Computing by Temporal Order: Asynchronous Cellular Automata

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Our concern is the behaviour of the elementary cellular automata with state set $\{0,1\}$ over the cell set $\mathbb{Z}/n\mathbb{Z}$ (one-dimensional finite wrap-around case), under *all* possible temporal rules (asynchronicity).

Over the torus $\mathbb{Z}/n\mathbb{Z}$ ($n \leq 10$),we will see that the ECA with Wolfram update rule 57 maps any $v \in \mathbb{F}_2^n$ to any $w \in \mathbb{F}_2^n$, varying the temporal rule.

We furthermore show that all even (element of the alternating group) bijective functions on the set $\mathbb{F}_2^n \cong \{0,\dots,2^n-1\}$, can be computed by ECA-57, by iterating it a sufficient number of times with varying temporal rules, at least for $n \leq 10$. We characterize the non-bijective functions computable by asynchronous rules.

The thread of all this is a novel paradigm:

The algorithm is neither hard-wired (in the ECA), nor in the program or data (initial configuration), but in the temporal order of updating cells, and temporal order is pattern-universal.

Keywords: Cellular automata, asynchronous, update rule, universality.

1 Introduction and Notation, Asynchronicity

We consider elementary cellular automata, *i.e.* with state set $S = \mathbb{F}_2 = \{0,1\}$ and update neighborhood (c_{i-1}, c_i, c_{i+1}) for cell c_i.

The cell index (site) i will come from $\mathbb{Z}/n\mathbb{Z}$ for some $n \geq 3$, *i.e.* we consider the finite one-dimensional torus, indices wrap around. In Section 2, we consider patterns "How universal can a mapping on $\mathbb{F}_2^n$ become?", and Section 3 covers functions $\mathbb{F}_2^n \ni v \to w \in \mathbb{F}_2^n$.

The 256 ECA's group into 88 classes under the symmetries 0/1 and left/right neighbor, see Appendix A. It is sufficient to consider one member per class.

The Wolfram rule ECA = $\sum_{k=0}^{7} 2^k \cdot p_k \in \{0,\dots,255\}$ defines the behavior. A cell with neighborhood $(c_{i-1}, c_i, c_{i+1}) \in \mathbb{F}_2^3$, summing up to $k := 4c_{i-1} + 2c_i + c_{i+1} \in \{0,\dots,7\}$ is replaced by $c_i^+ := p_k$.

Example 1 The behaviour of the ECA with Wolfram rule $57_{10} = \mathbf{00111001}_2$ is given in Table 1. We have that $0c_i1 \mapsto c_i$, all other cases $0c_i0, 1c_i0, 1c_i1 \mapsto \overline{c}_i$.

$$\begin{array}{ll} 111 \mapsto \mathbf{0}, & 011 \mapsto \mathbf{1}, \\ 110 \mapsto \mathbf{0}, & 010 \mapsto \mathbf{0}, \\ 101 \mapsto \mathbf{1}, & 001 \mapsto \mathbf{0}, \\ 100 \mapsto \mathbf{1}, & 000 \mapsto \mathbf{1}. \end{array}$$

Table 1: ECA-57

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1.1 State-of-the-Art

The study of asynchronous cellular automata started with Ingerson and Buvel's 1984 paper [2].

Lee *et al.* [3] give an asynchronous CA on the two-dimensional grid $\mathbb{Z} \times \mathbb{Z}$, which is Turing-universal.

Fatès *et al.* [1] consider ECA's with quiescent states ($000 \mapsto 0, 111 \mapsto 1$, *i.e.* with even Wolfram rule ≥ 128). They consider fully randomized ECA's.

A good overview is given in the thesis [4] by Sharkar.

Nevertheless, all these articles treat asynchronous CAs with randomized clocks.

Our concern is instead the (fully deterministic) behavior of a suitable ECA, with any fixed initial configuration, under *all* possible temporal sequences. There seems to be no work on the combined effect of all deterministic temporal rules, synchronous and asynchronous, so far.

Definition 1. Temporal Rules — Asynchronicity Rules

Let the set AS_n of asynchronicity rules over $\mathbb{Z}/n\mathbb{Z}$ consist of all words of length n over the alphabet $\{<, \equiv, >\}$ such that both $<$ and $>$ occur at least once. We also include the word "$\equiv \cdots \equiv$", the synchronous case, and have $AS_n = (\{<, \equiv, >\}^n \backslash (\{<, \equiv\}^n \cup \{\equiv, >\}^n)) \cup \{\equiv^n\}$ with $|AS_n| = 3^n - 2^{n+1} + 2$.

