

String Diagrams for Regular Logic (Extended Abstract)

Brendan Fong
MIT

David I. Spivak
MIT *

Regular logic can be regarded as the *internal language* of regular categories, but the logic itself is generally not given a categorical treatment. In this paper, we understand the syntax and proof rules of regular logic in terms of the free regular category $\text{FRg}(T)$ on a set T . From this point of view, regular theories are certain monoidal 2-functors from a suitable 2-category of contexts—the 2-category of relations in $\text{FRg}(T)$ —to that of posets. Such functors assign to each context the set of formulas in that context, ordered by entailment. We refer to such a 2-functor as a *regular calculus* because it naturally gives rise to a graphical string diagram calculus in the spirit of Joyal and Street. We shall show that every natural category has an associated regular calculus, and conversely from every regular calculus one can construct a regular category.

1 Introduction

Regular logic is the fragment of first order logic generated by equality ($=$), true (true), conjunction (\wedge), and existential quantification (\exists). A defining feature of this fragment is that it is expressive enough to define *functions* and *composition* of functions, or more generally of relations: given relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, their composite is given by the formula

$$R \circ S = \{(x, z) \mid \exists y. R(x, y) \wedge S(y, z)\}.$$

Indeed, regular logic is the internal language of regular categories, which may in turn be understood as a categorical characterization of the minimal structure needed to have a well-behaved notion of relation.

While regular categories put emphasis on the notion of *binary* relation, the existence of finite products allows them to handle *n*-ary relations—that is, subobjects of *n*-fold products—and their composition. To organize more complicated multi-way composites of relations, many fields have developed some notion of wiring diagram. A good amount of recent work, including but not limited to control theory [6, 1, 11], database theory and knowledge representation [5, 19], electrical engineering [2], and chemistry [3], all serve to demonstrate the link between these languages and categories for which the morphisms are relations.

A first goal of this paper is to clarify the link between regular logic and these various graphical languages. In doing so, we provide a new diagrammatic syntax for regular logic, which we refer to as *graphical regular logic*. Rather than pursue a direct translation with the classical syntax for first order logic, we demonstrate a tight connection between graphical regular logic and the notion of *regular category*. A second goal, then, is to repackage the structure of a regular category into terms that cleanly reflect its underlying logical theory. We call the resulting categorical structure a *regular calculus*. Regular calculi are based on free regular categories, so let's begin there.

We will show that the *free regular category* FRg on a singleton set can be obtained by freely adding a fresh terminal object to $\text{FinSet}^{\text{op}}$. Here is a depiction of a few objects in FRg :

$$0 \longleftarrow s \longleftarrow 1 \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} 2 \quad \dots \quad (1)$$

*Spivak and Fong acknowledge support from AFOSR grants FA9550-14-1-0031 and FA9550-17-1-0058.

The object s is the coequalizer of the two distinct maps $2 \rightrightarrows 1$, so in a sense it prevents the unique map $1 \rightarrow 0$ from being a regular epimorphism. Thus one may think of s as representing the *support* of an abstract object in a regular category. In \mathbf{Set} , the support of any object is either empty or singleton, but in general the concept is more refined. For example, the topos of sheaves on a space X is regular, and the support of a sheaf r is the union $U \subseteq X$ of all open sets on which $r(U)$ is nonempty.

For any small set T of *types* (also known as *sorts*), the free regular category on T is then the T -fold coproduct of regular categories $\mathbf{FRg}(T) := \bigsqcup_T \mathbf{FRg}$. That is, we have an adjunction

$$\mathbf{Set} \begin{array}{c} \xrightarrow{\mathbf{FRg}} \\ \xrightarrow{\cong} \\ \xleftarrow{\mathbf{Ob}} \end{array} \mathbf{RgCat} \tag{2}$$

which we will construct explicitly in Theorem 23. For any regular category \mathcal{R} , the counit provides a canonical regular functor, which we denote $\ulcorner - \urcorner: \mathbf{FRg}(\mathbf{Ob} \mathcal{R}) \rightarrow \mathcal{R}$. Note also that this extends to a 2-functor $\ulcorner - \urcorner: \mathbf{Rel}_{\mathbf{FRg}(\mathbf{Ob} \mathcal{R})} \rightarrow \mathbf{Rel}_{\mathcal{R}}$ between the associated relation bicategories.

Write $\mathbb{FRg}(T) := \mathbf{Rel}_{\mathbf{FRg}(T)}$ for this bicategory of relations. Just as \mathbf{FRg} is closely related to the opposite of the category of finite sets (see (1)), the objects in $\mathbb{FRg}(T)$ are, at a first approximation, much like finite sets \underline{n} equipped with a function $\underline{n} \rightarrow T$, and morphisms are much like corelations: equivalence relations on some coproduct $\underline{n} + \underline{n}'$. We draw objects and morphisms as on the left and right below:



The left-hand circle, equipped with its labeled ports and white dot, represents an object in $\mathbf{FRg}(T)$; we call this picture a *shell*. Here each port represents an element of the associated finite set $\underline{3}$, the white dot captures aspects related to the support object s of \mathbf{FRg} , and the labels x, y etc. are elements of T . In the right-hand morphism, the inner shell represents the domain, outer shell represents the codomain, and the things between them—the connected components of the wires and the white dots—represent the equivalence classes of the aforementioned equivalence relation.

A regular calculus lets us think of each object $\Gamma \in \mathbb{FRg}(T)$ —each shell—as a context for formulas in some regular theory, and of each morphism, i.e. each wiring diagram $\Gamma \rightarrow \Gamma'$, as a method for converting Γ -formulas to Γ' -formulas, using $=, \text{true}, \wedge,$ and \exists . We next want to think about how regular categories fit into this picture.

If \mathcal{R} is a regular category, formulas in the associated regular theory are given by relations $x \subseteq r_1 \times \cdots \times r_n$, where x and the r_i are objects in \mathcal{R} , i.e. $r_\bullet: \underline{n} \rightarrow \mathcal{R}$. Thus we could consider $\Gamma := r_\bullet$ as a context, and this brings us back to the free regular category $\mathbf{FRg}(\mathbf{Ob} \mathcal{R})$. The counit functor $\ulcorner - \urcorner: \mathbf{FRg}(\mathbf{Ob} \mathcal{R}) \rightarrow \mathcal{R}$ sends Γ to $\ulcorner \Gamma \urcorner := r_1 \times \cdots \times r_n$. A key feature of regular categories is that the subobjects $\mathbf{Sub}_{\mathcal{R}}(r_1 \times \cdots \times r_n)$ form a meet-semilattice, elements of which we call *predicates* in context Γ . As we shall see, the collection of all these semilattices, when related by the structure of $\mathbf{FRg}(\mathbf{Ob} \mathcal{R})$, includes enough data to recover the regular category \mathcal{R} itself.

Indeed, consider the commutative diagram

$$\begin{array}{ccc} \mathbf{FRg}(\mathbf{Ob} \mathcal{R}) & \xrightarrow{\ulcorner - \urcorner} & \mathcal{R} \\ \downarrow & & \downarrow \\ \mathbf{FRg}(\mathbf{Ob} \mathcal{R}) & \xrightarrow{\ulcorner - \urcorner} & \mathcal{R} \xrightarrow{\mathcal{R}(I, -)} \mathbb{P}\text{oset} \end{array}$$

where the vertical maps represent inclusions of a regular 1-category into its bicategory of relations, and the hom-2-functor $\mathcal{R}(I, -)$ sends each object $r \in \text{Ob } \mathcal{R} = \text{Ob } \mathcal{R}$ to the subobject lattice $\text{Sub}_{\mathcal{R}}(r) = \mathcal{R}(I, r)$. We can denote the composite of the bottom maps as

$$\text{Sub}_{\mathcal{R}}^{\ulcorner - \urcorner}: \mathbb{FRg}(\text{Ob } \mathcal{R}) \longrightarrow \mathbb{P}\text{oset}. \quad (3)$$

The domain $\mathbb{FRg}(\text{Ob } \mathcal{R})$ is a category of contexts and the functor $\text{Sub}_{\mathcal{R}}^{\ulcorner - \urcorner}$ assigns the poset of predicates to each context Γ .

As mentioned, the regular category \mathcal{R} may be reconstructed—up to equivalence—from the contexts $\Gamma \in \mathbb{FRg}(\text{Ob } \mathcal{R})$ and their predicate posets $\text{Sub}_{\mathcal{R}}^{\ulcorner - \urcorner}$ as in Eq. (3), once we give the abstract structure by which they hang together. The question is, given any set T , what extra structure do we need on a functor

$$P: \mathbb{FRg}(T) \longrightarrow \mathbb{P}\text{oset}$$

in order to construct a regular category from it?

Whatever the required structure on P is, of course $\text{Sub}_{\mathcal{R}}^{\ulcorner - \urcorner}$ needs to have that structure. First of all, $\text{Sub}_{\mathcal{R}}^{\ulcorner - \urcorner}$ is a 2-functor, and it happens to be the composite of $\mathbb{R}\text{el}^{\ulcorner - \urcorner}$ and $\text{Sub}_{\mathcal{R}}$. It is not hard to check that the 2-functor $\ulcorner - \urcorner$ is strong monoidal, whereas the 2-functor $\mathcal{R}(I, -)$ is only lax monoidal: given objects $r_1, r_2 \in \mathcal{R}$ the induced monotone map $\times: \text{Sub}_{\mathcal{R}}(r_1) \times \text{Sub}_{\mathcal{R}}(r_2) \rightarrow \text{Sub}_{\mathcal{R}}(r_1 \times r_2)$ is not an isomorphism. However, $\text{Sub}_{\mathcal{R}}^{\ulcorner - \urcorner}$ has a bit more structure than merely being a lax functor: each laxator has a left adjoint

$$1 \begin{array}{c} \xrightarrow{\text{true}} \\ \leftarrow \leftarrow \\ \downarrow ! \end{array} \text{Sub}_{\mathcal{R}}(1) \quad \text{Sub}_{\mathcal{R}}(r_1) \times \text{Sub}_{\mathcal{R}}(r_2) \begin{array}{c} \xrightarrow{\times} \\ \leftarrow \leftarrow \\ \downarrow \langle \text{im}_1, \text{im}_2 \rangle \end{array} \text{Sub}_{\mathcal{R}}(r_1 \times r_2).$$

Abstractly, if \mathcal{R} and \mathcal{P} are monoidal 2-categories, we say that a lax monoidal functor $\mathcal{R} \rightarrow \mathcal{P}$ is *ajax* (“adjoint-lax”) if its laxators ρ and $\rho_{v, v'}$ are right adjoints in \mathcal{P} . Thus we have seen that the monoidal functor $\text{Sub}_{\mathcal{R}}^{\ulcorner - \urcorner}: \mathbb{FRg}(\text{Ob } \mathcal{R}) \rightarrow \mathbb{P}\text{oset}$ is ajax. This is precisely the structure required to reconstruct a regular category.

Ajax functors have the important property that they preserve adjoint monoids. An *adjoint monoid* is an object with both monoid and comonoid structures, such that the monoid maps are right adjoint to their corresponding comonoid maps. In particular, we will see that each object in $\mathbb{FRg}(T)$ has a canonical adjoint monoid structure, and that adjoint monoids in $\mathbb{P}\text{oset}$ are exactly meet-semilattices. This guarantees that ajax functors $\mathbb{FRg}(T) \rightarrow \mathbb{P}\text{oset}$ send objects in $\mathbb{FRg}(T)$ —contexts—to meet-semilattices.

We now come to our main definition.

Definition 1. A *regular calculus* is a pair (T, P) where T is a set and $P: \mathbb{FRg}(T) \rightarrow \mathbb{P}\text{oset}$ is an ajax 2-functor.

Regular calculi have a natural notion of morphism, comprising a function between the sets of types and natural transformation between the ajax functors. We denote the category of regular calculi by RgCalc . The goal of this paper is to demonstrate there are functors

$$\text{RgCalc} \begin{array}{c} \xrightarrow{\text{syn}} \\ \leftarrow \leftarrow \\ \downarrow \text{prd} \end{array} \text{RgCat}.$$

and, moreover, that for each regular category \mathcal{R} we have an equivalence $\text{syn}(\text{prd}(\mathcal{R})) \simeq \mathcal{R}$.

Related work Regular categories were first defined by Barr [4], as a way to elucidate the structure present in abelian categories. Shortly thereafter, Freyd and Scedrov were the first to make the connection to regular logic. Similarly to the present work, they focused on the structure of the bicategory of relations, seeking an axiomatization through the notion of an allegory [14]. Carboni and Walters also sought to axiomatize these objects, defining functionally complete cartesian bicategories of relations [8]; see also [13] for our reinterpretation of this work. Both allegories and bicategories of relations take the structure of a regular category, and decompress it into a (locally posetal) 2-categorical expression. While regular calculi have similar features to both allegories and cartesian bicategories, such as emphasizing that the hom-posets are meet-semilattices or that there are adjoint monoid structures on each object, they represent this data in terms of a *functor* rather than a category.

In the world of databases, regular formulas correspond to conjunctive queries, and entailment corresponds to query containment. A well-known theorem of Chandra and Merlin states that (conjunctive) query containment is decidable; their proof translates logical expressions into graphical representations [9]. In more recent work, Bonchi, Seeber, and Sobociński show that the Chandra–Merlin approach permits an elegant formalization in terms of the Carboni–Walters axioms for bicategories of relations [5]. Patterson has also considered bicategories of relations, and their Joyal–Street string calculus [16], as a graphical way of capturing the regular logical aspects knowledge representation [19].

Presenting regular categories using monoidal maps $\mathbb{FRg}(T) \rightarrow \mathbb{P}oset$ fits into an emerging pattern. In [21] it was shown that lax monoidal functors $1\text{-Cob}_T \rightarrow \mathbb{S}et$ present traced monoidal categories, and in [12] it was shown that lax monoidal functors $\mathbb{C}ospan_T \rightarrow \mathbb{S}et$ present hypergraph categories. But now in all three cases, the domain of the functor represents a particular language of string diagrams, and the codomain represents a choice of enriching category. The present paper can be seen as an extension of that work, showing that regular categories are something like poset-enriched hypergraph categories.

Outline Section 2 briefly reviews the notion of a regular category \mathcal{R} , emphasizing in particular the construction of the symmetric monoidal po-category $\mathbb{R}el_{\mathcal{R}}$ of relations in \mathcal{R} . In Section 3 we introduce the notions of adjoint monoid and ajax monoidal functor, showing in particular that the subobjects functor of a regular po-category is ajax. In Section 4 we give our main definition: a regular calculus is an ajax functor from a free regular po-category to that of posets. We then give a fully faithful functor $\mathbf{prd}: \mathbb{R}gCat \rightarrow \mathbb{R}gCalc$, from regular categories to regular calculi. In Section 5, we define the *graphical terms* of a regular calculus, and give rules for composing and reasoning with these. In Section 6, we conclude by outlining how to construct a regular category from a regular calculus, defining the functor $\mathbf{syn}: \mathbb{R}gCalc \rightarrow \mathbb{R}gCat$.

1.0.1 Notation

Let us fix some notation. Most is standard, but we highlight in particular our use of \circledast for composition, of the term *po-category* for locally posetal 2-category, and of an arrow \Rightarrow pointing in the direction of the left adjoint to signify an adjunction.

- We typically denote composition in diagrammatic order, so the composite of $f: A \rightarrow B$ and $g: B \rightarrow C$ is $f \circledast g: A \rightarrow C$. We often denote the identity morphism $\text{id}_c: c \rightarrow c$ on an object $c \in \mathcal{C}$ simply by the name of the object, c . Thus if $f: c \rightarrow d$, we have $(c \circledast f) = f = (f \circledast d)$.
- We may denote the terminal object of any category by \star , and the associated map from an object c as $!: c \rightarrow \star$, but we denote the top element of any poset P by $\text{true} \in P$.
- We denote the universal map into a product by $\langle f, g \rangle$ and the universal map out of a coproduct by $[f, g]$.

- Given a natural number $n \in \mathbb{N}$, define $\underline{n} := \{1, 2, \dots, n\} \in \text{FinSet}$; in particular $\underline{0} = \emptyset$.
- Given a lax monoidal functor $F : \mathcal{C} \rightarrow \mathcal{D}$, we denote the laxators by $\rho : I \rightarrow F(I)$ and $\rho_{c,c'} : F(c) \otimes F(c') \rightarrow F(c \otimes c')$ for any $c, c' \in \mathcal{C}$. We use the same notation for longer lists, e.g. we write $\rho_{c,c',c''}$ for the canonical map $F(c) \otimes F(c') \otimes F(c'') \rightarrow F(c \otimes c' \otimes c'')$.
- We write

$$c \begin{array}{c} \xrightarrow{L} \\ \Rightarrow \\ \xleftarrow{R} \end{array} d$$

to denote an adjunction, where the \Rightarrow points in the direction of the left adjoint. We sometimes write $L \dashv R$ inline, but are careful to avoid the \vdash symbol in this context; *the symbol \vdash always means entailment*. We denote the category with the same objects of \mathcal{C} and with left adjoints as morphisms as $\text{LAdj}(\mathcal{C})$.

