r-l-1}^{|M^{l*}|} \;\le\; \mathbf{2}_{r-l-1}^{\mathbf{2}_l^{|M|}} \;=\; \mathbf{2}_{r-1}^{|M|}.$$ □

On the other hand, Ketema and Simonsen [9] showed that an upper bound on the size of confluence diagrams in λ-calculus is $\mathsf{bl}(l, |M|, r)$ for $P \twoheadleftarrow^l M \twoheadrightarrow^r Q$. The valley size a and b of $P \twoheadrightarrow^a N \twoheadleftarrow^b Q$ for some N is bounded by $\mathsf{bl}(l, |M|, r)$ as follows:

$$\mathsf{bl}(l, |M|, r) = \begin{cases} |M|^{2^{[2^l+l+2]}}, & \text{for } r = 1 \\ |M|^{2^{[2^{\mathsf{bl}(l,|M|,r-1)}+\mathsf{bl}(l,|M|,r-1)+r+1]}}, & \text{for } r > 1 \end{cases}$$

Their proof method is based on the use of the so-called Strip Lemma, and in this sense our first method CR-red is rather similar to theirs. However, for a large term M, bl can give a shorter reduction length than that by CR-red from the shape of the functions. The reason can be expounded as follows: From given terms, we explicitly constructed a common reduct via $*$-translation, so that more redexes than a set of residuals can be reduced, compared with those of bl. To overcome this point, an improved version of Theorem 2 is introduced such that $*$-translation is applied only when new redexes are indeed reduced.

The basic idea of the second method V-size is essentially the same as the proof given in [11]. In summary, the functions bl and CR-red including a common reduct are respectively defined by induction on the length of one side of the peak, and V-size is by induction on that of both sides of the peak. All the functions belong to the fourth level of the Grzegorczyk hierarchy.

4.3 Quantitative analysis of Church-Rosser for β-equality

Let $M_0 =_\beta^k M_k$ with length $k = l + r$ where $l = \sharp l[0, k]$ and $r = \sharp r[0, k]$, and M be $\mathsf{TermSize}(M_0 =_\beta^k M_k)$. Then we show a bound function $\mathsf{CR\text{-}eq}(M_0 =_\beta^k M_k) = \langle m, M_0^{r*}, n\rangle$ such that $M_0 \twoheadrightarrow^a M_0^{r*}$ and $M_0^{r*} \twoheadleftarrow^b M_k$ with $a \le m$ and $b \le n$. This analysis reveals the size of the valley described in Lemma 7.

Definition 16 *Given $M_0 =^k_\beta M_k$ with length $k = l + r$ where $l = \sharp l[0,k]$ and $r = \sharp r[0,k]$. Let* M *be* $\mathsf{TermSize}(M_0 =^k_\beta M_k)$. *A measure function* $\mathsf{CR\text{-}eq}$ *is defined by induction on the length of* $=^k_\beta$, *where* $\cdot$ *denotes an arbitrary term.*

1. $\mathsf{CR\text{-}eq}(M_0 \leftarrow \cdot) = \langle 0, M_0^{0*}, 1\rangle$; $\mathsf{CR\text{-}eq}(M_0 \to \cdot) = \langle \frac{1}{2}|M_0|, M_0^*, \frac{1}{2}|M_0|^2\rangle$
2. $\mathsf{CR\text{-}eq}(M_0 =^k_\beta \cdot \leftarrow \cdot) = \texttt{let } \langle a, M_0^{r*}, b\rangle \texttt{ be } \mathsf{CR\text{-}eq}(M_0 =^k_\beta \cdot) \texttt{ in } \langle a, M_0^{r*}, b+1\rangle$
3. $\mathsf{CR\text{-}eq}(M_0 =^k_\beta \cdot \to \cdot) = \texttt{let } \langle a, M_0^{r*}, b\rangle \texttt{ be } \mathsf{CR\text{-}eq}(M_0 =^k_\beta \cdot) \texttt{ in } \langle a + \dfrac{1}{2}\mathbf{2}_r^{|M_0|}, M_0^{(r+1)*}, \dfrac{1}{2}\mathsf{M} + 2^{\mathsf{M}^{2^b}}\rangle$

Note that in the definition of $\mathsf{CR\text{-}eq}$, as shown by the use of $\cdot$, we use no information on N such that $M_0 =_\beta N$, but only by the use of the length of $=_\beta$ and case analysis of $\to$ or $\leftarrow$. From Definition 12 and Proposition 1, $\mathsf{TermSize}(M_0 =_\beta M_k)$ is well-defined by induction on $=_\beta$. From the definition above, $\mathsf{CR\text{-}eq}$ is also a function in the fourth level of the Grzegorczyk hierarchy (non-elementary).