Given a rule $\text{AS} = \text{AS}_0 \cdots \text{AS}_{n-1}$, $\text{AS}_i =$ "$<$", "$\equiv$", and "$>$", resp., defines that cell c_i updates after, simultaneously with, resp. before $c_{i+1}, \forall 0 \leq i \leq n-2$. AS_{n-1} refers to cell c_{n-1} with respect to c_0.

For any partition $(S_1, \ldots S_m)$ of the cell sites, *i.e.* $\dot\cup_{k=1}^m S_k = \{0, \ldots, n-1\}$, let its temporal rule be $\text{AS}_i = <, \equiv, >$, resp., if $i \in S_{\iota(i)}$, $i+1 \in S_{\iota(i+1)}$, and $\iota(i)$ is $>, =, <$, resp., than $\iota(i+1)$ (we say that site i is "bigger" if it comes before $i+1$, hence dominates it).

With the exception of $\equiv^n$ (synchronous case), both $<$ and $>$ must occur at least once, since otherwise, by wrapping-around, each cell would update only after itself and the temporal rule would thus not be well-defined, *e.g.* $\equiv<\equiv$ leads to c_1 with c_2 after c_3 with c_1, so c_1 after, and thus before, itself.

Example 2 Let $n = 4$, and AS = "$<\equiv>>$": Cell 0 updates after cell 1, 1 with 2, 2 before 3, and 3 before 0. Hence the temporal order is $(1,2|3|0)$, first 1 and 2 simultaneously, then 3, finally cell 0, *i.e.* $S_1 = \{1,2\}, S_2 = \{3\}, S_3 = \{0\}$. Analogously, "$<><>$" leads to $(1,3|0,2)$, and "$>\equiv><$" leads to $(0|1,2|3)$.

One might be inclined to partition the n cells into sets $S_1, \ldots, S_m \subset \mathbb{Z}/n\mathbb{Z}$, and update those in S_1 first, then cells from S_2 and so forth. This, however, is too fine-grained:

Theorem 1

Consider two partitions $(S_1, \ldots, S_m)$ and $(S'_1, \ldots, S'_{m'})$ of the cell set $\mathbb{Z}/n\mathbb{Z}$ and define functions a, b, c, a', b', c' such that

$$\forall i \in \{0, \ldots, n-1\} : i-1 \in S_{a(i)}, i \in S_{b(i)}, i+1 \in S_{c(i)}, \ldots$$

$$\ldots i-1 \in S'_{a'(i)}, i \in S'_{b'(i)}, i+1 \in S'_{c'(i)}.$$

Then, if $\text{sgn}(a(i) - b(i)) = \text{sgn}(a'(i) - b'(i))$ and $\text{sgn}(b(i) - c(i)) = \text{sgn}(b'(i) - c'(i)), \forall i$, i.e. the relative update order of cells $i-1, i, i+1$ is the same for S and S', then updating according to S or according to S' leads to the same result, and this is described by the following asynchronicity rule (Table 2).

Proof. By construction. Since the relative temporal order of cell c_i with respect to c_{i-1} and c_{i+1} is the same for (S_k) and (S'_k) by $\text{sgn}(a(i) - b(i)) = \text{sgn}(a'(i) - b'(i))$ and $\text{sgn}(b(i) - c(i)) = \text{sgn}(b'(i) - c'(i))$, both partitions lead to the same overall behaviour, which is described by AS. □

The construction by the theorem shows that the $\text{AS} \in AS_n$ are sufficient to distinguish the behaviour. On the other hand, all these AS are necessary and can lead to different behaviour (at least for some ECA's), since any $\text{AS}_i \neq \text{AS'}_i$ will lead to a different order of updating cells c_i and c_{i+1}.

$\mathrm{sgn}(a-b)$	$\mathrm{sgn}(b-c)$	AS_{i-1}	AS_i	
-1	-1	$>$	$>$	a before b before c
-1	0	$>$	$\equiv$	a before b with c
-1	$+1$	$>$	$<$	a and c before b
0	-1	$\equiv$	$>$	a with b before c
0	0	$\equiv$	$\equiv$	a with b with c
0	$+1$	$\equiv$	$<$	c before a with b
$+1$	-1	$<$	$>$	a before b and c
$+1$	0	$<$	$\equiv$	b with c before a
$+1$	$+1$	$<$	$<$	c before b before a

Table 2: Local asynchronicity

Example 3 For $n=6$, “$<><><>$” requires the odd cells $1,3,5$ to update before the even ones $0,2,4$. There are 13 partitions of three elements, *e.g.* $(1,3,5),(1|3,5)$, $(1,5|3)$, and $(5|1|3)$, and thus $13^2 = 169$ partitions (S_k) for this AS.