Symmetric monoidal po-categories. We use the term *po-category* to mean locally posetal 2-category, i.e. a category enriched in partially ordered sets (posets). *Po-functors* are, of course, poset-enriched functors (functors that preserve the local order). The set of po-functors $\mathcal{C} \rightarrow \mathcal{D}$ itself has a natural order, where $F \leq G$ iff $F(c) \leq G(c)$ for all $c \in \mathcal{C}$. We define $\mathbb{P}\text{ocat}$ to be the po-category of po-categories and po-functors.

We use $\mathbb{X}\text{yz}$ —with first character made blackboard bold—to denote named po-categories and $\mathbb{X}\text{yz}$ for named 1-categories. We rely fairly heavily on this; for example our notations for the free regular category and the free regular po-category on a set T differ only in this way: $\mathbb{F}\text{Rg}(T)$ vs. $\mathbb{F}\mathbb{R}\text{g}(T)$.

A po-category is, in particular, a (strict) 2-category, and po-functors are (strict) 2-functors. As such there is a forgetful functor $\mathbb{P}\text{ocat} \rightarrow \text{Cat}$ sending each po-category and po-functor to its underlying 1-category and 1-functor. A *symmetric monoidal po-category* is a po-category \mathcal{C} together with po-functors $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and $I : \star \rightarrow \mathcal{C}$ whose underlying 1-structures form a symmetric monoidal category.

The symmetric monoidal po-category $\mathbb{P}\text{oset}$ has posets P as objects, monotone maps $f : P \rightarrow Q$ as morphisms, and order given by $f \leq g$ iff $f(p) \leq g(p)$ for all p . Its monoidal structure is given by cartesian product $P \times Q$, with the terminal poset 1 the monoidal unit.

2 Background on regular categories

Regular categories are, roughly speaking, categories that have a good notion of relation. They were first defined by Barr [4] to isolate important aspects of abelian categories. The reader who is unacquainted with regular categories and/or regular logic may see [7] for more details and proofs.

2.1 Definition of regular categories and functors

Definition 2 (Barr). A *regular category* is a category \mathcal{R} with the following properties:

1. it has all finite limits;
2. the kernel pair of any morphism $f : r \rightarrow s$ admits a coequalizer $r \times_s r \rightrightarrows r \rightarrow \text{coeq}(f)$, which we denote $\text{im}(f) := \text{coeq}(f)$ and call the *image* of f ; and
3. the pullback—along any map—of a regular epimorphism (a coequalizer of any parallel pair) is again a regular epimorphism.

A *regular functor* is a functor between regular categories that preserves finite limits and regular epis. We write RgCat for the category of regular categories.

Lemma 3. For any $f: r \rightarrow r'$, the universal map $\text{im}(f) \rightarrow r'$ is monic. Thus every map f can be factored into a regular epimorphism followed by a monomorphism: $r \rightarrow \text{im}(f) \rightarrow r'$, and this constitutes an orthogonal factorization system. In particular, image factorization is unique up to isomorphism.

Definition 4. The *support* of an object r in a regular category is the image $r \rightarrow \text{Supp}(r) \rightarrow \star$ of its unique map to the terminal object.

Definition 5. A *subobject* of an object r in a category is an isomorphism class of monomorphisms $r' \rightarrow r$, where morphisms between monomorphisms are as in the slice category over r . This defines a partially ordered set $\text{Sub}(r)$. We write $r' \subseteq r$ to denote the equivalence class represented by $r' \rightarrow r$.

Proposition 6. Any morphism $f: r \rightarrow s$ in a regular category \mathcal{R} induces an adjunction

$$\text{Sub}(r) \begin{array}{c} \xrightarrow{f_!} \\ \rightleftarrows \\ \xleftarrow{f^*} \end{array} \text{Sub}(s). \tag{4}$$

This extends to a functor $\text{Sub}: \mathcal{R} \rightarrow \text{LAdj}(\mathbb{P}\text{oset})$.

2.2 The relations po-category construction

A regular category \mathcal{R} has exactly the structure and properties necessary to construct a po-category of relations, or *relations po-category*.

Definition 7. Let \mathcal{R} be a regular category; its *relations po-category* $\mathbb{R}\text{el}_{\mathcal{R}}$ is the po-category with the same objects as \mathcal{R} but whose morphisms, written $x: r \rightarrow s$, are relations $x \subseteq r \times s$ in \mathcal{R} equipped with the subobject ordering $x \leq x'$ iff $x \subseteq x'$. The composite $x \circ y$ with a relation $y: s \rightarrow t$ is obtained by pulling back over s and image factorizing the map to $r \times t$:

$$\begin{array}{ccccc} & & x \times_s y & & \\ & \swarrow & \downarrow & \searrow & \\ & x & x \circ y & y & \\ & \downarrow & \downarrow & \downarrow & \\ & r \times s & r \times t & s \times t & \\ & \swarrow & \downarrow & \searrow & \\ r & & s & & t \end{array} \tag{5}$$

$\mathbb{R}\text{el}_{\mathcal{R}}$ also inherits a symmetric monoidal structure $I := 1$ and $r_1 \otimes r_2 := r_1 \times r_2$ from the cartesian monoidal structure on \mathcal{R} .

Given a regular functor $\mathcal{F}: \mathcal{R} \rightarrow \mathcal{R}'$, mapping a relation $x \subseteq r \times s$ to its factorization $\mathcal{F}(x) \rightarrow \mathbb{R}\text{el}_{\mathcal{F}}(x) \rightarrow \mathcal{F}(r \times s) \cong \mathcal{F}(r) \times \mathcal{F}(s)$ induces a (strong) symmetric monoidal po-functor $\mathbb{R}\text{el}_{\mathcal{F}}: \mathbb{R}\text{el}_{\mathcal{R}} \rightarrow \mathbb{R}\text{el}_{\mathcal{R}'}$. We refer to this po-functor as the *relations po-functor* of \mathcal{F} .

It is straightforward to check that the composition rule Eq. (5) is unital and associative using the pullback stability of factorizations, and to check that $\mathbb{R}\text{el}_{\mathcal{F}}$ is indeed a symmetric monoidal po-functor using the fact that a regular functor $\mathcal{F}: \mathcal{R} \rightarrow \mathcal{R}'$ preserves pullbacks and image factorizations. Direct proofs in the literature of these two facts seem difficult to find, but see for example [15, Theorem 2.3] and [10, Proposition 4.1] respectively.

The relations po-category is just a repackaging of the data of the regular category: any regular category can be recovered, at least up to isomorphism, by looking at the adjunctions in its relations po-category. (This result is standard, but we provide a proof in Appendix A.)

Lemma 8 (Fundamental lemma of regular categories). *Let \mathcal{R} be a regular category. Then there is an identity-on-objects isomorphism*

$$\mathcal{R} \rightarrow \text{LAdj}(\mathbb{R}\text{el}_{\mathcal{R}}).$$

In particular, a relation $x: r \twoheadrightarrow s$ is a left adjoint iff it is the graph $x = \langle \text{id}_r, f \rangle$ of a morphism $f: r \rightarrow s$ in \mathcal{R} .

Remark 9. It follows from the proof of Lemma 8 that $x: r \twoheadrightarrow s$ is a right adjoint iff it is the co-graph $\langle f, \text{id}_s \rangle$ of a morphism $f: s \rightarrow r$. Furthermore, since any morphism $x = \langle g, f \rangle: r \twoheadrightarrow s$ in \mathcal{R} can be written as $x = \langle g, \text{id}_x \rangle \circ \langle \text{id}_x, f \rangle$, it follows that every morphism in \mathcal{R} can be written as the composite of a right adjoint followed by a left adjoint.

Similarly, any regular functor can be recovered as the action of its relations po-functor on left adjoints. Although we do not assume it below, it is a result of Carboni and Walters that every strong symmetric monoidal functor $\mathbb{R}\text{el}_{\mathcal{R}} \rightarrow \mathbb{R}\text{el}_{\mathcal{R}'}$ is the relations po-functor associated to a regular functor $\mathcal{F}: \mathcal{R} \rightarrow \mathcal{R}'$ [8]. Indeed, this foreshadows the rephrasing of regular structure in terms of monoidal structure, which runs through this paper. In any case, this motivates the following definition.

Definition 10. A po-category is called a *regular po-category* if it is isomorphic to the relations po-category $\mathbb{R}\text{el}_{\mathcal{R}}$ of some regular category \mathcal{R} .

A strong symmetric monoidal po-functor between regular po-categories is called a *regular po-functor* if it is isomorphic to the relations po-functor $\mathbb{R}\text{el}_{\mathcal{F}}$ associated to a regular functor \mathcal{F} . We write RgPocat for the category of regular po-categories.

By the fundamental lemma (Lemma 8), we now have an equivalence of categories:

$$\text{RgCat} \begin{array}{c} \xrightarrow{\mathbb{R}\text{el}_-} \\ \cong \\ \xleftarrow{\text{LAdj}} \end{array} \text{RgPocat}. \quad (6)$$

3 Adjoint monoids and adjoint-lax functors

The poset of subobjects of an object in a regular category is always a meet-semilattice. We characterize these as precisely the adjoint monoids in $\mathbb{P}\text{oset}$. Every regular po-category \mathcal{R} is isomorphic to its own po-category of adjoint monoids $\mathcal{R} \cong \text{AdjMon}(\mathcal{R})$.

3.1 Definition of ajax functor and adjoint monoid

Definition 11. Let \mathcal{C} and \mathcal{D} be monoidal po-categories. An *adjoint-lax* or *ajax* po-functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a lax symmetric monoidal po-functor for which the laxators are right adjoints.

We denote the laxators by ρ and their left adjoints by λ :

$$I \begin{array}{c} \xrightarrow{\rho} \\ \Leftarrow \\ \xleftarrow{\lambda} \end{array} F(I) \quad \text{and} \quad F(c) \otimes F(c') \begin{array}{c} \xrightarrow{\rho_{c,c'}} \\ \Leftarrow \\ \xleftarrow{\lambda_{c,c'}} \end{array} F(c \otimes c').$$

Warning 12. The notion of ajax functor has a dual notion of op-ajax functor: an oplax functor $\mathcal{C} \rightarrow \mathcal{D}$ for which the op-laxators are left adjoints. These two notions *do not coincide!* The laxator naturality squares are asked to strictly commute in an ajax functor, and this property only implies that their mate squares, the corresponding oplaxator naturality squares weakly commute.

Here is an obvious, but useful, consequence of the definition.

Lemma 13. *Every strong monoidal functor between monoidal po-categories is ajax. The composite of ajax po-functors is ajax.*

Recall that 1 is the terminal monoidal po-category.

Proposition 14. *Let $(\mathcal{C}, I, \otimes)$ be a monoidal po-category. There is a bijection between:*

1. *The set of ajax functors $1 \rightarrow \mathcal{C}$,*
2. *The set of commutative monoid objects (c, μ, η) such that μ and η are right adjoints, and*
3. *The set of cocommutative comonoid objects (c, δ, ε) such that δ and ε are left adjoints.*

In particular, if $(c, \rho, \lambda): 1 \rightarrow \mathcal{C}$ is an ajax functor then the corresponding monoid and comonoid structures on c are given by $\eta = \rho$, $\mu = \rho_{1,1}$ and $\varepsilon = \lambda$, $\delta = \lambda_{1,1}$. This motivates the following definition.

Definition 15. Let $(\mathcal{C}, I, \otimes)$ be a monoidal po-category. An *adjoint commutative monoid* (or simply *adjoint monoid*) in \mathcal{C} is a commutative monoid object (c, μ, η) in \mathcal{C} such that μ and η are right adjoints.

Adjoint monoids are a slight weakening of the *internal meet-semilattice* notion from theoretical computer science; see [20, Chapter 5] and references therein.

Since the composite of ajax functors is ajax, we have:

Proposition 16. *Ajax functors send adjoint monoids to adjoint monoids.*

Recall that in a cartesian monoidal category every object has a unique comonoid structure, in fact a commutative comonoid, given by the universal properties of terminal objects and products. This yields the following examples.

Example 17. A poset $P \in \mathbb{P}\text{oset}$ is an adjoint monoid iff it is a meet-semilattice, in which case $\eta = \text{true}$ and $\mu = \wedge$.

Example 18. Let \mathcal{R} be a regular category. Every object $r \in \mathcal{R}$ in its relations po-category has a unique adjoint monoid structure. Indeed, since \mathcal{R} is cartesian monoidal, there is a unique cocommutative comonoid structure on every object. The fundamental lemma (Lemma 8) then gives an isomorphism $\mathcal{R} \cong \text{LAdj}(\mathcal{R})$, and the statement follows by Proposition 14 (2) \Leftrightarrow (3).

3.2 The subobjects-functor is ajax

Let \mathcal{R} be a regular category and recall the subobjects functor $\text{Sub}: \mathcal{R} \rightarrow \text{LAdj}(\mathbb{P}\text{oset})$ from Proposition 6. It extends to a po-functor $\text{Sub}: \mathcal{R} \rightarrow \mathbb{P}\text{oset}$, where $\mathcal{R} = \mathbb{R}\text{el}_{\mathcal{R}}$ is the relations po-category. To be explicit, write a relation $A \subseteq r \times r'$ as a span $r \xleftarrow{f} A \xrightarrow{f'} r'$. Then the map $\text{Sub}(A): \text{Sub}(r) \rightarrow \text{Sub}(r')$ applied to a subobject $\varphi \subseteq r$ is given pulling back and then taking the image:

$$\begin{array}{ccccc}
 \varphi & \longleftarrow & \cdot & \longrightarrow & \cdot \\
 \downarrow & & \lrcorner & \downarrow & \downarrow \\
 r & \longleftarrow & A & \longrightarrow & r'
 \end{array}$$

That is, $\text{Sub}(A) = f_! \circ g^*$. This po-functor is representable: $\text{Sub}(-) = \mathcal{R}(I, -)$, where I is the terminal object in \mathcal{R} . It is straightforward to show, moreover, that its monoidal structure maps have left adjoints, and so this functor is ajax (see Appendix B).

Theorem 19. *The po-functor $\text{Sub}_{\mathcal{R}}: \mathcal{R} \rightarrow \mathbb{P}\text{oset}$ is ajax for any regular po-category \mathcal{R} .*

Since ajax functors send adjoint monoids to adjoint monoids, we also have:

Corollary 20. *The po-functor $\text{Sub}_{\mathcal{R}}: \mathcal{R} \rightarrow \mathbb{P}\text{oset}$ sends each object $r \in \mathcal{R}$ to a meet-semilattice.*

4 Free regular categories and regular calculi

We now construct the free regular category $\text{FRg}(\mathbb{T})$ —as well as the free regular po-category $\mathbb{FRg}(\mathbb{T})$ —on a set \mathbb{T} . This allows us to define a *regular calculus* to be an ajax po-functor $\mathbb{FRg}(\mathbb{T}) \rightarrow \mathbb{P}\text{oset}$.

4.1 The free regular category and po-category on a set

Write $\mathbb{P}_f(\mathbb{T})$ for the poset of finite subsets of \mathbb{T} ; this, or equally its opposite category $\mathbb{P}_f(\mathbb{T})^{\text{op}}$, is a free \wedge -semilattice on \mathbb{T} . Write also FinSet for the category of finite sets and functions. Note that $\text{FinSet}^{\text{op}}$ is the free category with finite limits on one object. The free regular category on \mathbb{T} arises when these two structures interact.

Note that for any \mathbb{T} there is an inclusion of categories $\text{inc}: \mathbb{P}_f(\mathbb{T}) \rightarrow \text{FinSet}$.

Definition 21. Define $\text{FRg}(\mathbb{T}) := (\mathbb{P}_f(\mathbb{T})^{\text{op}} \downarrow \text{FinSet}^{\text{op}})$ to be the comma category

$$\begin{array}{ccccc}
 & & \text{FRg}(\mathbb{T}) & & \\
 & \text{Supp} & \swarrow & \searrow & \text{Vars} \\
 \mathbb{P}_f(\mathbb{T})^{\text{op}} & & & & \text{FinSet}^{\text{op}} \\
 & \text{inc} & \searrow & \swarrow & \text{id} \\
 & & \text{FinSet}^{\text{op}} & &
 \end{array}
 \quad \begin{array}{c} \\ \\ \xrightarrow{\text{Tp}} \\ \\ \end{array}$$

for any set \mathbb{T} . We refer to objects $\Gamma \in \text{FRg}(\mathbb{T})$ as *contexts*.

We can unpack a context Γ into a quasi-traditional form, e.g. as

$$\Gamma = x_1 : \tau_1, \dots, x_n : \tau_n \mid \tau'_1, \dots, \tau'_m$$

which has a finite set of *variables*, $\text{Vars}(\Gamma) = \{x_1, \dots, x_n\}$, *support* set $\text{Supp}(\Gamma) = \{\tau_1, \dots, \tau_n, \tau'_1, \dots, \tau'_m\}$, and which has the *typing* function $\text{Tp}(x_i) = \tau_i$. The notion of support does not typically have a place in traditional logical contexts, but we include it because $\text{Supp}(\Gamma)$ has a definite place in objects of the free regular category.