Proposition 9 (Church-Rosser for β-equality) *If $M_0 =^k_\beta M_k$ with length $k = l + r$ where $l = \sharp l[0,k]$ and $r = \sharp r[0,k]$, then we have* $\mathsf{CR\text{-}eq}(M_0 =^k_\beta M_k) = \langle m, M_0^{r*}, n\rangle$ *such that $M_0 \twoheadrightarrow^a M_0^{r*}$ and $M_0^{r*} \twoheadleftarrow^b M_k$ with $a \le m$ and $b \le n$.*

Proof. By induction on the length of $=_\beta^{(l+r)}$. The outline of the proof is the same as that of Lemma 7.

1. Base cases of $k = 1$:
   - $\mathsf{CR\text{-}eq}(M_0 \leftarrow \cdot) = \langle 0, M_0^{0*}, 1\rangle$:
     We have $M_0 \equiv M_0^{0*} \leftarrow M_1$ for some M_1.
   - $\mathsf{CR\text{-}eq}(M_0 \to \cdot) = \langle \frac{1}{2}|M_0|, M_0^*, \frac{1}{2}|M_0|^2\rangle$:
     We have $M_0 \to M_1$ for some M_1, and then $M_0 \twoheadrightarrow^a M_0^*$ with $a \le \frac{1}{2}|M_0|$ and $M_0^* \twoheadleftarrow^b M_1$ with $b \le \mathsf{Rev}(|M_0|, 1) = \frac{1}{2}|M_0|^2$.
2. Step cases:
   - $\mathsf{CR\text{-}eq}(M_0 =^k_\beta \cdot \leftarrow \cdot) = \texttt{let } \langle a, M_0^{r*}, b\rangle \texttt{ be } \mathsf{CR\text{-}eq}(M_0 =^k_\beta \cdot) \texttt{ in } \langle a, M_0^{r*}, b+1\rangle$:
     From the induction hypothesis, we have $M_0 \twoheadrightarrow^m M_0^{r*}$ with $m \le a$ and $M_0^{r*} \twoheadleftarrow^n M_2 \leftarrow M_3$ for some M_2, M_3 with $n \le b$. Then we have the same common reduct M_0^{r*} and $n+1 \le b+1$ from $M_0^{r*} \twoheadleftarrow^{n+1} M_3$.
   - $\mathsf{CR\text{-}eq}(M_0 =^k_\beta \cdot \to \cdot) = \texttt{let } \langle a, M_0^{r*}, b\rangle \texttt{ be } \mathsf{CR\text{-}eq}(M_0 =^k_\beta \cdot) \texttt{ in } \langle a + \frac{1}{2}\mathbf{2}_r^{|M_0|}, M_0^{(r+1)*}, \frac{1}{2}\mathsf{M} + 2^{\mathsf{M}^{2^b}}\rangle$:
     From the induction hypothesis, we have $M_0 \twoheadrightarrow^m M_0^{r*}$ with $m \le a$ and $M_0^{r*} \twoheadleftarrow^n M_2 \to M_3$ for some M_2, M_3 with $n \le b$. We also have $M_2^* \twoheadleftarrow^c M_3$ with $c \le \frac{1}{2}|M_2| \le \frac{1}{2}\mathsf{M}$, and then $M_0^{(r+1)*} \twoheadleftarrow^d M_2^*$ where

     $$d \;\le\; \mathsf{Mon}(|M_2|, b, 1) \;\le\; \mathsf{Mon}(\mathsf{M}, b, 1) \;=\; 2^{\mathsf{M}^{2^b}}.$$

     Hence, we have a common reduct $M_0^{(r+1)*}$ such that $M_0 \twoheadrightarrow^m M_0^{r*} \twoheadrightarrow^e M_0^{(r+1)*}$ where

     $$m + e \;\le\; a + \frac{1}{2}|M_0^{r*}| \;\le\; a + \frac{1}{2}\mathbf{2}_r^{|M_0|}.$$ □

Example 2 *The Church numerals $\mathbf{c}_n = \lambda f x. f^n(x)$ are defined as usual due to Rosser [1], where we write $F^0(M) = M$, and $F^{n+1}(M) = F(F^n(M))$. We define N_i such that $N_1 = \mathbf{c}_2$, and $N_{n+1} = N_n\mathbf{c}_2$. We also define $M_1 = \mathbf{c}_1 p(N_n pq)$ and $M_2 = N_n p(\mathbf{c}_1 pq)$ with fresh variables p and q for $n \ge 4$. We might have*