Definition 2. By $\mathrm{ECA}_{\mathrm{AS}}(v) = w$, we mean that the elementary CA with rule ECA maps $v \in \{0,1\}^n$ to $w \in \{0,1\}^n$ via the temporal sequence AS.

Example 4 ECA-$57_{<><>}(1000) = 1110$, in two steps: $1\underline{0}0\underline{0} \mapsto \underline{1}1\underline{0}0 \mapsto 1110$, where underlined cells are active in the next step.

2 The Finite Torus $\mathbb{Z}/n\mathbb{Z}$: Patterns

In this section, we work on the torus $\mathbb{Z}/n\mathbb{Z}$, and consider all ECA's for all initial configurations. We apply a fixed temporal rule $\mathrm{AS} \in AS_n$ repeatedly, τ times, and ask, whether these 5 pattern universality properties hold:

$$\left.\begin{array}{llll} (o) & \exists v \in \mathbb{F}_2^n, & \forall w \in \mathbb{F}_2^n, & \exists \tau \in \mathbb{N}, \dots \\ (i) & \forall v \in \mathbb{F}_2^n, & \forall w \in \mathbb{F}_2^n, & \exists \tau \in \mathbb{N}, \dots \\ (ii) & \forall v \in \mathbb{F}_2^n, & \exists \tau \in \mathbb{N}, & \forall w \in \mathbb{F}_2^n, \dots \\ (iii) & \exists \tau \in \mathbb{N}, & \forall v \in \mathbb{F}_2^n, & \forall w \in \mathbb{F}_2^n, \dots \\ (iv) & \exists \tau_0 \in \mathbb{N}, & \forall \tau \geq \tau_0, & \forall v, w \in \mathbb{F}_2^n, \dots \end{array}\right\} \exists \mathrm{AS} \in AS_n : \mathrm{ECA}_{\mathrm{AS}}^{\tau}(v) = w.$$

All results are experimental *i.e.* derived from exhaustive computer simulations for the stated lengths.

We start with

(o) $\exists v \in \mathbb{F}_2^n, \forall w \in \mathbb{F}_2^n, \exists \tau \in \mathbb{N}, \exists \mathrm{AS} \in AS_n : \mathrm{ECA}_{\mathrm{AS}}^{\tau}(v) = w$. That is from some v we eventually reach any w. We give the largest number of w's reached for some v, for $n = 4,8$, and 12. To satisfy (o), these must be $(16,256,4096)$.

The 3 ECA families 0 (1,1,1), 200 (1,1,1), and 204 (1,1,1) are resilient to asynchronicity. They have a constant result, for all n.

ECA-51 (2,2,2) varies between at most two results.

The next 49 ECA families are ordered by increasing image size for $n = 12$:

140	(2,6,16),	160	(12,130,1182),	164	(13,197,2930),	108	(16,256,4052),
136	(2,9,27),	2	(11,211,1477),	24	(15,211,2961),	56	(16,256,4066),
128	(2,16,49),	72	(11,131,1499),	34	(13,209,2998),	74	(15,255,4071),
132	(4,18,81),	76	(11,131,1499),	130	(14,211,3160),	73	(16,256,4084),
32	(7,31,127),	172	(11,137,1506),	94	(16,216,3448),	33	(16,256,4092),
8	(5,45,320),	168	(12,147,1601),	152	(14,237,3561),	10	(13,253,4093),
4	(7,47,322),	13	(16,168,1792),	138	(13,238,3751),	134	(15,255,4093),
12	(7,47,322),	232	(12,156,1830),	104	(14,232,3824),	42	(15,255,4093),
28	(11,91,641),	77	(12,156,1830),	162	(16,250,3970),	35	(16,256,4094),
29	(12,92,642),	142	(12,140,1848),	170	(16,256,3976),	43	(16,256,4094),
44	(12,100,870),	78	(15,167,1851),	15	(16,256,3976),		
156	(4,64,1024),	36	(14,162,1943),	150	(12,240,4032),		
40	(11,119,1052),	5	(16,216,2542),	1	(16,256,4051),		

The 4 ECA families 6, 14, 18 [for $n \geq 7$], 50 [for $n \geq 4$], miss exactly one pattern, leading to $2^n - 1$ in general.