Working in the skeleton of $\text{FRg}(\mathbb{T})$, we can assume that each cardinality has a unique set of variables, e.g. $\underline{n} = \{1, \dots, n\}$. Here is an equivalent but more concrete description of the free regular category on \mathbb{T} :

$$\begin{aligned}
 \text{Ob FRg}(\mathbb{T}) &:= \{(n, S, \tau) \mid n \in \mathbb{N}, S \subseteq \mathbb{T} \text{ finite}, \tau: \underline{n} \rightarrow S\} \\
 \text{FRg}(\mathbb{T})(\Gamma, \Gamma') &:= \left\{ f: \underline{n}' \rightarrow \underline{n} \mid \begin{array}{c} \underline{n} \xrightarrow{\tau} S \\ f \uparrow \quad \cup \\ \underline{n}' \xrightarrow{\tau'} S' \end{array} \begin{array}{c} \subseteq \\ \cup \\ \subseteq \end{array} \mathbb{T} \right\} \quad (7)
 \end{aligned}$$

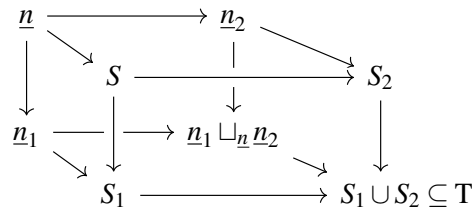
It is straightforward to check the following.

Proposition 22. $\text{FRg}(\mathbb{T})$ is a regular category, with the following explicit descriptions.

terminal: $0 \xrightarrow{!} \emptyset \subseteq \mathbb{T}$ is terminal. We denote it 0 .

product: The product of $\Gamma = (n, S, \tau)$ and $\Gamma' = (n', S', \tau')$ is $(n+n', S \cup S', [\tau, \tau'])$. We denote it $\Gamma \oplus \Gamma'$.

pullback: The pullback of a diagram $(n_1, S_1, \tau_1) \rightarrow (n, S, \tau) \leftarrow (n_2, S_2, \tau_2)$ is obtained as a pushout (and union) in FinSet :



monos: A map $f: (n_1, S_1, \tau_1) \rightarrow (n_2, S_2, \tau_2)$ is monic iff the function $\underline{f}: \underline{n}_2 \rightarrow \underline{n}_1$ is surjective.

regular epis: A map $f: (n_1, S_1, \tau_1) \rightarrow (n_2, S_2, \tau_2)$ is regular epic iff both: the corresponding function $\underline{f}: \underline{n}_2 \rightarrow \underline{n}_1$ is injective and $S_2 = S_1$.

Theorem 23. The category $\text{FRg}(T)$ is the free regular category on T , i.e. there is an adjunction

$$\text{Set} \begin{array}{c} \xrightarrow{\text{FRg}(-)} \\ \rightleftarrows \\ \xleftarrow{\text{Ob}} \end{array} \text{RgCat}.$$

For a proof, see Appendix C.

Remark 24. The free finite limit category on a single generator is $\text{FinSet}^{\text{op}}$, and there the unique map $\underline{n} \rightarrow \emptyset$ is a regular epimorphism for every object \underline{n} . Consequently, $\text{FinSet}^{\text{op}}$ has another universal property: it is the free regular category in which every object is inhabited. Of course the same holds for any set T : the free finite limit category is also the free “fully inhabited” regular category. It is equivalent to the result of inverting the map $(\emptyset, S, !) \rightarrow (\emptyset, \emptyset, !)$ in $\text{FRg}(T)$ for every $S \in \mathbb{P}_f(T)$.

Because $(\text{FinSet}_T)^{\text{op}}$ is very similar to—but far more familiar than— $\text{FRg}(T)$, it can be useful for intuition to replace $\text{FRg}(T)$ with $\text{FinSet}^{\text{op}}$ throughout this story; the only cost is the assumption of inhabitedness, which is a common assumption in classical logic.

Since $\text{FRg}(T)$ is a regular category, we may construct its po-category of relations. It should be no surprise that these are the free regular po-categories.

Corollary 25. The po-category $\mathbb{FRg}(T) := \mathbb{R}\text{el}_{\text{FRg}(T)}$ is the free regular po-category on the set T . That is, there is an adjunction

$$\text{Set} \begin{array}{c} \xrightarrow{\mathbb{FRg}(-)} \\ \rightleftarrows \\ \xleftarrow{\text{Ob}} \end{array} \text{RgPocat}.$$

Free regular po-categories will form the foundation of our graphical calculus for regular logic; we give an explicit description in Section 5.

For any regular category \mathcal{R} , the counit map of the adjunction in Theorem 23 gives a regular functor $\lceil - \rceil: \text{FRg}(\text{Ob } \mathcal{R}) \rightarrow \mathcal{R}$ that sends a context $\Gamma = (n, S, \tau)$ to the product

$$\lceil \Gamma \rceil := \prod_{i \in \underline{n}} \langle \tau(i) \rangle \times \prod_{s \in S} \text{Supp}(s). \tag{8}$$

For any regular po-category \mathcal{R} , the counit map of the adjunction in Corollary 25 gives a morphism of regular po-categories that we again denote $\lceil - \rceil: \mathbb{FRg}(\text{Ob } \mathcal{R}) \rightarrow \mathcal{R}$. This is a strong monoidal functor because the counit of Theorem 23 preserves finite products.

4.2 Regular calculi

In this section we introduce regular calculi. This is a new category-theoretic way to look at the kinds of logical moves—and the relationships between them—found in regular logic.

Definition 26. A *regular calculus* is a pair (T, P) where T is a set and $P: \mathbb{FRg}(T) \rightarrow \mathbb{P}\text{oset}$ is an ajax po-functor. For any object $\Gamma \in \mathbb{FRg}(T)$, we denote the order in the poset $P(\Gamma)$ using the \vdash_{Γ} or \vdash symbol (rather than \leq).

A *morphism* $(T, P) \rightarrow (T', P')$ of regular calculi is a pair (F, F^{\sharp}) where $F: T \rightarrow T'$ is a function and F^{\sharp} is a monoidal natural transformation

$$\begin{array}{ccc} T & \mathbb{FRg}(T) & \xrightarrow{P} \\ F \downarrow & \mathbb{FRg}(F) \downarrow & F^{\sharp} \Downarrow \\ T' & \mathbb{FRg}(T') & \xrightarrow{P'} \mathbb{P}\text{oset} \end{array}$$

that is strict in every respect: all the required coherence diagrams of posets commute on the nose. We denote the category of regular calculi by RgCalc .

Notation 27 (Adjoint notation $(f_!$ and f^*) in regular calculi). It will be convenient to define notation mimicking that in Eq. (4) for P 's action on adjoints in $\mathbb{R}\text{el}_{\mathcal{R}}$. Given an ajax po-functor $P: \mathbb{R}\text{el}_{\mathcal{R}} \rightarrow \mathbb{P}\text{oset}$, we can take adjoints and use the fundamental lemma (Lemma 8) to obtain the diagram below:

$$\begin{array}{ccccc} & & f \mapsto f_! & & \\ & & \curvearrowright & & \\ \mathcal{R} & \xrightarrow{\cong} & \text{LAdj}(\mathbb{R}\text{el}_{\mathcal{R}}) & \xrightarrow{\text{LAdj}(P)} & \text{LAdj}(\mathbb{P}\text{oset}) \\ & & \downarrow \cong & & \downarrow \cong \\ \mathcal{R} & \xrightarrow{\cong} & \text{RAdj}(\mathbb{R}\text{el}_{\mathcal{R}})^{\text{op}} & \xrightarrow{\text{RAdj}(P)^{\text{op}}} & \text{RAdj}(\mathbb{P}\text{oset})^{\text{op}} \\ & & \curvearrowleft & & \\ & & f \mapsto f^* & & \end{array}$$

That is, for any $f: r \rightarrow r'$ in \mathcal{R} we have an adjunction $f_! \dashv f^*$ between posets $P(r)$ and $P(r')$. In particular, since $\mathbb{FRg}(T)$ has finite products (denoted using 0 and \oplus), we will speak of projection maps $\pi_i: (\Gamma_1 \oplus \Gamma_2) \rightarrow \Gamma_i$, for $i = 1, 2$, diagonal maps $\delta_r: r \rightarrow r \oplus r$, and the unique map $\varepsilon_r: r \rightarrow 0$. Each determines an adjunction as above.

Remark 28 (Regular calculi send objects to meet-semilattices). If $P: \mathbb{FRg}(T) \rightarrow \mathbb{P}\text{oset}$ is a regular calculus, i.e. an ajax po-functor, then by Corollary 20 the poset $P(\Gamma)$ is a meet-semilattice for each object $\Gamma \in \mathcal{R}$. Explicitly, its top element and meet are given by the composites of right adjoints shown here:

$$1 \xrightarrow[\! \leftarrow \!]{\rho} P(0) \xrightarrow[\! \leftarrow \!]{\varepsilon_{\Gamma}^*} P(\Gamma) \quad \text{and} \quad P(\Gamma) \times P(\Gamma) \xrightarrow[\! \leftarrow \!]{\rho_{\Gamma, \Gamma}} P(\Gamma \oplus \Gamma) \xrightarrow[\! \leftarrow \!]{\delta_{\Gamma}^*} P(\Gamma). \quad (9)$$

4.3 The predicates functor $\text{prd}: \text{RgCat} \rightarrow \text{RgCalc}$

Let \mathcal{R} be a regular category and let $\mathcal{R} := \mathbb{R}\text{el}_{\mathcal{R}}$ denote its relations po-category; note that $\text{Ob } \mathcal{R} = \text{Ob } \mathcal{R}$. We have a counit map $\ulcorner - \urcorner: \mathbb{FRg}(\text{Ob } \mathcal{R}) \rightarrow \mathcal{R}$ from Corollary 25, and it is a strong monoidal functor. We can compose it with the “subobjects” functor $\text{Sub}_{\mathcal{R}} := \mathcal{R}(I, -): \mathcal{R} \rightarrow \mathbb{P}\text{oset}$. The result is a po-functor

$$\text{Sub}_{\mathcal{R}} \ulcorner - \urcorner: \mathbb{FRg}(\text{Ob } \mathcal{R}) \rightarrow \mathbb{P}\text{oset} \quad (10)$$

which assigns to each context Γ the poset of *predicates* in Γ . By Lemma 13 and Theorem 19, the po-functor $\text{Sub}_{\mathcal{R}}^{\ulcorner - \urcorner}$, is ajax, so $(\text{Ob } \mathcal{R}, \text{Sub}_{\mathcal{R}}^{\ulcorner - \urcorner})$ is a regular calculus.

Proposition 29. *The mapping from Eq. (10) extends to a faithful functor*

$$\mathbf{prd}: \text{RgCat} \rightarrow \text{RgCalc}.$$

Proof. Given an object \mathcal{R} of RgCat —that is, given a regular category—we define its image to be $\mathbf{prd}(\mathcal{R}) := (\text{Ob } \mathcal{R}, \text{Sub}_{\mathcal{R}}^{\ulcorner - \urcorner})$. As mentioned above, $\text{Sub}_{\mathcal{R}}^{\ulcorner - \urcorner}$ is ajax, so $\mathbf{prd}(\mathcal{R})$ is a regular calculus. We need to say how \mathbf{prd} behaves on morphisms.

A regular functor $\mathcal{F}: \mathcal{R} \rightarrow \mathcal{R}'$ induces a function $\text{Ob } \mathcal{F}: \text{Ob } \mathcal{R} \rightarrow \text{Ob } \mathcal{R}'$ and hence a morphism $\overline{\mathcal{F}} := \mathbb{F}\text{Rg}(\text{Ob } \mathcal{F}): \mathbb{F}\text{Rg}(\text{Ob } \mathcal{R}) \rightarrow \mathbb{F}\text{Rg}(\text{Ob } \mathcal{R}')$. We need to construct a (strict) monoidal natural transformation $\mathcal{F}^\# : \text{Sub}_{\mathcal{R}}^{\ulcorner - \urcorner} \rightarrow (\overline{\mathcal{F}} \circ \text{Sub}_{\mathcal{R}'}^{\ulcorner - \urcorner})$.

Let $\Gamma \in \mathbb{F}\text{Rg}(\text{Ob } \mathcal{R})$ be a context. The left-hand square in the following diagram commutes by the naturality of the counit $\ulcorner - \urcorner$, and we have a map $\mathbb{R}\text{el}_{\mathcal{F}}(I, -): \mathbb{R}\text{el}_{\mathcal{R}}(I, -) \rightarrow \mathbb{R}\text{el}_{\mathcal{R}'}(I, -)$ because $\mathcal{F}(I) = I$. We define $\mathcal{F}^\#$ to be the composite 2-cell, which we denote $\text{Sub}_{\mathcal{F}}^{\ulcorner - \urcorner}$:

$$\begin{array}{ccccc}
 \text{Ob } \mathcal{R} & \mathbb{F}\text{Rg}(\text{Ob } \mathcal{R}) & \xrightarrow{\ulcorner - \urcorner} & \mathbb{R}\text{el}_{\mathcal{R}} & \xrightarrow{\mathbb{R}\text{el}_{\mathcal{R}}(I, -)} \\
 \text{Ob } \mathcal{F} \downarrow & \overline{\mathcal{F}} \downarrow & & \mathbb{R}\text{el}_{\mathcal{F}} \downarrow & \Downarrow \mathbb{R}\text{el}_{\mathcal{F}}(I, -) \\
 \text{Ob } \mathcal{R}' & \mathbb{F}\text{Rg}(\text{Ob } \mathcal{R}') & \xrightarrow{\ulcorner - \urcorner} & \mathbb{R}\text{el}_{\mathcal{R}'} & \xrightarrow{\mathbb{R}\text{el}_{\mathcal{R}'}(I, -)} \\
 & & & & \text{Poset}
 \end{array}$$

Thus we define \mathbf{prd} on morphisms by $\mathbf{prd}(\mathcal{F}) = (\text{Ob } \mathcal{F}, \text{Sub}_{\mathcal{F}}^{\ulcorner - \urcorner})$; it is easy to check that \mathbf{prd} preserves identities and compositions. It remains to check that it is faithful, so let $\mathcal{F}, \mathcal{G}: \mathcal{R} \rightarrow \mathcal{R}'$ be regular functors and suppose $\mathbf{prd}(\mathcal{F}) = \mathbf{prd}(\mathcal{G})$. There is agreement on objects $\text{Ob } \mathcal{F} = \text{Ob } \mathcal{G}$, so let $f: r_1 \rightarrow r_2$ be a morphism in \mathcal{R} and consider the its graph $\hat{f} := \langle \text{id}_r, f \rangle \subseteq r_1 \times r_2$. Write $(r_1, r_2) := (2, \{r_1, r_2\}, \cong) \in \mathbb{F}\text{Rg}(\mathbb{T})$. From the fact that $\text{Sub}_{\mathcal{F}}^{\ulcorner - \urcorner} r_1, r_2^{\ulcorner \hat{f} \urcorner} = \text{Sub}_{\mathcal{G}}^{\ulcorner - \urcorner} r_1, r_2^{\ulcorner \hat{f} \urcorner}$ it follows that $\mathcal{F}(f) = \mathcal{G}(f)$, completing the proof. \square

The goal for the rest of this paper is to construct a functor \mathbf{prd} in the opposite direction. Our construction will rely on the fact that regular calculi can be incarnated as a sort of *graphical calculus* for regular logic reasoning.

5 Graphical regular logic

A key advantage of the regular calculus perspective on regular categories and regular logic is that it suggests a graphical notation for relations in regular categories, as well as how they behave under base-change and co-base-change. This is the promised graphical regular logic. In this section we develop this graphical formalism, first by giving a graphical description of the free regular po-category on a set, and then by defining the notion of graphical term, showing how these represent elements of posets, and explaining how to reason with them.

5.1 Depicting free regular po-categories $\mathbb{F}\text{Rg}(\mathbb{T})$

Since the po-categories $\mathbb{F}\text{Rg}(\mathbb{T})$ form the foundation of our diagrammatic language for regular logic, we begin our exploration of graphical regular logic by giving an explicit description of the objects, morphisms, 2-cells, and composition in $\mathbb{F}\text{Rg}(\mathbb{T})$ in terms of wiring diagrams.

Notation 30. By definition, an object of $\mathbb{FRg}(\mathbb{T})$ is simply a context $\Gamma = (\underline{n} \xrightarrow{\tau} S \subseteq \mathbb{T})$ of $\mathbb{FRg}(\mathbb{T})$. We represent a context graphically by a circle with n ports around the exterior, with i th port annotated by the value $\tau(i)$, and with a white dot at the base annotated by the remaining elements of the support $S \setminus \text{im } \tau$.¹

$$\begin{array}{c}
 \tau(2) \\
 \vdots \\
 \tau(1) \quad \bullet \quad \tau(n) \\
 \downarrow \\
 S \setminus \text{im } \tau
 \end{array}
 \quad (11)$$

Our convention will be for the ports to be numbered clockwise from the left of the circle, unless otherwise indicated, and to omit the white dot if $S = \text{im } \tau$. We refer to such an annotated circle as a *shell*.

As a syntactic shorthand for the shell in (11), we may combine all the ports and the white dot into a single wire labeled with the context $\Gamma \in \mathbb{FRg}(\mathbb{T})$: $\Gamma \bullet$.