$M_1 =_\beta M_2$, but the length of $=_\beta$ is not trivial. From the fact that $N_n \twoheadrightarrow^a \lambda f \lambda x.f^{\mathbf{2}^1_n}(x)$ with $a \leq \mathbf{2}^1_n$, indeed we prove $M_1 =_\beta M_2$ as follows:

$M_1 \twoheadrightarrow \mathbf{c}_1 p((\lambda f \lambda x.f^{\mathbf{2}^1_n}(x))pq) \twoheadrightarrow^2 \mathbf{c}_1 p(p^{\mathbf{2}^1_n}(q)) \twoheadrightarrow^2 p(p^{\mathbf{2}^1_n}(q))$, and similarly $p^{\mathbf{2}^1_n}(p(q)) \twoheadleftarrow M_2$. Hence, the length of $=_\beta$ is at most $2 \times (4 + \mathbf{2}^1_n)$, and the size of the common reduct is $1 + 2 \times (\mathbf{2}^1_{n+1} + 1)$, although $|M_1| = |M_2| = 8n + 1$. The example suggests that there is plenty of room for improvement of the upper bound. Note that $M_1 \twoheadrightarrow p^{\mathbf{2}^1_n+1}(q) \twoheadleftarrow M_2$ is regarded as a base case in the sense of Example 1.

5 Concluding remarks and further work

The main lemma revealed that a common contractum P from M_0 and M_k with $M_0 =^k_\beta M_k$ can be determined by (i) M_0 and the number of occurrences of $\rightarrow$ in $=_\beta$, and also by (ii) M_k and that of $\leftarrow$. In general, we have 2^k patterns of reduction graph for $=^k_\beta$ as a combination of $\rightarrow$ and $\leftarrow$ with length k. This lemma means that 2^k patterns of graph can be grouped into $(k+1)$ classes with ${}_kC_i$ patterns $(i = 0, \ldots, k)$, like Pascal's triangle. As demonstrated by Example 1, we have common contractums $\langle M_0^{(k-i)*}, M_k^{i*} \rangle$ for each class $(i = 0, \ldots, k)$, contrary to an exponential size of the patterns of reduction graph. Moreover, Corollary 1 provides an optimum common contractum $M_r^{m_l *}$ for $M_0 =^k_\beta M_k$ in terms of Takahashi translation, which is one of important consequences of the main lemma.

The main lemma depends only on Proposition 4 and Lemma 5, which can be expounded geometrically as parallel and flipped properties respectively. Hence, if there exists an arbitrary reduction strategy $*$ that satisfies both properties, then the main lemma can be established. In fact, the main lemma holds even for $\beta\eta$-equality, because for $\beta\eta$-reduction, under an inside-out development we still have Lemma 5, Proposition 4, and Proposition 2 without bounds as observed already in [11]. This implies that under a general framework with such a strategy, it is possible to analyze quantitative properties of rewriting systems in the exactly same way, and indeed λ-calculus with $\beta\eta$-reduction and weakly orthogonal higher-order rewriting systems [17, 5] are instances of these systems. Moreover, this general approach is available as well for compositional Z [13] that is an extension of the so-called Z property [5] (property of a reduction strategy that is cofinal and monotonic), which makes it possible to apply a divide and conquer method for proving confluence.

In order to analyze reduction length of the Church-Rosser theorem, we provided measure functions Len, TermSize, Mon, and Rev. In terms of the measure functions, bound functions are obtained for the theorem for β-reduction and β-equality, explicitly together with common contractums. A bound on the valley size for the theorem for β-equality is obtained by induction on the length of $=_\beta$. Compared with [9], the use of TermSize is important to set bounds to the size of terms, in particular, for the theorem for β-equality. Given $M =_\beta N$, then there exists some constant TermSize$(M =_\beta N)$, and under the constant bound functions can be provided by induction only on the length of $=_\beta$ with neither information on M nor N, including the size of a common contractum.

In addition, based on Corollary 1, it is also possible to analyze the valley size of $M_0 =^{(l+r)}_\beta M_{l+r}$ in terms of $M_r^{m_l *}$: In the base case of $m_l = 0$, the valley size is bounded simply by l and r, for instance, see Example 2; in the maximum case of $m_l = \min\{l, r\}$, the valley size is at most that of the theorem for β-reduction as observed in Example 1; and this analysis will be discussed elsewhere.

Towards a tight bound, our bound depends essentially on Proposition 2 and Lemma 4. Proposition 2 provides an optimal reduction, since we adopted the so-called minimal complete development [8, 10, 15]. For the bound on the size of M^*, Lemma 4 can be proved, in general, under some function $f(x)$ such that $f(x) \times f(y) \leq f(x+y)$, which may lead to a non-elementary recursive function, as described by Len.

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