Finally, the 31 ECA families
3, 7, 9, 11, 19, 22, 23, 25, 26, 27, 30, 37, 38, 41, 45, 46, 54, 57, 58, 60, 62, 90,
105 [$n \neq 0 \mod 4$], 106, 110, 122, 126, 146, 154, 178 [$n \neq 3$], 184 [$n \neq 3$],
satisfy property (o) (for $3 \leq n \leq 12$).

(i) $\forall v, w \in \mathbb{F}_2^n, \exists \text{AS} \in AS_n, \exists \tau \in \mathbb{N} : \text{ECA}^{\tau}_{\text{AS}}(v) = w$. From the 31 families satisfying (o), most fall short for some v. We give the smallest number of w reachable from some v, for $n = 4, 8$, and 12, this should be (16,256,4096) to satisfy (i).

Eighteen ECA families are insensitive (or resilient) to asynchronicity for at least some v, the same w resulting for all AS. Hence, $(1,1,1)$ patterns are reached:
22, 26, 30, 38, 46, 54, 58, 60, 62, 90, 106, 110, 122, 126, 146, 154, 178, 184.
ECA family 7 reaches $2^n - 1$ for $n \neq 0 \mod 3$ and only 1 pattern for $n \equiv 0 \mod 3$.
ECA family 45 has $2^n - 1$ patterns for odd n, 1 for even n.
Six ECA families get near the full 2^n for all w: 3 (15,233,3411), 9(12,243,3963), 11 (15,233,3515), 25 (16,251,4031), 27 (16,253,4052), 43 (12,236,3554).

The following 6 ECA families satisfy (i) at least for certain $[n]$ ($3 \leq n \leq 12$ considered):

19 [3-12], 23 [3,5,7,9,11], 37 [4-5,7-8,10-11], 41[3,5,7-12], 57[3-12], 105 [3,5-7,9-11] all generate 2^n patterns for these $[n]$.

$(ii) - (iv)$ From now on, we will consider the 6 ECA families satisfying (i): 19, 23 ($n \not\equiv 0 \mod 2$), 37 ($n \not\equiv 0 \mod 3$), 41, 57, and 105 ($n \not\equiv 0 \mod 4$).

(ii) $\forall v \in \mathbb{F}_2^n, \exists \tau \in \mathbb{N}, \forall w \in \mathbb{F}_2^n, \exists \text{AS} \in AS_n : \text{ECA}^{\tau}_{\text{AS}}(v) = w$; *i.e.* for fixed v, all w are reached at the same time.

We considered τ up to 20000, and obtain:
ECA-23: No v has any $\tau \leq 20000$ to satisfy (ii).
ECA-19,-37,-41: For some v, there is no $\tau \leq 20000$ to satisfy (ii).
ECA-57 satisfies (ii), for $n \geq 5$ and all v. The largest τ required is 28 for $n = 5$; 14 for $n = 6$; 10 for $7 \leq n \leq 13$; and 9 for $n = 14$ and 15.
ECA-105 satisfies (ii) for odd $n \geq 7$ and all v. The largest τ required is 30 for $n = 7$; 16 for $n = 9, 11, 13$; and 8 for $n = 15$.

In general, the time τ decreases with n, since the number of patterns, 2^n, increases slower than the number of asynchronicities, 3^n–2^{n+1}+1, and thus for larger n, AS_n is more likely to satisfy (ii) early on.

(iii) $\exists\tau \in \mathbb{N}, \forall v, w \in \mathbb{F}_2^n, \exists \textsc{as} \in AS_n : \mathrm{ECA}^{\tau}_{\mathrm{AS}}(v) = w$. All transductions $v \mapsto w$ are done in the same time.

From the result of (ii), we can infer that at most ECA-57 and ECA-105 can satisfy (iii).
ECA-57 has a joint τ_n at which all transductions $v \mapsto w$ are satisfied simultaneously in these cases: $\tau_5 = 445, \tau_7 = 70, \tau_8 = 242, \tau_9 = 35, \tau_{10} = \cdots = \tau_{14} = 13$, $\tau_{15} = 10$.
For ECA-105, we have $\tau_7 = 570, \tau_9 = 14, \tau_{11} = \tau_{13} = 6$, and $\tau_{15} = 8$.