Example 31. Let $\Gamma = (n, S, \tau)$ be the context with arity $n = 3$, support $S = \{w, x, y, z\} \subseteq \mathbb{T}$, and typing $\tau: \underline{3} \rightarrow S$ given by $\tau(1) = \tau(3) = y$, $\tau(2) = z$. It can be depicted by the shell



The hom-posets of $\mathbb{FRg}(\mathbb{T}) = \mathbb{Rel}_{\mathbb{FRg}(\mathbb{T})}$ are the subobject posets $\mathbb{FRg}(\mathbb{T})(\Gamma, \Gamma') = \text{Sub}(\Gamma \oplus \Gamma')$. Explicitly, a morphism $\omega: \Gamma_1 \rightarrow \Gamma_{\text{out}}$ is represented by a monomorphism $\Gamma_{\omega} = (n_{\omega} \xrightarrow{\tau_{\omega}} S_{\omega} \subseteq \mathbb{T}) \rightarrow \Gamma_1 \oplus \Gamma_{\text{out}}$ in $\mathbb{FRg}(\mathbb{T})$, and hence specified by a surjection ω (see Proposition 22) such that

$$\begin{array}{ccc}
 \underline{n}_{\omega} & \xrightarrow{\tau_{\omega}} & S_{\omega} \\
 \omega \uparrow & & \cup \\
 \underline{n}_1 + \underline{n}_{\text{out}} & \xrightarrow{\tau_1 + \tau_{\text{out}}} & S_1 \cup S_{\text{out}}
 \end{array}$$

commutes. We depict ω using a *wiring diagram*. More generally, wiring diagrams will give graphical representations of morphisms $\omega: \Gamma_1 \oplus \cdots \oplus \Gamma_k \rightarrow \Gamma_{\text{out}}$.

Notation 32. Suppose we have a morphism $\omega: \Gamma_1 \oplus \cdots \oplus \Gamma_k \rightarrow \Gamma_{\text{out}}$ in $\mathbb{FRg}(\mathbb{T})$. We depict ω as follows.

1. Draw the shell for Γ_{out} .
2. Draw the object Γ_i , for $i = 1, \dots, k$, as non-overlapping shells inside the Γ_{out} shell.
3. For each $i \in \underline{n}_{\omega}$, draw a black dot anywhere in the region interior to the Γ_{out} shell but exterior to all the Γ_i shells, and annotate it by the value $\tau_{\omega}(i)$.
4. Draw a white dot in the same region, annotated by all elements of S_{ω} not already present in the diagram.
5. For each element $(i, j) \in \sum_{i=1, \dots, k, \text{out}} \underline{n}_i$, draw a wire connecting the j th port on the object Γ_i to the black dot $\omega(i, j)$.

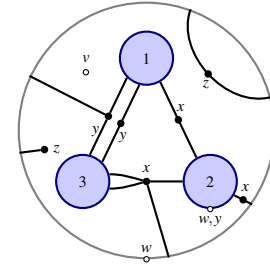
Just as for objects, we may neglect to draw a white dot when $\text{im } \tau = S$.

For a more compact notation, we may also neglect to explicitly draw the object Γ_{out} , leaving it implicit as comprising the wires left dangling on the boundary of the diagram.

¹By the idempotence of support contexts Eq. (16), one may equivalently include the whole support, S .

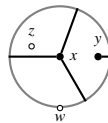
Example 33. Here is the set-theoretic data of a morphism $\omega: \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \rightarrow \Gamma_{\text{out}}$, together with its wiring diagram depiction:

$$\begin{aligned} \Gamma_1 &= (3, \{x, y\}, \tau_1) \text{ where } \tau_1(1) = x, \tau_1(2) = \tau_1(3) = y; \\ \Gamma_2 &= (3, \{w, x, y\}, \tau_2) \text{ where } \tau_2(1) = \tau_2(2) = \tau_2(3) = x; \\ \Gamma_3 &= (4, \{x, y\}, \tau_3) \text{ where } \tau_3(1) = \tau_3(2) = y, \tau_3(3) = \tau_3(4) = x; \\ \Gamma_{\text{out}} &= (6, \{w, x, y, z\}, \tau_{\text{out}}) \text{ where } \tau_{\text{out}}(1) = y, \\ &\quad \tau_{\text{out}}(2) = \tau_{\text{out}}(3) = \tau_{\text{out}}(6) = z, \tau_{\text{out}}(4) = \tau_{\text{out}}(5) = x; \\ \Gamma_{\omega} &= (7, \{v, w, x, y, z\}, \tau_{\omega}) \text{ where } \tau_{\omega}(1) = \tau_{\omega}(2) = y, \\ &\quad \tau_{\omega}(3) = \tau_{\omega}(7) = z, \tau_{\omega}(4) = \tau_{\omega}(5) = \tau_{\omega}(6) = x; \end{aligned}$$

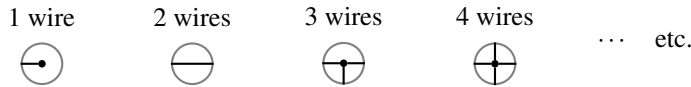


$$\begin{aligned} f(1, 1) &= 4, f(1, 2) = 2, f(1, 3) = 1, f(2, 1) = 6, f(2, 2) = 4, \\ f(2, 3) &= 5, f(3, 1) = 1, f(3, 2) = 2, f(3, 3) = f(3, 4) = 6, \\ f(\text{out}, 1) &= 1, f(\text{out}, 2) = 3, f(\text{out}, 3) = 3, f(\text{out}, 4) = 5 \\ f(\text{out}, 5) &= 6, f(\text{out}, 6) = 7. \end{aligned}$$

Example 34. Note that we may have $k = 0$, in which case there are no inner shells. For example, the following has $\Gamma_{\omega} = (2, \{x, y, z, w\}, 1 \mapsto x, 2 \mapsto y)$.



Remark 35. When multiple wires meet at a point, our convention will be to draw a dot iff the number of wires is different from two.



When wires intersect and we do not draw a black dot, the intended interpretation is that the wires are *not connected*: $\oplus \neq \oplus$. Of course this is bound to happen when the graph is non-planar.

The following examples give a flavor of how composition, monoidal product, and 2-cells are represented using this graphical notation.

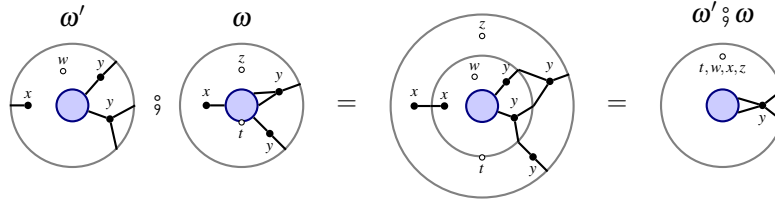
Example 36 (Composition as substitution). Composition of morphisms is described by *nesting* of wiring diagrams. Let $\omega': \Gamma' \rightarrow \Gamma_1$ and $\omega: \Gamma_1 \rightarrow \Gamma_{\text{out}}$ be morphisms in $\mathbb{F}Rg(\mathbb{T})$. Then the composite relation $\omega' \circ \omega: \Gamma' \rightarrow \Gamma_{\text{out}}$ is given by

1. drawing the wiring diagram for ω' inside the inner circle of the diagram for ω ,
2. erasing the object Γ_1 ,
3. amalgamating any connected black dots into a single black dot, and
4. removing all components not connected to the objects Γ' or Γ_{out} , and adding a single white dot annotated by the set containing all elements of \mathbb{T} present in these components, but not present elsewhere in the diagram.

Note that step 3 corresponds to taking pullbacks in $\mathbb{F}Rg(\mathbb{T})$ (pushouts in FinSet), while step 4 corresponds to epi-mono factorization.

As a shorthand for composition, we simply draw one wiring diagram directly substituted into another,

as per step 1. For example, we have

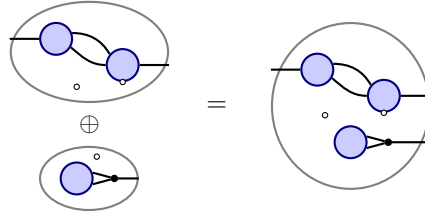


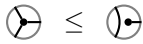



For the more general k -ary or operadic case, we may obtain the composite

$$(\Gamma_1 \oplus \cdots \oplus \Gamma_{i-1} \oplus \omega' \oplus \Gamma_{i+1} \oplus \cdots \oplus \Gamma_k) \circ \omega$$

of any two morphisms $\omega': \Gamma'_1 \oplus \cdots \oplus \Gamma'_k \rightarrow \Gamma_i$ and $\omega: \Gamma_1 \oplus \cdots \oplus \Gamma_k \rightarrow \Gamma_{\text{out}}$ by substituting the wiring diagram for ω' into the i th inner circle of the diagram for ω , and following a procedure similar to that in Example 36.

Example 37 (Monoidal product as juxtaposition). The monoidal product of two morphisms in $\mathbb{FRg}(\mathbb{T})$ is simply their juxtaposition, merging the labels on the floating white dots as appropriate. For example, leaving off labels, we might have:



Example 38 (2-cells as breaking wires and removing white dots). Let $\omega, \omega': \Gamma_1 \oplus \cdots \oplus \Gamma_k \rightarrow \Gamma_{\text{out}}$ be morphisms in $\mathbb{FRg}(\mathbb{T}) = \mathbb{Rel}_{\mathbb{FRg}(\mathbb{T})}$. By definition, there exists a 2-cell $\omega \leq \omega'$ if there is a monomorphism $m: \Gamma_\omega \rightarrow \Gamma_{\omega'}$ in $\mathbb{FRg}(\mathbb{T})$ such that $m \circ \omega' = \omega$ holds in $\mathbb{FRg}(\mathbb{T})$. By Proposition 22, this data consists of a surjection of finite sets $m: \underline{n}'_{\omega'} \rightarrow \underline{n}_\omega$ and an inclusion $S_{\omega'} \subseteq S_\omega$. In diagrams, the former means 2-cells may break wires, and the latter means they may remove annotations from the inner white dot (or remove it completely). For example, we have 2-cells:  \leq  and  \leq .

5.2 Graphical terms

Given a regular calculus $P: \mathbb{FRg}(\mathbb{T}) \rightarrow \mathbb{Poset}$, we give graphical representations of its predicates, i.e. the elements in $P(\Gamma)$ for various contexts $\Gamma \in \mathbb{FRg}(\mathbb{T})$. Here's how it works.

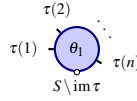
Definition 39. A P -graphical term $(\theta_1, \dots, \theta_k; \omega)$ in an ajax po-functor $P: \mathbb{FRg}(\mathbb{T}) \rightarrow \mathbb{Poset}$ is a morphism $\omega: \Gamma_1 \oplus \cdots \oplus \Gamma_k \rightarrow \Gamma_{\text{out}}$ in $\mathbb{FRg}(\mathbb{T})$ together with, for each $i = 1, \dots, k$, an element $\theta_i \in P(\Gamma_i)$.

We say that the graphical term $t = (\theta_1, \dots, \theta_k; \omega)$ represents the poset element

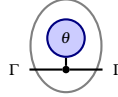
$$\llbracket t \rrbracket := (P(\omega) \circ \rho)(\theta_1, \dots, \theta_k) \in P(\Gamma_{\text{out}})$$

where ρ is the k -ary laxator. If t and t' are graphical terms, we write $t \vdash t'$ when $\llbracket t \rrbracket \vdash \llbracket t' \rrbracket$, and $t = t'$ when $\llbracket t \rrbracket = \llbracket t' \rrbracket$.

Notation 40. We draw a graphical term $(\theta_1, \dots, \theta_k; \omega)$ by annotating the i th inner shell with its corresponding poset element θ_i . In the case that $k = 1$ and ω is the identity morphism, we may simply draw the object Γ_1 annotated by θ_1 :



Example 41. Recall that we have a diagonal map $\delta: \Gamma \rightarrow \Gamma \oplus \Gamma$ in $\text{FRg}(\mathbb{T}) \subseteq \mathbb{FRg}(\mathbb{T})$. Given $\theta \in P(\Gamma)$, the element $(\delta)_!(\theta) \in P(\Gamma \oplus \Gamma)$ is represented by the graphical term



Example 42. When $\mathbb{T} = \emptyset$ is empty, $\text{FRg}(\emptyset)$ is the terminal category. By Proposition 14, an ajax po-functor $P: \mathbb{FRg}(\mathbb{T}) \rightarrow \mathbb{P}\text{oset}$ then simply chooses a \wedge -semilattice $P(0)$. The po-category \mathbb{IntRel}_P is that \wedge -semilattice considered as a one object po-category: it has a unique object whose poset of endomorphisms is $P(0)$. The diagrammatic language has no wires, since there is only the monoidal unit in $\text{FRg}(\emptyset)$. The semantics of an arbitrary graphical term $(\theta_1, \dots, \theta_k; \text{id})$ is simply the meet $\theta_1 \wedge \dots \wedge \theta_k$.

Remark 43. Graphical terms are an alternate syntax for regular logic. While we will not dwell on the translation, a graphical term $(\theta_1, \dots, \theta_k; \omega)$ represents the regular formula

$$\exists_{j \in \{1, \dots, k, \omega\}} \bigwedge_{i \in \underline{n}_j} x_{ij}. \bigwedge_{j \in \{1, \dots, k\}} \theta_k(x_{ij}) \wedge \bigwedge_{j \in \{1, \dots, k, \text{out}\}} \bigwedge_{i \in \underline{n}_j} (x_{ij} = x_{\omega(i)j}).$$

This formula creates a variable of each element of \underline{n}_j , where $j \in \{1, \dots, k, \text{out}, \omega\}$, equates any two variables with the same image under ω , takes the conjunction with all the formulas θ_j , and the existentially quantifies over all variables except those in Γ_{out} . In particular, if we were to take $\omega: \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \rightarrow \Gamma_{\text{out}}$ as in Example 33, the resulting graphical term would simplify to the formula

$$\psi(y, z, z', x, x', z'') = \exists \tilde{x}, \tilde{y}. \theta_1(\tilde{x}, \tilde{y}, y) \wedge \theta_2(\tilde{x}, x, x') \wedge \theta_3(y, \tilde{y}, x', x') \wedge (z = z') \wedge (z'' = z'').$$

Remark 44. Note that $\mathbb{P}\text{oset}$ is a subcategory of $\mathbb{C}\text{at}$. This allows us to take the monoidal Grothendieck construction $\int P$ of $P: \mathbb{FRg}(\mathbb{T}) \rightarrow \mathbb{P}\text{oset}$, [18]. A P -graphical term is an object in the comma category $\int P \downarrow \mathbb{FRg}(\mathbb{T})$. This perspective lends structure to the various operations on diagrams belows; we, however, pursue it no further here.

5.3 Reasoning with graphical terms

The following rules for reasoning with diagrams express the (2-)functoriality and monoidality of P .

Proposition 45. Let $(\theta_1, \dots, \theta_k; \omega)$ be a graphical term, where $\theta_i \in P(\Gamma_i)$.

(i) (Monotonicity) Suppose $\theta_i \vdash \theta'_i$ for some i . Then

$$\llbracket (\theta_1, \dots, \theta_i, \dots, \theta_k; \omega) \rrbracket \vdash \llbracket (\theta_1, \dots, \theta'_i, \dots, \theta_k; \omega) \rrbracket.$$

(ii) (Breaking) Suppose $\omega \leq \omega'$ in $\mathbb{FRg}(\mathbb{T})$. Then

$$\llbracket (\theta_1, \dots, \theta_k; \omega) \rrbracket \vdash \llbracket (\theta_1, \dots, \theta_k; \omega') \rrbracket.$$

(iii) (Nesting) Suppose $\theta_i = \llbracket (\theta'_1, \dots, \theta'_\ell; \omega') \rrbracket$ for some i . Then

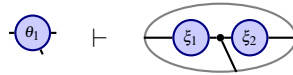
$$\llbracket (\theta_1, \dots, \theta_k; \omega) \rrbracket = \llbracket (\theta_1, \dots, \theta_{i-1}, \theta'_1, \dots, \theta'_\ell, \theta_{i+1}, \dots, \theta_k; (\Gamma_1 \oplus \dots \oplus \Gamma_{i-1} \oplus \omega' \oplus \Gamma_{i+1} \oplus \dots \oplus \Gamma_k) \circlearrowleft \omega) \rrbracket.$$

Proof. Statement (i) is the monotonicity of the map $P(\omega) \circlearrowleft \rho$, while (ii) is the 2-functoriality of P . Statement (iii) follows from the monoidality and 1-functoriality of P . In particular, it is the commutativity of the following diagram. Using the braiding we can assume without loss of generality that $i = k$.

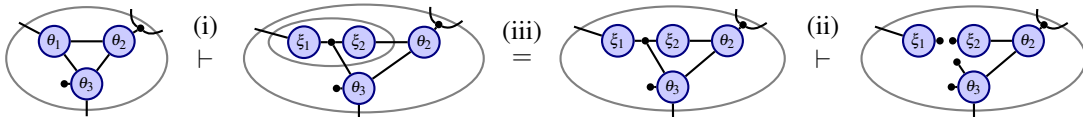
$$\begin{array}{ccccc} \prod_{j=1}^{k-1} P(\Gamma_j) \times \prod_{j=1}^{\ell} P(\Gamma'_j) & & & & \\ \text{id} \times \rho \downarrow & \searrow \rho & & & \\ \prod_{j=1}^{k-1} P(\Gamma_j) \times P\left(\bigoplus_{j=1}^{\ell} \Gamma'_j\right) & \xrightarrow{\rho} & P\left(\bigoplus_{j=1}^{k-1} \Gamma_j \oplus \bigoplus_{j=1}^{\ell} \Gamma'_j\right) & & \\ \Pi_P(\Gamma_j) \times P(\omega) \downarrow & & P(\bigoplus_{j=1}^{k-1} \Gamma_j + \omega') \downarrow & \searrow P((\bigoplus_{j=1}^{k-1} \Gamma_j + \omega') \circlearrowleft \omega) & \\ \prod_{j=1}^k P(\Gamma_j) & \xrightarrow{\rho} & P\left(\bigoplus_{j=1}^k \Gamma_j\right) & \xrightarrow{P(\omega)} & P(\Gamma_{\text{out}}) \end{array}$$

The upper triangle commutes by coherence laws for ρ , the square commutes by naturality of ρ , and the right hand triangle commutes by functoriality of P . \square

Example 46. Proposition 45 is perhaps more quickly grasped through a graphical example of these facts in action. Suppose we have the entailment



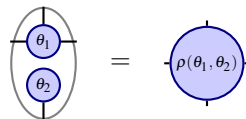
Then using monotonicity, nesting, and then breaking we can deduce the entailment



We'll see many further examples of such reasoning in the subsequent sections of this paper, as we prove that we can construct a regular category from a regular calculus.

Example 47. The nesting rule in Proposition 45 has two particularly important cases. The first occurs when we consider wiring diagrams themselves as poset elements. More precisely, if $f: \Gamma_1 \rightarrow \Gamma_{\text{out}}$ is a morphism in $\text{FRg}(\mathbb{T})$, and $\hat{f} := \langle \text{id}_{\Gamma_1}, f \rangle$ is its graph, then taking $i = k = 1$, $\ell = 0$, $\theta = \llbracket (\hat{f}) \rrbracket$, $\omega = \Gamma_{\text{out}}$ (the identity) and $\omega' = \hat{f}$ in (iii) gives $\llbracket (\theta; \Gamma_{\text{out}}) \rrbracket = \llbracket (\hat{f}) \rrbracket$. Note that this equates a graphical term with inner object Γ_{out} and annotation θ with a term that has no inner object at all; see e.g. Example 34.

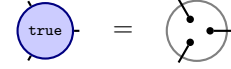
The second important case is that of 'exterior AND'. If we take $i = k = 1$, $\ell = 2$, and $\omega = \omega' = \Gamma_1 \oplus \Gamma_2$, then $\llbracket (\theta'_1, \theta'_2; \Gamma_1 \oplus \Gamma_2) \rrbracket = \llbracket (\rho(\theta'_1, \theta'_2); \Gamma_1 \oplus \Gamma_2) \rrbracket$. In pictures, this means we can take any two circles, say $\theta_1 \in P(\Gamma_1)$ and $\theta_2 \in P(\Gamma_2)$, and merge them, labelling the merged circle with $\rho_{\Gamma_1, \Gamma_2}(\theta_1, \theta_2)$:



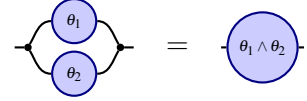
The meet-semilattice structure permits an intuitive graphical interpretation. Indeed, the definitions of `true` and `meet` (see Eq. (9)) immediately yield the following proposition. In the following proposition, the graphical terms on right are illustrative examples of the equalities stated on the left.

Proposition 48. *For all contexts Γ in $\mathbb{F}\text{Rg}(\mathbb{T})$ and $\theta, \theta' \in P(\Gamma)$, we have*

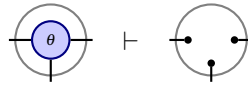
(i) (*True is removable*) $\llbracket (\text{true}_\Gamma; \Gamma) \rrbracket = \llbracket (; \varepsilon_\Gamma) \rrbracket$



(ii) (*Meets are merges*) $\llbracket (\theta_1 \wedge \theta_2; \Gamma) \rrbracket = \llbracket (\theta_1, \theta_2; \delta_\Gamma) \rrbracket$.



Example 49 (Discarding). Note that Proposition 48(i) and the monotonicity of diagrams (Proposition 45(i)) further imply that for all $\theta \in P(\Gamma)$ we have $\llbracket (\theta; \Gamma) \rrbracket \vdash \llbracket (; \varepsilon_\Gamma) \rrbracket$:



6 The syntactic category

Finally, we show that given a regular calculus, we can construct a regular category, and that this construction extends to a functor $\mathbf{syn}: \text{RgCalc} \rightarrow \text{RgCat}$, that acts as a one-sided weak inverse to \mathbf{prd} .

6.1 Internal relations and internal functions

Definition 50. Given objects Γ_1, Γ_2 and $\varphi_1 \in P(\Gamma_1)$ and $\varphi_2 \in P(\Gamma_2)$, we define the poset $\mathbb{I}ntRel_P(\varphi_1, \varphi_2)$ of *P-internal relations from φ_1 to φ_2* to be the subposet

$$\mathbb{I}ntRel_P(\varphi_1, \varphi_2) := \{ \theta \in P(\Gamma_1 \oplus \Gamma_2) \mid (\pi_1)_! \theta \vdash_{\Gamma_1} \varphi_1 \text{ and } (\pi_2)_! \theta \vdash_{\Gamma_2} \varphi_2 \} \subseteq P(\Gamma_1 \oplus \Gamma_2).$$

An internal relation θ may be represented by the graphical term $r_1 \text{---} \theta \text{---} r_2$ together with the two entailments



Note that when this definition is applied to the regular calculus $\mathbf{prd}(\mathcal{R})$ associated to a regular category \mathcal{R} , it recovers the usual notion of relation between objects in \mathcal{R} .

Proposition 51. *Let \mathcal{R} be a regular category, let $\Gamma_1, \Gamma_2 \in \text{FRg}(\text{Ob } \mathcal{R})$ be contexts, and suppose given $r_1 \in \text{Sub}_{\mathcal{R}} \lceil \Gamma_1 \rceil$ and $r_2 \in \text{Sub}_{\mathcal{R}} \lceil \Gamma_2 \rceil$. There is a natural isomorphism*

$$\mathbb{I}ntRel_{\mathbf{prd}(\mathcal{R})}((\Gamma_1, r_1), (\Gamma_2, r_2)) \cong \mathbb{R}el_{\mathcal{R}}(r_1, r_2).$$

Proof. Let $g_1 := \lceil \Gamma_1 \rceil$ and $g_2 := \lceil \Gamma_2 \rceil$ so we have $r_1 \subseteq g_1$ and $r_2 \subseteq g_2$; see Eq. (8). By Definition 50 and Proposition 29, a $\mathbf{prd}(\mathcal{R})$ -internal relation between them is an element $t \subseteq g_1 \times g_2$ such that there exist dotted arrows making the following diagram commute:

$$\begin{array}{ccccc} r_1 & \xleftarrow{\dots\dots\dots} & t & \xrightarrow{\dots\dots\dots} & r_2 \\ \downarrow & & \downarrow & & \downarrow \\ g_1 & \longleftarrow & g_1 \times g_2 & \longrightarrow & g_2 \end{array}$$

The composite $t \rightarrow r_1 \times r_2 \rightarrow g_1 \times g_2$ is monic, so we have that $t \subseteq r_1 \times r_2$. The result follows. \square

Theorem 52. Let $P: \mathbb{FRg}(\mathbb{T}) \rightarrow \mathbb{P}\text{oset}$ be a regular calculus. Then there exists a po-category \mathbb{IntRel}_P whose objects are pairs (Γ, φ) , where Γ is an object of $\mathbb{FRg}(\mathbb{T})$ and $\varphi \in P(\Gamma)$, and with hom-posets $(\Gamma_1, \varphi_1) \rightarrow (\Gamma_2, \varphi_2)$ given by $\mathbb{IntRel}_P(\varphi_1, \varphi_2)$.

The composition rule is given as follows. For objects $\Gamma_1, \Gamma_2, \Gamma_3$ in $\mathbb{FRg}(\mathbb{T})$, let

$$\text{comp}_{\Gamma_1, \Gamma_2, \Gamma_3} := \begin{array}{c} \text{---} \Gamma_1 \text{---} \text{---} \Gamma_2 \text{---} \text{---} \Gamma_3 \text{---} \\ \text{---} \text{---} \end{array}$$

It is a morphism $(\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_2 \oplus \Gamma_3) \rightarrow (\Gamma_1 \oplus \Gamma_3)$ in $\mathbb{FRg}(\mathbb{T})$. We then define

$$(-) \circ (-): P(\Gamma_1 \oplus \Gamma_2) \times P(\Gamma_2 \oplus \Gamma_3) \xrightarrow{\rho} P(\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_2 \oplus \Gamma_3) \xrightarrow{P(\text{comp})} P(\Gamma_1 \oplus \Gamma_3). \quad (12)$$

Definition 53. Given a regular calculus (\mathbb{T}, P) , where $P: \mathbb{FRg}(\mathbb{T}) \rightarrow \mathbb{P}\text{oset}$, we define the category \mathcal{R}_P of P -internal functions to be the category of left adjoints in \mathbb{IntRel}_P :

$$\mathcal{R}_P := \text{LAdj}(\mathbb{IntRel}_P). \quad (13)$$

In more detail, suppose given elements $\varphi_1 \in P(\Gamma_1)$ and $\varphi_2 \in P(\Gamma_2)$. We say that an internal relation $\theta \in \mathbb{IntRel}_P(\varphi_1, \varphi_2) \subseteq P(\Gamma_1 \oplus \Gamma_2)$ is an *internal function* if there exists an internal relation ξ such that

$$\begin{array}{c} \varphi_1 \\ \downarrow \end{array} \vdash \begin{array}{c} \theta \\ \text{---} \end{array} \begin{array}{c} \xi \\ \downarrow \end{array} \quad \text{and} \quad \begin{array}{c} \xi \\ \downarrow \end{array} \begin{array}{c} \theta \\ \text{---} \end{array} \vdash \begin{array}{c} \varphi_2 \\ \downarrow \end{array}.$$

The category \mathcal{R}_P has the same objects (Γ, φ) as \mathbb{IntRel}_P , and morphisms given by internal functions.

The central result of this paper is that internal functions form a regular category.

Theorem 54. For any regular calculus $P: \mathbb{FRg}(\mathbb{T}) \rightarrow \mathbb{P}\text{oset}$, the category \mathcal{R}_P of internal functions in \mathbb{IntRel}_P is regular.

The proof, found in Appendix E is divided into three parts: in Appendix E.1 we explore properties of internal functions, in Appendix E.2 we show \mathcal{R}_P has finite limits, and in Appendix E.3 we show it has pullback stable image factorizations.

6.2 The functor $\mathbf{syn}: \text{RgCalc} \rightarrow \text{RgCat}$

We want to define a functor $\mathbf{syn}: \text{RgCalc} \rightarrow \text{RgCat}$. On objects, this is now easy: given a regular calculus $(\mathbb{T}, P) \in \text{RgCalc}$, define $\mathbf{syn}(\mathbb{T}, P) := \mathcal{R}_P$ to be the syntactic category as in Definition 53. For morphisms, suppose given $(F, F^\sharp): (\mathbb{T}, P) \rightarrow (\mathbb{T}', P')$:

$$\begin{array}{ccc} \mathbb{T} & \mathbb{FRg}(\mathbb{T}) & \xrightarrow{P} \\ \downarrow F & \downarrow \bar{F} & \searrow F^\sharp \\ \mathbb{T}' & \mathbb{FRg}(\mathbb{T}') & \xrightarrow{P'} \end{array} \quad \mathbb{P}\text{oset}$$

where again $\bar{F} := \mathbb{FRg}(F)$. We define $\mathcal{F} := \mathbf{syn}(F, F^\sharp): \mathcal{R}_P \rightarrow \mathcal{R}_{P'}$ on an object $(\Gamma, \varphi) \in \mathcal{R}_P$ by

$$\mathcal{F}(\Gamma, \varphi) := \left(\bar{F}(\Gamma), F^\sharp_\Gamma(\varphi) \right) \in \mathcal{R}_{P'}. \quad (14)$$

and on a morphism $\theta: (\Gamma_1, \varphi_1) \rightarrow (\Gamma_2, \varphi_2)$ by

$$\mathcal{F}(\theta) := F^\sharp_{\Gamma_1 \oplus \Gamma_2}(\theta). \quad (15)$$

Theorem 55. *The assignment $\mathbf{syn}(\mathbb{T}, P) := \mathcal{R}_P$ on objects, and Eqs. (14) and (15) on morphisms, constitutes a functor $\mathbf{syn}: \mathbf{RgCalc} \rightarrow \mathbf{RgCat}$.*

To prove this, we first need the following lemma.

Lemma 56. *F^\sharp preserves semantics of graphical terms.*

More precisely, given any P -graphical term $(\theta_1, \dots, \theta_k; \omega)$, the morphism (F, F^\sharp) induces a P' -graphical term $(F^\sharp\theta_1, \dots, F^\sharp\theta_k; \overline{F}(\omega))$; we call this its image under F^\sharp . The image obeys

$$F^\sharp\llbracket(\theta_1, \dots, \theta_k; \omega)\rrbracket = \llbracket(F^\sharp\theta_1, \dots, F^\sharp\theta_k; \overline{F}(\omega))\rrbracket.$$

Furthermore, given the entailment $(\theta_1, \dots, \theta_k; \omega) \vdash (\theta'_1, \dots, \theta'_{k'}; \omega')$, it follows that

$$(F^\sharp\theta_1, \dots, F^\sharp\theta_k; \overline{F}(\omega)) \vdash (F^\sharp\theta'_1, \dots, F^\sharp\theta'_{k'}; \overline{F}(\omega')).$$

Proof. The naturality and monoidality of (F, F^\sharp) imply:

$$\begin{aligned} F^\sharp\llbracket(\theta_1, \dots, \theta_k; \omega)\rrbracket &= F^\sharp(P(\omega))(\rho(\theta_1, \dots, \theta_k)) \\ &= P'(\overline{F}(\omega))(F^\sharp(\rho(\theta_1, \dots, \theta_k))) \\ &= P'(\overline{F}(\omega))(\rho(F^\sharp\theta_1, \dots, F^\sharp\theta_k)) \\ &= \llbracket(F^\sharp\theta_1, \dots, F^\sharp\theta_k; \overline{F}(\omega))\rrbracket. \end{aligned}$$

The second claim then follows from the monotonicity of components in F^\sharp . \square

Proof of Theorem 55. First we must check that our data type-checks. We have already shown that \mathcal{R}_P is a regular category, so it remains to show that \mathcal{F} is a regular functor. This is a consequence of Lemma 56.

In particular, recall from Definition 53 that morphisms in \mathcal{R}_P can be represented by P -graphical terms obeying certain entailments. It was shown in Appendices E.2 and E.3 that composition, identities, finite limits, and regular epis can also be described in this way. Lemma 56 implies that given a P -graphical term, its image under F^\sharp preserves entailments and equalities. Thus \mathcal{F} sends internal functions to internal functions of the required domain and codomain, preserves composition, identities, finite limits, and regular epis, and hence is a regular functor.

It is then immediate from the definition (Eqs. (14) and (15)) that \mathbf{syn} preserves identity morphisms and composition, and so \mathbf{syn} is indeed a functor. \square

Finally, we note that each regular category is equivalent to the syntactic category of its regular calculus.

Proposition 57. *For any regular category \mathcal{R} , there is a natural equivalence of categories*

$$\varepsilon: \mathbf{syn}(\mathbf{prd}(\mathcal{R})) \xrightarrow{\cong} \mathcal{R}.$$

Proof. We will define functors $\varepsilon: \mathcal{R}_{\mathbf{prd}(\mathcal{R})} \rightleftarrows \mathcal{R}: \varepsilon'$ and show that they constitute an equivalence. We have $\mathbf{Ob}(\mathcal{R}_{\mathbf{prd}(\mathcal{R})}) = \{(\Gamma, r) \mid \Gamma \in \mathbf{FRg}(\mathbf{Ob}\mathcal{R}), r \in \mathbf{Sub}_{\mathcal{R}}\Gamma^\top\}$, so put

$$\varepsilon(\Gamma, r) := r, \quad \text{and} \quad \varepsilon'(r) := (\langle r \rangle, r),$$

where $\langle r \rangle$ is the unary context on r and $r \subseteq r = \lceil \langle r \rangle \rceil$ is the top element. Given also (Γ', r') , we have an isomorphism of hom-sets

$$\mathcal{R}_{\mathbf{prd}(\mathcal{R})}((\Gamma, r), (\Gamma', r')) \cong \mathbf{LAdj}(\mathbf{Rel}_{\mathcal{R}})(r, r') \cong \mathcal{R}(r, r'),$$

by Definition 53, Proposition 51, , and Lemma 8. Hence, we define ε and ε' on morphisms to be the corresponding mutually-inverse maps. Obviously, ε and ε' are fully faithful functors, and $\varepsilon' \circ \varepsilon = \mathbf{id}_{\mathcal{R}}$, so ε is essentially surjective. \square

7 Future Work

Having constructed the functors **prd** and **syn**, and a potential counit map $\mathbf{syn}(\mathbf{prd}(\mathcal{R})) \rightarrow \mathcal{R}$, one might hope we have an adjunction

$$\mathrm{RgCalc} \begin{array}{c} \xrightarrow{\mathbf{syn}} \\ \Leftrightarrow \\ \xleftarrow{\mathbf{prd}} \end{array} \mathrm{RgCat}.$$

This would allow us to understand RgCat as essentially a reflective subcategory of RgCalc , in the sense that for any regular category \mathcal{R} , the counit map $\mathbf{syn}(\mathbf{prd}(\mathcal{R})) \rightarrow \mathcal{R}$ is an equivalence of categories.