(iv) $\exists\tau_0 \in \mathbb{N}, \forall\tau \geq \tau_0, \forall v, w \in \mathbb{F}_2^n, \exists \textsc{as} \in AS_n : \mathrm{ECA}^{\tau}_{\mathrm{AS}}(v) = w$. Eventually all transductions $v \mapsto w$ can be done at all times.

Theorem 2

There is no $\tau_0 \in \mathbb{N}, \forall\tau \geq \tau_0, \forall v, w \in \mathbb{F}_2^n, \exists \textsc{as} \in AS_n : \mathrm{ECA}^{\tau}_{\mathrm{AS}}(v) = w$*, i.e.* (iv) *can not be satisfied.*

Proof. Consider the case $w = v$.

For each rule AS, applying AS repeatedly, starting at v, we will either return to v at some time, which is the period length $per(\textsc{as})$, the length of the cycle of AS containing v, or else v is in a preperiod and will never be reached again. Therefore, either v is in the preperiod and thus will not reappear, or else there is no preperiod, and $v = w$ appears exactly after $k \cdot per(\textsc{as}), \forall k$, applications of AS.

Let now $\mathrm{PER} = lcm_{\mathrm{AS}}(per(\textsc{as}))$, where AS runs over those temporal rules without preperiod. Apparently, after $k \cdot \mathrm{PER}(v), \forall k$, applications of AS, we return to v, for all these rules without preperiod simultaneously. After $k \cdot \mathrm{PER}(v) \pm 1$ steps, $\forall k$, we are not at v (unless the period is 1, and thus v is a fixed point). Hence, $v \mapsto v$ is impossible for all these timesteps $k \cdot \mathrm{PER}(v) \pm 1$, and there is no such τ_0.

Finally, in the case that v is a fixed point under AS, no cell changes its contents for this temporal rule and thus only $w = v$, but no $w \neq v$ is ever reached. □

Definition 3. We call an elementary cellular automaton pattern-universal at length n, if it is able to convert any pattern v in $\mathbb{F}_2^n$ into any other, *i.e.* satisfies property (i) ($\forall v, w \in \mathbb{F}_2^n, \exists \textsc{as} \in AS_n, \exists\tau \in \mathbb{N} : \mathrm{ECA}^{\tau}_{\mathrm{AS}}(v) = w$).

If an ECA is pattern-universal for all $n \geq n_0$, for some n_0, it is called uniformly pattern-universal.

Corollary ECA's from the 6 families $19, 23, 37, 41, 57$, and 105 are pattern-universal for the lengths indicated in property (i) above.

We conjecture that ECA's from families 19, 41, and 57 are uniformly pattern-universal.

3 The Finite Torus $\mathbb{Z}/n\mathbb{Z}$: Functions

In Section 2, we focussed on transductions $v \mapsto w$, which — in general — used different temporal rules for different v's and w's, but for each pair (v, w) stayed with the same rule, applied repeatedly.

In this section, we are interested in functions $\mathbb{F}_2^n \ni v \mapsto f(v) = w \in \mathbb{F}_2^n$, which use the same temporal rule sequence for all v, but — necessary to generate enough variation — concatenate several different temporal rules.

We may restrict ourselves to ECA families 19, 23, 37, 41, 57, and 105. Given a function on $\mathbb{F}_2^n$ defined by the values $w(v) \in \mathbb{F}_2^n, \forall v$, our question is thus:

$$\exists k \in \mathbb{N}, \exists \textsc{as}_1, \ldots \textsc{as}_k \in AS_n, \forall v \in \mathbb{F}_2^n : \mathrm{ECA}_{\mathrm{AS}_k}(\ldots(\mathrm{ECA}_{\mathrm{AS}_1}(v))\ldots) = w(v)?$$

3.1 Bijective Functions

We first consider bijective functions on $\mathbb{F}_2^n$. In this case the equivalent group-theoretic statement is:

Do the $\mathrm{ECA}_{\mathrm{AS}} \in AS_n$ *(written as permutations on the set* $\{0, 1, \ldots, 2^n - 1\}$*) generate the full symmetric group* S_{2^n} ?

To answer this question, we used the program GAP (Graphs, Algorithms, Programming) from RWTH Aachen (Prof. Neubüser's group) and St. Andrews University [5]. Thank you!