Unfortunately, this is not true. It is, however, very nearly so. A candidate unit map exists, and indeed one triangle axiom is satisfied, but the other only holds up equivalence. To state the structure precisely, we must instead move one dimension higher, examining a 2-adjunction between the 2-category of regular categories and the 2-category of regular calculi. We shall leave this to a future, expanded version of this paper.

Acknowledgments

The authors thank Christina Vasilakopoulou and Paolo Perrone for comments that have improved this article.

References

- [1] John C Baez & Jason Erbele (2015): *Categories in control*. *Theory and Applications of Categories* 30(24), pp. 836–881.
- [2] John C. Baez & Brendan Fong (2018): *A compositional framework for passive linear networks*. *Theory and Applications of Categories* 33(38), pp. 1158–1222.
- [3] John C Baez & Blake S Pollard (2017): *A compositional framework for reaction networks*. *Reviews in Mathematical Physics* 29(09), p. 1750028, doi:10.1142/S0129055X17500283.
- [4] Michael Barr (1971): *Exact categories*. In: *Exact categories and categories of sheaves*, Springer, pp. 1–120, doi:10.1007/BFb0058579.
- [5] Filippo Bonchi, Jens Seeber & Paweł Sobociński (2018): *Graphical conjunctive queries*. *arXiv preprint arXiv:1804.07626*.
- [6] Filippo Bonchi, Paweł Sobociński & Fabio Zanasi (2014): *A categorical semantics of signal flow graphs*. In: *International Conference on Concurrency Theory*, Springer, pp. 435–450, doi:10.1007/978-3-662-44584-6_30.
- [7] Carsten Butz (1998): *Regular categories and regular logic*. *BRICS Lecture Series LS-98-2*.
- [8] A. Carboni & R.F.C. Walters (1987): *Cartesian bicategories I*. *Journal of Pure and Applied Algebra* 49(1), pp. 11 – 32, doi:10.1016/0022-4049(87)90121-6.
- [9] Ashok K Chandra & Philip M Merlin (1977): *Optimal implementation of conjunctive queries in relational data bases*. In: *Proceedings of the ninth annual ACM symposium on Theory of computing*, ACM, pp. 77–90, doi:10.1145/800105.803397.
- [10] Brendan Fong (2018): *Decorated corelations*. *Theory and Applications of Categories* 33(22), pp. 608–643.
- [11] Brendan Fong, Paweł Sobociński & Paolo Rapisarda (2016): *A categorical approach to open and interconnected dynamical systems*. In: *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, ACM, pp. 495–504, doi:10.1145/2933575.2934556.

- [12] Brendan Fong & David I Spivak (2019): *Hypergraph categories*. *Journal of Pure and Applied Algebra* 223(11), pp. 4746–4777, doi:10.1016/j.jpaa.2019.02.014.
- [13] Brendan Fong & David I Spivak (2019): *Regular and relational categories: Revisiting 'Cartesian bicategories I'*. *arXiv preprint arXiv:1909.00069*.
- [14] Peter J Freyd & Andre Scedrov (1990): *Categories, allegories*. Elsevier.
- [15] Romaine Jayewardene & Oswald Wyler (2000): *Categories of Relations and Functional Relations*. *Applied Categorical Structures* 8(1), pp. 279–305, doi:10.1023/A:1008651524610.
- [16] André Joyal & Ross Street (1991): *The geometry of tensor calculus. I*. *Adv. Math.* 88(1), pp. 55–112, doi:10.1016/0001-8708(91)90003-P.
- [17] Saunders MacLane & Ieke Moerdijk (2012): *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media.
- [18] Joe Moeller & Christina Vasilakopoulou (2018): *Monoidal grothendieck construction*. *arXiv preprint arXiv:1809.00727*.
- [19] Evan Patterson (2017): *Knowledge representation in bicategories of relations*. *arXiv preprint arXiv:1706.00526*.
- [20] Andrea Schalk (1994): *Algebras for generalized power constructions*. *Bulletin of the European Association for Theoretical Computer Science* 53, pp. 491–491, doi:10.1016/0167-6423(91)90037-X.
- [21] David I. Spivak, Patrick Schultz & Dylan Rupel (2016): *String diagrams for traced and compact categories are oriented 1-cobordisms*. *Journal of Pure and Applied Algebra*, doi:10.1016/j.jpaa.2016.10.009.

A Proof of the fundamental lemma of regular categories

Proof of Lemma 8. This fact is well known, but since it is crucial to what follows, we provide a proof here. We shall show that there is an identity-on-objects, full, and faithful functor from \mathcal{R} to its relations po-category $\mathbb{R}el_{\mathcal{R}}$, which maps a morphism $f: r \rightarrow s$ to its graph $\langle \text{id}_r, f \rangle \subseteq r \times s$. Indeed, it is straightforward to check that any pair of the form $\langle \text{id}_r, f \rangle \dashv \langle f, \text{id}_s \rangle$ is an adjunction, and subsequently that the proposed map is functorial.

To show that it is full and faithful, we characterize the adjunctions $x \dashv x'$ in $\mathbb{R}el_{\mathcal{R}}$. Suppose we have $x \xrightarrow{\langle g, f \rangle} r \times s$ and $x' \xrightarrow{\langle f', g' \rangle} s \times r$ with unit $i: r \rightarrow (x \circledast x')$ and counit $j: (x' \circledast x) \rightarrow s$. This gives rise to the following diagram (equations shown right):

$$\begin{array}{ccccc}
 x \times_s x' & \xrightarrow{\pi'_s} & x' & & \\
 \downarrow \pi_s & \swarrow i & \leftarrow g' & & \uparrow \pi'_r \\
 & r & & & \\
 & \swarrow g & \leftarrow f' & & \\
 & s & & & \\
 x & \xrightarrow{f} & s & \xleftarrow{j} & x \times_r x' \\
 & \swarrow f & \leftarrow j & & \downarrow \pi_r
 \end{array}
 \quad
 \begin{array}{l}
 i \circledast \pi_s \circledast g = \text{id}_r = i \circledast \pi'_s \circledast g' \\
 \pi_r \circledast f = j = \pi'_r \circledast f'
 \end{array}$$

We shall show that g and g' are isomorphisms, and that $f' = g' \circledast g^{-1} \circledast f$.

We first show that $i \circledast \pi_s$ is inverse to g . Since the unit already gives that $i \circledast \pi_s \circledast g = \text{id}_r$, it suffices to show that $g \circledast i \circledast \pi_s = \text{id}_x$. Moreover, since $\langle g, f \rangle: x \rightarrow r \times s$ is monic and $g = (g \circledast i \circledast \pi_s) \circledast g$, it suffices to show that $f = (g \circledast i \circledast \pi_s) \circledast f$. This is a diagram chase: since $g = g \circledast i \circledast \pi'_s \circledast g'$, we can define a morphism $q := \langle \text{id}_x, g \circledast i \circledast \pi'_s \rangle: x \rightarrow x \times_r x'$, and we conclude

$$f = q \circledast \pi_r \circledast f = q \circledast \pi'_r \circledast f' = g \circledast i \circledast \pi'_s \circledast f' = g \circledast i \circledast \pi_s \circledast f.$$

Similarly, we see that $i \circledast \pi'_s$ is inverse to g' , and hence obtain $f' = g' \circledast g^{-1} \circledast f$.

Note that this implies the adjunction $x \dashv x'$ is isomorphic to the adjunction $\langle 1_r, (g^{-1} \circledast f) \rangle \dashv \langle (g^{-1} \circledast f), \text{id}_s \rangle$. Thus the proposed functor is full. Faithfulness amounts to the fact that the existence of a morphism $\langle 1_r, f \rangle \rightarrow \langle 1_r, f' \rangle$ implies $f = f'$. This proves the lemma. \square

B Proof that the subobjects functor is ajax

Proof of Theorem 19. The functor $\text{Sub}_{\mathcal{R}}(-) = \mathcal{R}(I, -)$ has a canonical lax monoidal structure since $I \otimes I \cong I$. We need to show the laxators \otimes and id_I have left adjoints in $\mathbb{P}oset$. The first is easy: id_I is the top element in $\mathcal{R}(I, I)$ and thus a right adjoint since there is a unique map $\mathcal{R}(I, I) \rightarrow 1$.

Now suppose given $r_1, r_2 \in \mathcal{R}$, and consider the morphisms $\pi_i: r_1 \otimes r_2 \rightarrow r_i$ and $\delta: r \rightarrow r \otimes r$ corresponding to the i th projection and the diagonal in \mathcal{R} . Composition with the π_i induces a monotone map $\lambda_{r_1, r_2}: \mathcal{R}(I, r_1 \otimes r_2) \rightarrow \mathcal{R}(I, r_1) \times \mathcal{R}(I, r_2)$, natural in r_1, r_2 . It remains to show that each λ_{r_1, r_2} is indeed a left adjoint,

$$\mathcal{R}(I, r_1 \otimes r_2) \xrightleftharpoons[\otimes]{\lambda_{r_1, r_2}} \mathcal{R}(I, r_1) \times \mathcal{R}(I, r_2) .$$

For the unit, given $g: I \rightarrow r_1 \otimes r_2$, we have

$$g = g \circledast \delta_{r_1 \otimes r_2} \circledast ((r_1 \otimes r_2) \otimes \epsilon_{r_1 \otimes r_2})$$

$$\begin{aligned}
 &= g \circ \delta_{r_1 \otimes r_2} \circ ((r_1 \otimes \varepsilon_{r_2}) \otimes (\varepsilon_{r_1} \otimes r_2)) \\
 &\leq \delta_I \circ (g \otimes g) \circ ((r_1 \otimes \varepsilon_{r_2}) \otimes (\varepsilon_{r_1} \otimes r_2)) \\
 &= (g \circ (r_1 \otimes \varepsilon_{r_2})) \otimes (g \circ (\varepsilon_{r_1} \otimes r_2)).
 \end{aligned}$$

For the counit, given $f_1: I \rightarrow r_1$ and $f_2: I \rightarrow r_2$, it suffices to show that $(f_1 \otimes f_2) \circ (r_1 \otimes \varepsilon_{r_2}) \leq f_1$, since the other projection is similar. And this holds because

$$(f_1 \otimes f_2) \circ (r_1 \otimes \varepsilon_{r_2}) = f_1 \otimes (f_2 \circ \varepsilon_{r_2}) \leq f_1 \otimes \varepsilon_I = f_1. \quad \square$$

C Proof that $\text{FRg}(\mathbb{T})$ is the free regular category on \mathbb{T}

We shall prove the theorem on the following page. To set up the proof, we first develop the notion of a unary support context.

Lemma 58. *Suppose \mathcal{C} , \mathcal{D} , and \mathcal{E} have I -shaped limits, for some small category I , and suppose that $f: \mathcal{C} \rightarrow \mathcal{E}$ and $g: \mathcal{D} \rightarrow \mathcal{E}$ preserve I -shaped limits. Then the comma category $\mathcal{B} := (\mathcal{C} \downarrow \mathcal{D})$ has I -shaped limits, and they are preserved and reflected by the projection $(\pi_1, \pi_2): \mathcal{B} \rightarrow \mathcal{C} \times \mathcal{D}$.*

Proposition 59. *Let $\mathcal{R} \rightarrow \mathcal{T} \leftarrow \mathcal{S}$ be regular functors. Then the comma category $\mathcal{B} := (\mathcal{R} \downarrow \mathcal{S})$ is regular, and the projection $\mathcal{B} \rightarrow \mathcal{R} \times \mathcal{S}$ preserves and reflects finite limits and regular epimorphisms. In particular, $\text{FRg}(\mathbb{T})$ is regular for any \mathbb{T} .*

Proof. It is well-known that $\text{FinSet}^{\text{op}}$ is regular, and the finite powerset $\mathbb{P}_f(\mathbb{T})$ is regular because it has finite meets and, because it is a poset, regular epis are equalities. Hence the second statement follows from the first. Since the opposite of a comma category is the comma category of the opposites of its defining data, Lemma 58 shows that \mathcal{B} has finite limits and coequalizers of kernel pairs, and that regular epis are stable under pullback. \square

Remark 60. As mentioned in Proposition 22, we denote the product of Γ_1 and Γ_2 by $\Gamma_1 \oplus \Gamma_2$. This is reminiscent of the notation for products in an abelian category. However, it is not quite analogous: in an abelian category the product $V \oplus W$ is a biproduct—i.e. also a coproduct—and this is not the case in $\text{FRg}(\mathbb{T})$. We use the \oplus notation to remind us that

$$(n, S, \tau) \oplus (n', S', \tau') \cong (n + n', S \cup S', [\tau, \tau']).$$

Remark 61. Note that one should think of the support $S = \text{Supp}(\Gamma)$ of a context Γ as a kind of constraint, because the larger S is, the smaller Γ is. Indeed, for any $n \in \mathbb{N}$ and context $\tau: \underline{n} \rightarrow S$, if one composes with an inclusion $S \subseteq S' \subseteq \mathbb{T}$ on the level of support, the result is a monic map in $\text{FRg}(\mathbb{T})$ going the other way,

$$(\underline{n} \xrightarrow{\tau} S \subseteq S') \mapsto (\underline{n} \xrightarrow{\tau} S).$$

Definition 62. Given a map $f: \Gamma \rightarrow \Gamma'$, we denote the corresponding function as $\underline{f}: \underline{n}' \rightarrow \underline{n}$. Say that a context $\Gamma = (n, S, \tau)$:

- is a *unary context* if it is of the form $(1, \{s\}, !)$, i.e. if it has arity $n = 1$ and full support $|S| = 1$; we denote it simply as $\langle s \rangle$.
- is a *unary support context* if it is of the form $(0, \{s\}, !)$; i.e. if it has $n = 0$ and $|S| = 1$; we abuse notation to denote this $\text{Supp}(s)$.

Example 63. Suppose $T = \{s\}$ is unary. When $n = 0$, the map τ is unique, and we either have $S = \emptyset$ or $S = \{s\}$. Thus we recover the description from Eq. (1), though in the present terms it looks like this:

$$(0, \emptyset) \longleftarrow (0, \{s\}) \longleftarrow (1, \{s\}) \xleftrightarrow{\quad} (2, \{s\}) \quad \dots$$

Example 64. For any set T , the poset of subobjects of 0 in $\text{FRg}(T)$ is the free meet-semilattice on T , i.e. the finite powerset $\mathbb{P}_f(T)$. This follows from Proposition 22.

Recall from Definition 4 that the support of an object in a regular category is the image of its unique map to the terminal object.

Corollary 65. *Every unary support context is the support of a unary context.*

Proof. Given any unary support context $\text{Supp}(s)$, the explicit descriptions in Proposition 22 make it easy to check that $\langle s \rangle \rightarrow \text{Supp}(s) \rightarrow 0$ is the image factorization of the unique map $\langle s \rangle \rightarrow 0$. \square

Corollary 66. *Every object $\Gamma = (n, S, \tau) \in \text{FRg}(T)$ can be written as the product of n -many unary contexts and $|S|$ -many unary support contexts, and morphisms in $\text{FRg}(T)$ correspond to projections and diagonals.*

Proof. It follows directly from Proposition 22 that $\Gamma = \prod_{i \in \underline{n}} \langle \tau(i) \rangle \times \prod_{s \in S} \text{Supp}(s)$. In particular, it will be useful to note the idempotence of support contexts:

$$\text{Supp}(s) \times \text{Supp}(s) = \text{Supp}(s). \quad (16)$$

If $f: \Gamma \rightarrow \Gamma'$ is a morphism as in Eq. (7), then the corresponding map

$$\begin{array}{c} \prod_{i \in \underline{n}} \langle \tau(i) \rangle \times \prod_{s \in S} \text{Supp}(s) \\ \downarrow f \\ \prod_{i' \in \underline{n}'} \langle \tau'(i') \rangle \times \prod_{s' \in S'} \text{Supp}(s') \end{array}$$

acts coordinatewise according to $f: \underline{n}' \rightarrow \underline{n}$ and $S' \subseteq S$. \square

Proof of Theorem 23. We denote the unit component for a set T by $\langle - \rangle: T \rightarrow \text{Ob FRg}(T)$; it is given by unary contexts, $\langle t \rangle = (1, \{t\}, !)$. We denote the counit component $\ulcorner - \urcorner: \text{FRg}(\text{Ob } \mathcal{R}) \rightarrow \mathcal{R}$ for a regular category \mathcal{R} ; it is roughly-speaking given by products and supports in \mathcal{R} (see Definition 4). More precisely, given a context $\Gamma = (n, S, \tau) \in \text{FRg}(\text{Ob } \mathcal{R})$, we put

$$\ulcorner \Gamma \urcorner := \prod_{i \in \underline{n}} \tau(i) \times \prod_{s \in S} \text{Supp}(s),$$

By the universal property of products, a morphism $f: \Gamma \rightarrow \Gamma'$, i.e. a function $f: \underline{n}' \rightarrow \underline{n}$ as in Eq. (7) naturally induces a map $\ulcorner f \urcorner: \ulcorner \Gamma \urcorner \rightarrow \ulcorner \Gamma' \urcorner$, so $\ulcorner - \urcorner$ is a functor. We need to check that it is regular and for this we use Proposition 22.

For preservation of finite limits, first observe that $\ulcorner - \urcorner$ preserves the terminal object because the empty product in \mathcal{R} is terminal. For pullbacks we need to check that for every pushout diagram as to the left, the diagram to the right is a pullback:

$$\begin{array}{ccc} \underline{n} & \longrightarrow & \underline{n}_2 \\ \downarrow & \ulcorner & \downarrow \\ \underline{n}_1 & \longrightarrow & \underline{n}' \xrightarrow{\tau'} T \end{array} \qquad \begin{array}{ccc} \prod_{i \in \underline{n}} \langle \tau'(i) \rangle & \longleftarrow & \prod_{i_2 \in \underline{n}_2} \langle \tau'(i_2) \rangle \\ \uparrow & \ulcorner & \uparrow \\ \prod_{i_1 \in \underline{n}_1} \langle \tau'(i_1) \rangle & \longleftarrow & \prod_{i' \in \underline{n}'} \langle \tau'(i') \rangle \end{array}$$

where $\underline{n}' \cong \underline{n}_1 \sqcup_{\underline{n}} \underline{n}_2$ and $\tau' : \underline{n}' \rightarrow T$ is the induced map. This follows from the well-known fact that \mathbf{FinSet} is the free finite-colimit completion of a singleton [17], and fact that the slice category \mathbf{FinSet}/T is the free finite-colimit completion of the set T .

Finally, suppose $f : (n_1, S_1, \tau_1) \rightarrow (n_2, S_2, \tau_2)$ is a regular epi in $\mathbf{FRg}(T)$; i.e. the corresponding function $\underline{f} : \underline{n}_2 \rightarrow \underline{n}_1$ is monic and $S_1 = S_2$. Letting $\underline{n}' := n_1 - n_2$, we use Corollary 66 and Eq. (16) to write $\ulcorner f \urcorner$ as follows:

$$\begin{array}{c} \prod_{i \in \underline{n}'} \langle \tau_1(i) \rangle \times \prod_{i \in \underline{n}_2} \langle \tau_2(i) \rangle \times \prod_{s \in S} \text{Supp}(s) \\ \downarrow \ulcorner f \urcorner \\ \prod_{i \in \underline{n}'} \text{Supp}(\tau_1(i)) \times \prod_{i \in \underline{n}_2} \langle \tau_2(i) \rangle \times \prod_{s \in S} \text{Supp}(s). \end{array}$$

Since for each $i \in \underline{n}'$ the map $\tau_1(i) \rightarrow \text{Supp}(\tau_1(i))$ is a regular epi and regular epis are closed under finite products in a regular category, this shows that $\ulcorner f \urcorner$ is again a regular epi. Hence $\ulcorner - \urcorner$ is a regular functor. The triangle identities are straightforward: the first is that for any $r \in \text{Ob } \mathcal{R}$, the product of a unary context $\langle r \rangle$ is just r . The second follows from Corollary 66. \square

Remark 67. Given a function $f : T \rightarrow T'$, we can use Theorem 23 and the idempotence of support contexts to see that the induced regular functor $\mathbf{FRg}(f) : \mathbf{FRg}(T) \rightarrow \mathbf{FRg}(T')$ sends $\Gamma = (\underline{n} \xrightarrow{\tau} S \subseteq T)$ to the composite

$$\mathbf{FRg}(f)(\Gamma) = (\underline{n} \xrightarrow{\tau} S \xrightarrow{f|_S} f(S) \subseteq T'),$$

where $S \xrightarrow{f} f(S) \hookrightarrow T'$ is the image factorization of f restricted to S .

D Proofs regarding the po-category of internal relations

The main point of this appendix is to prove the po-category of internal relations is well-defined. We tackle this in stages, culminating in the proof of Theorem 52 on 222.

Lemma 68. *The composite of internal relations is an internal relation. That is, let $\varphi_1 \in P(\Gamma_1)$, $\varphi_2 \in P(\Gamma_2)$, and $\varphi_3 \in P(\Gamma_3)$. Then given $\theta_{12} \in \mathbb{IntRel}_P(\varphi_1, \varphi_2)$ and $\theta_{23} \in \mathbb{IntRel}_P(\varphi_2, \varphi_3)$, the element $(\theta_{12} \circ \theta_{23}) \in P(\Gamma_1 \oplus \Gamma_3)$ is in $\mathbb{IntRel}_P(\varphi_1, \varphi_3)$.*

Proof. We must prove $(\pi_1)_!(\theta_{12} \circ \theta_{23}) \vdash \varphi_1$ and $(\pi_2)_!(\theta_{12} \circ \theta_{23}) \vdash \varphi_3$. We prove the first; the second follows similarly. This is not hard, we simply use Example 49 and then that θ_{12} obeys Definition 50:

$$\text{---} \textcircled{\theta_{12}} \textcircled{\theta_{23}} \text{---} \vdash_{\Gamma_1} \textcircled{\theta_{12}} \text{---} \vdash_{\Gamma_1} \textcircled{\varphi_1} \quad \square$$

Given an object $\Gamma \in \mathbf{FRg}(T)$ and $\varphi \in P(\Gamma)$, define $\text{id}_\varphi := (\delta_\Gamma)_!(\varphi)$ in $P(\Gamma \oplus \Gamma)$. Here it is graphically.

$$\text{id}_\varphi := \Gamma \textcircled{\varphi} \Gamma \quad (17)$$

Lemma 69. *For any $\Gamma \in \mathbf{FRg}(T)$ and $\varphi \in P(\Gamma)$, the element $\text{id}_\varphi \in P(\Gamma \oplus \Gamma)$ is an element of $\mathbb{IntRel}_P(\varphi, \varphi)$.*

Proof. By Proposition 45(iii), composing the nested graphical term on the left is precisely the graphical term on the right (and similarly for the codomain):

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \vdash_{\Gamma} \text{---} \quad \square$$

In what follows, we often elide details about—and graphical notation that indicates—nesting and contexts.

Lemma 70. The map \circledast from Eq. (12) is unital with respect to id , i.e. $\theta \circledast \text{id} = \theta = \text{id} \circledast \theta$.

Proof. We prove that $(\theta \circledast \text{id}) = \theta$; the other unitality axiom is similar. The inequality $(\theta \circledast \text{id}) \vdash \theta$ follows from Example 49 and Proposition 45:

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \vdash \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array}$$

The reverse inequality $\theta \vdash (\theta \circledast \text{id})$ uses Proposition 48, Example 38, and Definition 50:

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \vdash \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \vdash \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad \square$$

Lemma 71. The map \circledast from Eq. (12) is associative, i.e. $(\theta_1 \circledast \theta_2) \circledast \theta_3 = \theta_1 \circledast (\theta_2 \circledast \theta_3)$.

Proof. This is immediate from Proposition 45(iii). Both sides can be represented by (nested versions of) the graphical term $\text{---} \theta_1 \theta_2 \theta_3 \text{---}$. \square

Proof of Theorem 52. Lemmas 70 and 71 show that we have a 1-category. Each homset $\text{IntRel}_P(\varphi_1, \varphi_2) \subseteq P(\Gamma_1, \Gamma_2)$ inherits a partial order from the poset $P(\Gamma_1, \Gamma_2)$. Moreover, composition is given by the monotone map

$$\text{IntRel}_P(\varphi_1, \varphi_2) \times \text{IntRel}_P(\varphi_2, \varphi_3) \xrightarrow{\rho} \text{IntRel}_P(\varphi_1, \varphi_2, \varphi_3) \xrightarrow{P(\text{comp})} \text{IntRel}_P(\varphi_1, \varphi_3).$$

We thus have a po-category. \square

Proposition 72. Let $\theta \in P(\Gamma_1 \oplus \Gamma_2)$, $\varphi_i \in P(\Gamma_i)$. Then θ is a relation $\varphi_1 \rightarrow \varphi_2$ if and only if

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad (18)$$

Proof. Any internal relation $\varphi_1 \rightarrow \varphi_2$ obeys the identity Eq. (18) by unitality, Lemma 70. Conversely, if θ obeys Eq. (18), then by Example 49

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \vdash \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array}$$

and similarly for φ_2 , proving that $\theta \in \text{IntRel}_P(\varphi_1, \varphi_2)$. \square

Definition 73. Write $\sigma_{\Gamma_1, \Gamma_2}: \Gamma_1 \oplus \Gamma_2 \rightarrow \Gamma_2 \oplus \Gamma_1$ for the braiding in $\text{FRg}(\mathbb{T})$, and define the map $(-)^{\dagger} := \sigma_{\Gamma_1, \Gamma_2}: P(\Gamma_1 \oplus \Gamma_2) \rightarrow P(\Gamma_2 \oplus \Gamma_1)$. We say that the *transpose* of a graphical term $(\theta; \Gamma_1 \oplus \Gamma_2)$ is the graphical term $(\theta^{\dagger}; \Gamma_2 \oplus \Gamma_1)$.

Remark 74. Note that transposes are given by “rotating the shell”:

$$\Gamma_2 \text{---} \theta^\dagger \text{---} \Gamma_1 = \text{---} \theta \text{---} \Gamma_1$$

In particular, for $\varphi \in P(\Gamma)$, we have $\llbracket (\varphi^\dagger; \Gamma) \rrbracket = \llbracket (\varphi; \Gamma) \rrbracket$. That is, both φ and φ^\dagger can be represented by the diagram $\text{---} \varphi \text{---}$.

E Proof that the category of internal functions is regular

E.1 Properties and examples of internal functions

Before we embark on the theorem, let’s get to know the category of internal functions a bit. We’ll first characterize functions in two ways: they’re the relations that have their own transposes as right adjoints, and they’re the relations that are total and deterministic. We’ll then note that the order inherited by functions as a subset of the poset of relations is just the discrete order, and give two important examples of functions: bijections and projections.

Note that we’ll sometimes use the shape $\Gamma_1 \text{---} \theta \text{---} \Gamma_2$ to denote an internal function $\theta \in P(\Gamma_1, \Gamma_2)$.

To obtain our characterizations of functions, we’ll need definitions of deterministic and total.

Definition 75. Let $\theta \in \text{IntRel}_P(\varphi_1, \varphi_2)$. We say that θ is

- *total* if $\text{---} \varphi_1 \text{---} \vdash \text{---} \theta \text{---}$, and
- *deterministic* if $\text{---} \theta \text{---} \vdash \text{---} \theta \text{---}$.

Remark 76. Note that by the domain of θ and discarding (Example 49) we always have

$$\text{---} \theta \text{---} = \text{---} \varphi_1 \text{---} \text{---} \theta \text{---} \vdash \text{---} \varphi_1 \text{---}$$

and that by meets (Proposition 48(ii)) and breaking (Proposition 45(ii)) we always have

$$\text{---} \theta \text{---} \vdash \text{---} \theta \text{---} \text{---} \theta \text{---} \vdash \text{---} \theta \text{---}$$

This means that in Definition 75 the two entailments are in fact equalities.

In what follows, we’ll often omit the transpose symbol \dagger (see Definition 73) from our diagrams when it can be deduced from the ambient contextual information.

Theorem 77. Let $\theta \in \text{IntRel}_P(\varphi_1, \varphi_2)$. Then the following are equivalent.

- $\theta \in \mathcal{R}_P$ is an internal function in the sense of Definition 53.
- θ has right adjoint θ^\dagger . That is, $\text{---} \varphi_1 \text{---} \vdash \text{---} \theta \text{---} \text{---} \theta \text{---}$ and $\text{---} \theta \text{---} \text{---} \theta \text{---} \vdash \text{---} \varphi_2 \text{---}$.
- θ is total and deterministic in the sense of Definition 75.

Proof. (i) \Leftrightarrow (ii): Clearly (ii) \Rightarrow (i). Conversely, assume θ has a right adjoint ξ . Note that the unit axiom implies $\varphi_1 \text{---} \vdash \text{---} \theta \text{---} \text{---} \xi \text{---} \vdash \text{---} \theta \text{---}$. Then using meets and breaking we have

$$\Gamma_2 \text{---} \xi \text{---} \Gamma_1 = \text{---} \xi \text{---} \text{---} \theta \text{---} = \text{---} \xi \text{---} \text{---} \theta \text{---} \text{---} \theta \text{---} \vdash \text{---} \xi \text{---} \text{---} \theta \text{---} \text{---} \theta \text{---} \vdash \Gamma_2 \text{---} \theta^\dagger \text{---} \Gamma_1$$

Similarly we can show $\theta \vdash \xi^\dagger$, and hence $\xi = \theta^\dagger$.

(ii) \Leftrightarrow (iii): We shall prove a stronger statement, that θ has a unit if and only if it is total, and that it has a counit if and only if it is deterministic.

First, (ii)-units iff (iii)-totalness. Using the unit of the adjunction we have

$$\Gamma_1 \dashv \varphi_1 \vdash \begin{array}{c} \theta \\ \theta \end{array} = \dashv \theta$$

Conversely, using totalness, meets, and breaking we have

$$\begin{array}{c} \varphi_1 \\ \vdash \end{array} = \begin{array}{c} \theta \\ \vdash \end{array} = \begin{array}{c} \theta \\ \theta \end{array} \vdash \dashv \theta \dashv \theta \dashv \Gamma_2$$

Next, (ii)-counits iff (iii)-determinism. We can use the counit of the adjunction to give

$$\begin{array}{c} \theta \\ \theta \end{array} = \begin{array}{c} \theta \\ \theta \\ \theta \\ \theta \end{array} \vdash \begin{array}{c} \theta \\ \theta \\ \theta \\ \theta \end{array} \vdash \begin{array}{c} \theta \\ \theta \end{array} \dashv \theta \dashv \theta = \dashv \theta \dashv \theta$$

Conversely, assuming determinism we get the counit, which concludes the proof:

$$\dashv \theta \dashv \theta \dashv \Gamma_1 = \begin{array}{c} \theta \\ \vdash \end{array} \vdash \begin{array}{c} \varphi_2 \\ \vdash \end{array}$$

□

Next, we describe how the order on relations restricts to the functions.

Proposition 78. *The order on functions is discrete.*

Proof. Suppose $\dashv \theta \vdash \dashv \theta'$. Then using the unit of θ and counit of θ' we have

$$\dashv \theta' \vdash \dashv \theta \dashv \theta \dashv \theta' \vdash \dashv \theta \dashv \theta' \dashv \theta' \vdash \dashv \theta$$

□

Finally, we note that bijections and projections are examples of functions.

Example 79. A P -internal bijection is an invertible P -internal relation. Note that every bijection is a function. We can also characterise bijections as the adjunctions whose unit and counit are the identity.

Proposition 80. *Suppose given $\varphi_1 \in P(\Gamma_1)$ and $\varphi_2 \in P(\Gamma_2)$ and a relation $\theta \in \mathbb{I}ntRel_P(\varphi_1, \varphi_2) \subseteq P(\Gamma_1 \oplus \Gamma_2)$. Define*

$$\pi_1 := (\delta_{\Gamma_1} \oplus \Gamma_2)_!(\theta) \quad \text{and} \quad \pi_2 := (\Gamma_2 \oplus \delta_{\Gamma_2})_!(\theta).$$

Then $\pi_i \in P(\Gamma_1 \oplus \Gamma_i \oplus \Gamma_2)$ are internal functions for $i = 1, 2$, i.e. $\pi_i \in \mathcal{R}_P(\theta, \varphi_i)$

Proof. We prove π_1 is a function; the argument for π_2 is similar. Note that π_1 is depicted by the graphical term

$$\begin{array}{c} \theta \\ \Gamma_2 \\ \Gamma_1 \dashv \end{array} \dashv \Gamma_1$$

By Proposition 48 and the fact that $\theta \in \mathbb{I}ntRel_P(\varphi_1, \varphi_2)$ we have

$$\begin{array}{c} \theta \\ \Gamma_2 \\ \Gamma_1 \dashv \end{array} \dashv \Gamma_1 = \begin{array}{c} \theta \\ \theta \\ \varphi_1 \end{array}$$

and hence by Proposition 72, $\pi_1 \in \mathbb{I}ntRel_P(\theta, \varphi_1)$.

Proving that π_1 is an adjunction in $\mathbb{I}ntRel_P(\theta, \varphi_1)$ again uses Proposition 48 and that $\theta \in \mathbb{I}ntRel_P(\varphi_1, \varphi_2)$, as well as Example 38:

$$\begin{array}{c} \theta \\ \Gamma_1 \text{---} \text{---} \Gamma_2 \\ \Gamma_1 \end{array} = \begin{array}{c} \theta \quad \theta \\ \text{---} \text{---} \\ \Gamma_1 \end{array} \vdash \begin{array}{c} \theta \quad \theta \\ \text{---} \text{---} \\ \Gamma_1 \end{array} \quad \text{and} \quad \begin{array}{c} \theta \quad \theta \\ \Gamma_1 \text{---} \text{---} \Gamma_1 \end{array} = \begin{array}{c} \theta \\ \text{---} \\ \Gamma_1 \end{array} \vdash \begin{array}{c} \varphi_1 \\ \text{---} \\ \Gamma_1 \end{array} \quad \square$$

Definition 81. With $\varphi_1, \varphi_2, \theta$ as in Proposition 80, we refer to the map $(\delta_{\Gamma_1 \oplus \Gamma_2})_! \in \mathcal{R}_P(\theta, \varphi_1)$ as the *left projection* and similarly to $(\Gamma_1 \oplus \delta_{\Gamma_2})_! \in \mathcal{R}_P(\theta, \varphi_2)$ as the *right projection*.