We ran GAP on some subsets of only 3 asynchronicity rules to show that $\mathrm{ECA}_{\mathrm{AS}} \in AS_n$ generates at least the alternating group A_{2^n}, for $3 \leq n \leq 11$.

Trying directly to obtain the group generated by the full set $\mathrm{ECA}_{\mathrm{AS}} \in AS_n$ overburdens GAP already from $n = 4$ on. Therefore, in order to check for the generation of S_{2^n}, it is then sufficient to exhibit at least one odd permutation, which is the case for $n = 3$, with the whole S_{2^3} generated — or to show that all permutations generated by $\mathrm{ECA}_{\mathrm{AS}} \in AS_n$ are even, which is the case for $4 \leq n \leq 11$, and thus only A_{2^n}, but not S_{2^n}, is generated in these cases.

Out of the 6 ECA families satisfying property (i), ECA-57 and ECA-105 are the only ones, which have a locally bijective update rule. Therefore, only these families must be considered. We immediately have that temporal rules avoiding the symbol "$\equiv$" are bijective, when the temporal rule is bijective, since different applications of that temporal rule do not interfere with each other. On the other hand, for $n \neq 3$, all temporal rules involving the symbol "$\equiv$" lead to non-bijective functions, see next subsection.

The rules excluding $\equiv$ define bijective functions, whenever the ECA itself is (locally) bijective, that is the application of such an temporal rule for a single cell yields bijectivity. Those temporal rules including $\equiv$ define the non-bijective functions. Hence, the only way to generate bijective functions for $n \geq 4$ is by using ECA-57 or ECA-105, and only applying temporal rules from $\{<,>\}^n$.

ECA-57: GAP tells us that the $2^n - 2$ temporal rules from $\{<,>\}^n \backslash \{<^n, >^n\}$ always yield at least the alternating group A_{2^n}, which is in fact generated already by 3 of the temporal rules, for $3 \leq n \leq 10$.

The case S_{2^n} vs. A_{2^n} is easiest checked by hand: Is there some odd permutation within the temporal rules? This is only the case for $n = 3$. For $4 \leq n \leq 10$, all temporal rules yield even permutations and thus can not generate the full S_{2^n}.

Hence, for $n = 3$, all bijective functions are generated through ECA-57 by concatenation of suitable temporal rules, while for $n \geq 4$, only the even permutations from A_{2^n} (that is half of the $2^n!$ bijective functions) are generated.

ECA-105: GAP tells us that all bijective temporal rules combined generate only fairly small groups: $< AS_3 >$ has order 24, $| < AS_4 > | = 48 = 4! \cdot 2^1$, $| < AS_5 > | = 1920 = 5! \cdot 2^4$, $| < AS_6 > | = 11520 = 6! \cdot 2^6$, $| < AS_7 > | = 322560 = 7! \cdot 2^6$, all are far below $|S_{2^n}| = 2^n!$, the number of bijective functions.

3.2 Non-Bijective Functions

We now turn to non-bijective functions. Then $Im(f) \subset \mathbb{F}_2^n$ with $|Im(f)|$ strictly less than 2^n.

We start with $n = 3$. The convex hull over all AS $\in AS_3$ has cardinality at least $2^3!/2 = 20160$ for the following ECA's, Table 3 (the other ECA with bijective update rule, ECA-105, generates only 344 functions):

ECA-25:	22496	ECA-46:	89110
ECA-110:	23166	ECA-41:	210493
ECA-30:	25258	ECA-38:	223102
ECA-3:	39155	ECA-27:	268034
ECA-57:	**40320**	ECA-35:	751760
ECA-11:	52934	ECA-54:	1.190.449
ECA-62:	62683	ECA-19:	3.519.992

Table 3: Image size for ECAs on $\mathbb{F}_2^3$

There are 8^8, about 16 Mio., functions on $\mathbb{F}_2^3$. Hence, for $n = 3$, none of the ECA's even generates a quarter of all functions. The case ECA-57 is special in that this ECA actually generates all bijective functions, but no non-bijective one, for $n = 3$.

In the sequel, $n \geq 4$, we consider only ECA-57, which has sufficiently many bijective functions, namely $2^n!/2$, at least for $4 \leq n \leq 10$. We will generate a considerable subset of all functions by suitably interleaving bijective and non-bijective temporal rules for ECA-57.