E.2 Finite limits in \mathcal{R}_P

We now show how to construct finite limits in the category \mathcal{R}_P of internal functions in P .

Lemma 82 (Terminal object). *The object $(0, \text{true}) \in \mathcal{R}_P$ is terminal.*

Proof. For any context Γ and element $\varphi \in P(\Gamma)$ we shall show $\varphi \in \mathcal{R}_P((\Gamma, \varphi), (0, \text{true})) \subseteq P(\Gamma \oplus 0)$ is the unique element. Note first that φ is indeed an internal function: it's an internal relation because $\varphi \vdash \varphi$ and $\pi_{2!}(\varphi) \vdash \text{true}$, and is an adjunction with counit given by the fact that true is the top element, and unit given by meets and breaking as follows

$$\begin{array}{c} \varphi \\ \text{---} \\ \Gamma \end{array} = \begin{array}{c} \varphi \\ \text{---} \\ \varphi \end{array} \vdash \begin{array}{c} \varphi \\ \text{---} \\ \varphi \end{array} = \begin{array}{c} \varphi \quad \varphi \\ \text{---} \end{array}$$

It remains to show uniqueness. If θ is an internal function then $\theta \vdash \varphi$, so it remains to show that $\varphi \vdash \theta$. But it is easy to verify: $\begin{array}{c} \varphi \\ \text{---} \end{array} \vdash \begin{array}{c} \theta \\ \text{---} \end{array} \begin{array}{c} \theta \\ \text{---} \end{array} \vdash \begin{array}{c} \theta \\ \text{---} \end{array}$. □

Lemma 83 (Pullbacks). *Let $\theta_1: (\Gamma_1, \varphi_1) \rightarrow (\Gamma, \varphi)$ and $\theta_2: (\Gamma_2, \varphi_2) \rightarrow (\Gamma, \varphi)$ be morphisms in \mathcal{R}_P . Let $\theta_{12} := (\theta_1 \circledast \theta_2)^\dagger$. Then the following is a pullback square in \mathcal{R}_P :*

$$\begin{array}{ccc} ((\Gamma_1 \oplus \Gamma_2), \theta_{12}) & \xrightarrow{(\Gamma_1 \oplus \delta_{\Gamma_2})_!(\theta_{12})} & (\Gamma_2, \varphi_2) \\ (\delta_{\Gamma_1 \oplus \Gamma_2})_!(\theta_{12}) \downarrow & & \downarrow \theta_2 \\ (\Gamma_1, \varphi_1) & \xrightarrow{\theta_1} & (\Gamma, \varphi) \end{array}$$

Proof. The graphical term for the proposed pullback $((\Gamma_1 \oplus \Gamma_2), \theta_{12})$ is shown left, and its proposed projection maps are shown middle and right:

$$\begin{array}{c} \theta_{12} \\ \text{---} \\ \Gamma_1 \oplus \Gamma_2 \end{array} := \begin{array}{c} \Gamma \\ \text{---} \\ \theta_1 \quad \theta_2 \end{array} \quad \begin{array}{c} \Gamma_1 \\ \text{---} \\ \theta_1 \end{array} \quad \begin{array}{c} \Gamma_2 \\ \text{---} \\ \theta_2 \end{array} \quad \begin{array}{c} \Gamma_1 \quad \Gamma_2 \\ \text{---} \\ \theta_1 \quad \theta_2 \end{array}$$

Both projections are internal functions by Proposition 80. The necessary diagram commutes, i.e. we have equalities

$$\begin{array}{c} \theta_1 \quad \theta_2 \\ \text{---} \\ \theta_1 \end{array} = \begin{array}{c} \Gamma \\ \text{---} \\ \theta_1 \quad \theta_2 \end{array} = \begin{array}{c} \theta_1 \quad \theta_2 \\ \text{---} \\ \theta_2 \end{array} \quad (19)$$

because functions are deterministic (Theorem 77).

Now we come to the universal property. Suppose given an object (Γ', φ') and morphisms $\theta'_1 : (\Gamma', \varphi') \rightarrow (\Gamma_1, \varphi_1)$ and $\theta'_2 : (\Gamma', \varphi') \rightarrow (\Gamma_2, \varphi_2)$ in \mathcal{R}_P , such that the $\theta'_1 \circ \theta_1 = \theta'_2 \circ \theta_2$. Let $\langle \theta'_1, \theta'_2 \rangle$ denote the following graphical term:

$$\text{---} \begin{array}{c} \theta'_1 \\ \Gamma \\ \theta'_2 \end{array} \text{---} \quad (20)$$

We give one half of the proof that $\langle \theta'_1, \theta'_2 \rangle \in \text{IntRel}_P(\varphi', \theta_{12})$, the other half being easier.

$$\text{---} \begin{array}{c} \theta'_1 \\ \theta'_2 \end{array} \text{---} \vdash \text{---} \begin{array}{c} \theta_1 \\ \theta_1 \\ \theta'_1 \\ \theta'_2 \end{array} \text{---} = \text{---} \begin{array}{c} \theta_1 \\ \theta_2 \\ \theta'_2 \\ \theta'_2 \end{array} \text{---} \vdash \text{---} \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \text{---}$$

Moreover, applying Theorem 77, a similarly straightforward diagrammatic argument shows $\langle \theta'_1, \theta'_2 \rangle \in \mathcal{R}_P(\varphi', \theta_{12})$. We next need to show that $\langle \theta'_1, \theta'_2 \rangle \circ (\delta_{\Gamma_1} \oplus \Gamma_2)_1(\theta_{12}) = \theta'_1$ and similarly for θ'_2 . This follows easily from Proposition 78 and the diagram

$$\begin{array}{c} \theta'_1 \\ \theta_1 \end{array} \begin{array}{c} \theta'_2 \\ \theta_2 \end{array} \vdash \text{---} \begin{array}{c} \theta'_1 \end{array} \text{---}$$

It only remains to show that this is unique. So suppose given $\theta' \in \mathcal{R}_P(\varphi', \theta_{12})$ with $\Gamma' \text{---} \begin{array}{c} \theta' \end{array} \Gamma_1 = \Gamma' \text{---} \begin{array}{c} \theta'_1 \end{array} \Gamma_1$

and $\Gamma' \text{---} \begin{array}{c} \theta' \end{array} \Gamma_2 = \Gamma' \text{---} \begin{array}{c} \theta'_2 \end{array} \Gamma_2$. Then by basic diagram manipulations, one shows that θ' must equal the graphical term in Eq. (20), as desired. \square

Proposition 84. *Suppose that $\text{---} \begin{array}{c} \theta \end{array} \text{---} \in \mathcal{R}_P(\varphi_1, \varphi_2)$ is an internal function. It is a monomorphism iff it satisfies $\begin{array}{c} \varphi_1 \\ \downarrow \end{array} = \text{---} \begin{array}{c} \theta \\ \theta \end{array} \text{---}$.*

Proof. Recall that a morphism is a monomorphism iff the projection maps of its pullbacks along itself are the identity maps. Using the characterization of the projection maps of the pullback of θ along itself (Lemma 83) and the graphical logic, the proposition is immediate. \square

Corollary 85 (Monomorphisms). *If $\varphi \vdash_{\Gamma} \varphi'$, then $\text{id}_{\varphi} \in P(\Gamma \oplus \Gamma)$ as in Eq. (17) is an element of $\mathcal{R}_P((\Gamma, \varphi), (\Gamma, \varphi'))$ and it is a monomorphism.*

Proof. Since meets merge circles, we have the equality

$$\begin{array}{c} \varphi \\ \downarrow \end{array} = \begin{array}{c} \varphi \\ \downarrow \end{array} \quad (21)$$


and it follows easily that $\text{id}_{\varphi} \in \mathcal{R}_P(\varphi, \varphi')$. But this also proves that id_{φ} is a monomorphism, by Proposition 84. \square

Remark 86 (Equalizers). Given parallel arrows $\theta, \theta' : (\Gamma_1, \varphi_1) \rightarrow (\Gamma_2, \varphi_2)$, their equalizing object (Γ_1, e) is the following graphical term:

$$\Gamma_1 \text{---} \begin{array}{c} e \end{array} \text{---} = \begin{array}{c} \theta \\ \theta' \end{array}$$

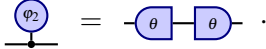
E.3 Image factorizations

We next discuss image factorizations, and show that they are stable under pullback.

Definition 87. Suppose that $\theta \in \mathcal{R}_P(\varphi_1, \varphi_2)$ is an internal function. Define its *image*, denoted $\text{im}(\theta) \in P(\Gamma_2)$ to be the graphical term  or, in symbols, $\varepsilon_{\Gamma_1}^* \circ \theta$.

We will now show that this has the usual properties of images, for example that θ is a regular epimorphism in \mathcal{R}_P iff it satisfies $\varphi_2 \vdash \text{im}(\theta)$.

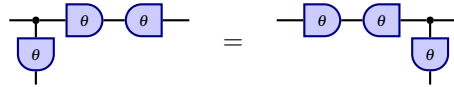
Proposition 88. Consider an element $\theta \in \mathcal{R}_P(\varphi_1, \varphi_2)$. The following are equivalent:

1. θ , considered as a morphism in \mathcal{R}_P , is a regular epimorphism,
2. $\varphi_2 \vdash_{\Gamma_2} \text{im}(\theta)$,
3. $\varphi_2 = \text{im}(\theta)$, and
4.  .

Proof. (1 \Rightarrow 2): It is straightforward to show that $\theta \in P(\Gamma_1, \Gamma_2)$ is an element of $\mathcal{R}_P(\varphi_1, \text{im}(\theta))$. Now supposing that θ is a regular epi, i.e. that the kernel pair diagram

$$\varphi_1 \times_{\varphi_2} \varphi_1 \rightrightarrows \varphi_1 \longrightarrow \varphi_2$$

is a coequalizer, it suffices to show that $\text{im}(\theta)$ also coequalizes the parallel pair:

$$\text{im}(\theta) \circ \theta = \theta \circ \text{im}(\theta) \tag{22}$$


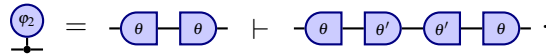
This follows directly from determinism.

(2 \Rightarrow 3): For any relation $\theta \in \text{IntRel}_P(\varphi_1, \varphi_2)$ we always have the converse $\text{im}(\theta) \vdash \varphi_2$.

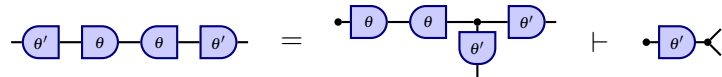
(3 \Rightarrow 4): By determinism of θ , we have $\varphi_2 \llcorner = \theta \llcorner = \left(\begin{smallmatrix} \theta \\ \theta \end{smallmatrix} \right) \llcorner$

(4 \Rightarrow 1): Assuming 4, we need to show that $\varphi_1 \times_{\varphi_2} \varphi_1 \rightrightarrows \varphi_1 \rightarrow \varphi_2$ is a coequalizer. It is easy to show that φ_2 coequalizes the parallel pair; this is basically Eq. (22) again. So let $\theta' : \varphi_1 \rightarrow \varphi'_2$ coequalize the parallel pair, and define $\xi \in \text{IntRel}_P(\varphi_2, \varphi'_2)$ by $\xi := \theta^\dagger \circ \theta'$. We need to show that ξ is a function and that $\theta \circ \xi = \theta'$.

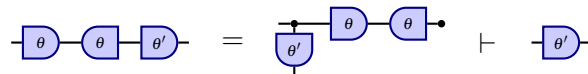
We obtain $\text{id}_{\varphi_2} \vdash \xi \circ \xi^\dagger$ using (4) and the fact that θ' is a function:

$$\text{id}_{\varphi_2} = \theta \circ \theta \vdash \theta \circ \theta' \circ \theta' \circ \theta \tag{23}$$


We obtain $\xi^\dagger \circ \xi \vdash \text{id}_{\varphi'_2}$ as follows:

$$\theta' \circ \theta \circ \theta \circ \theta' = \theta \circ \theta \circ \theta' \circ \theta' \circ \theta \llcorner \vdash \theta' \llcorner$$


where the first equality comes from the fact that θ' coequalizes the parallel pair, and the second is discarding and determinism of θ' . Finally, $\theta' \vdash \theta \circ \theta^\dagger \circ \theta = \theta \circ \xi$ follows easily from θ being a function. The converse $\theta \circ \xi \vdash \theta'$ follows from the fact that θ' coequalizes the parallel pair:

$$\theta \circ \theta \circ \theta' = \theta' \circ \theta \circ \theta \vdash \theta'$$


□

Lemma 89 (Image factorizations). *Any morphism $\theta: (\Gamma', \varphi') \rightarrow (\Gamma, \varphi)$ can be factored into a regular epimorphism followed by a monomorphism; the image object is $(\Gamma, \varepsilon_{\Gamma'}^* \circ \theta)$.*

Proof. The image factorization of θ is given by

$$\Gamma' \dashv \theta \dashv \Gamma = \text{Diagram showing the image factorization of } \theta \text{ as a regular epimorphism followed by a monomorphism.}$$

The graphical representation of the image object $(\Gamma, \varepsilon_{\Gamma'}^* \circ \theta)$ is $\dashv \theta \dashv$. It is immediate from Proposition 88 that θ is a regular epimorphism $(\Gamma', \varphi') \rightarrow (\Gamma, \varepsilon_{\Gamma'}^* \circ \theta)$, and from Corollary 85 that $(\delta_{\Gamma})_!(\varepsilon_{\Gamma'}^* \circ \theta)$ is a monomorphism $(\Gamma, \varepsilon_{\Gamma'}^* \circ \theta) \rightarrow (\Gamma, \varphi)$. \square

Lemma 90 (Pullback stability of image factorizations). *The pullback of a regular epimorphism along any morphism is again a regular epimorphism in \mathcal{R}_P .*

Proof. Suppose that $\xi: \varphi_1 \rightarrow \varphi$ is a regular epimorphism and that $\theta: \varphi_2 \rightarrow \varphi$ is any morphism. Then the pullback $\theta \times_{\varphi} \xi \rightarrow \varphi_2$ is a regular epimorphism by Proposition 88 and the following reasoning:

$$\text{Diagram showing the pullback of a regular epimorphism along a morphism.}$$

It is now straightforward to observe that \mathcal{R}_P is a regular category.

Proof of Theorem 54. By Lemmas 82 and 83, \mathcal{R}_P has all finite limits, and by Lemmas 89 and 90, it has pullback-stable image factorizations. \square

E.4 Subobject lattices in \mathcal{R}_P

To round out the picture, we also state the following two results.

Proposition 91. *Let (\mathbb{T}, P) be a regular calculus, let $\Gamma \in \text{FRg}(\mathbb{T})$ be a context, and let $s \in P(\Gamma)$. There is an isomorphism of posets*

$$\{t \in P(\Gamma) \mid t \leq s\} \cong \text{Sub}_{\mathcal{R}_P}(\Gamma, s),$$

with each element $t \leq s$ mapped to the subobject $P(\delta_t)(t) = \dashv t \dashv: (\Gamma, t) \rightarrow (\Gamma, s)$.

Proof. The proposed map indeed sends each t to a subobject by the characterization of monomorphisms in Corollary 85. To see that it is surjective, note that given a monomorphism $\theta: (\Gamma', s') \rightarrow (\Gamma, s)$ in \mathcal{R}_P , Lemma 89 (characterizing image factorizations) shows that it is isomorphic to the monomorphism

$$\dashv \theta \dashv: (\Gamma, \dashv \theta \dashv) \rightarrow (\Gamma, s)$$

where $\dashv \theta \dashv = P(\varepsilon_{\Gamma} \oplus \Gamma)(\theta)$.

To see that it is injective, suppose we have a map θ of monomorphisms

$$\begin{array}{ccc} (\Gamma, t') & \xrightarrow{P(\delta_t)(t')} & (\Gamma, s) \\ \theta \downarrow & & \nearrow \\ (\Gamma, t) & \xrightarrow{P(\delta_t)(t)} & (\Gamma, s) \end{array}$$

Note that this implies that

$$\text{im } \theta \wedge t = \text{---} \square_{\theta} \text{---} \circlearrowleft_t \text{---} = \text{---} \circlearrowleft_{t'} \text{---} = t'$$

and hence that $t' \leq t \in P(\Gamma)$. Thus the subobjects (Γ, t) and (Γ, t') of (Γ, s) are isomorphic if and only if $t = t'$. This proves the proposition. \square

Corollary 92. *Let (\mathbb{T}, P) be a regular calculus. Then $\mathbb{I}ntRel_P$ is isomorphic to the po-category of relations in \mathcal{R}_P . In particular, $\mathbb{I}ntRel_P$ is a regular po-category.*

Proof. Observe that $\mathbb{I}ntRel_P$ and \mathcal{R}_P have the same set of objects by definition, and that by Proposition 91 for any two objects $(\Gamma, s), (\Gamma', s')$ the poset of relations $(\Gamma, s) \rightarrow (\Gamma', s')$ in \mathcal{R}_P is given by $\{\theta \in P(\Gamma \oplus \Gamma') \mid \theta \leq s \boxplus s'\}$. It remains to prove that the composition rule in $\mathbb{I}ntRel_P$ agrees with composition of relations in \mathcal{R}_P . Reasoning using graphical terms, this is a straightforward consequence of Lemma 83, which describes pullbacks in the category \mathcal{R}_P . \square