We now consider ECA-57 for a temporal rule with a single $\equiv$ on $\mathbb{Z}/n\mathbb{Z}, n \geq 4$.

Considering larger neighborhoods, with 2 cells changing simultaneously, also ECA-57 becomes non-surjective (we show the effect of $\text{AS}_1 =$ "$\equiv$" on the two middle cells for all configurations of 4 adjacent cells):

v	$\mapsto$	ECA-57(v)
0000	$\mapsto$	0110
0001	$\mapsto$	**0101**
1000	$\mapsto$	1110
1001	$\mapsto$	1101
0010	$\mapsto$	0000
0011	$\mapsto$	**0011**
1010	$\mapsto$	1100
1011	$\mapsto$	1111
0100	$\mapsto$	0010
0101	$\mapsto$	**0011**
1100	$\mapsto$	1010
1101	$\mapsto$	1011
0110	$\mapsto$	0100
0111	$\mapsto$	**0101**
1110	$\mapsto$	1000
1111	$\mapsto$	1001

Table 4: ECA-57: Effect of $\text{AS}_1 =$ "$\equiv$"

We obtain the patterns 0011 and 0101 twice, while missing 0001 and 0111. Hence the image is smaller than the full 2^4 by 2, or by a factor of 7/8.

Extending this neighborhood of $\equiv$ to any size n, and using only $<$ and $>$ for the other $n-1$ positions, before and after the $\equiv$ transition, ECA-57 behaves bijectively. Therefore, the whole image shrinks by just the factor 7/8, when applying $\equiv$ once.

Since all temporal rules without $\equiv$ are bijective, and inclusion of more than one $\equiv$ shrinks the image even further, we have the following result on the functions that can be represented by ECA-57:

Theorem 3

Let the patterns from $\{<,>\}^n \setminus \{<^n, >^n\}$ *generate at least the alternating group* A_{2^n} (*which is the case at least for* $3 \leq n \leq 10$).

Let $f : \mathbb{F}_2^n \to \mathbb{F}_2^n, n \geq 4$ *be any non-bijective function on at least* 4 *symbols. Let* $\#(w) = |\{v | f(v) = w\}|$ *be the number of configurations v leading to configuration w. Then f is representable by* ECA-57 *under asynchronicity, if and only if*

$$\sum_{w \in \mathbb{F}_2^n} \lfloor \#(w)/2 \rfloor \geq 2^{n-3}.$$

Proof. We first introduce the functions # on $\mathbb{F}_2^n$ and @ on $\mathbb{N}_0$:

The multiplicity $\#(w)$ tells us, how often w is reached, *i.e.* is the size of the preimage of $\{w\}$.

For $k \in \mathbb{N}_0$, let $@(k) \in \mathbb{N}_0$ be the number of results w appearing with multiplicity k, $@(k) = |\{w \in \mathbb{F}_2^n \colon \#(w) = k\}|$. In particular, $@(0) = 2^n - |Im(f)|$ is the number of words avoided by the image of f. We have $\sum_k k \cdot @(k) = 2^n$.

We make use of the temporal rule $\text{AS}^* := $ "$<\equiv> \cdots >$" which maps 2^{n-3} pairs (v_1, v_2) onto 2^{n-3} words w, and otherwise is 1-to-1. Hence, for AS^*, we have $@(1) = 6 \cdot 2^{n-3}, @(0) = @(2) = 2^{n-3}$. We generate f by a chain $f = \pi_k \circ \text{AS}^* \circ \cdots \circ \pi_2 \circ \text{AS}^* \circ \pi_1 \circ \text{AS}^*$, alternating AS^* and permutations $\pi_k \in S_{2^n}$.

For the second and every further application of AS^*, we will join $2^{n-3} - 1$ words v_1 with $\#(v_1) > 0$ to $2^{n-3} - 1$ words v_2 with $\#(v_2) = 0$, hence without changing the distribution @. We also map $6 \cdot 2^{n-3}$ words 1-to-1, and finally we join two multiplicities $\#(v_1), \#(v_2)$ by mapping v_1, v_2 onto the same w, the actual effect of this application of AS^*. The new values $@^+$ are thus $@^+(\#(v_1)) = @(\#(v_1)) - 1, @^+(\#(v_2)) = @(\#(v_2)) - 1, @^+(\#(v_1) + \#(v_2)) = @(\#(v_1) + \#(v_2)) + 1$, and $@^+(k) = @(k)$ otherwise. In this way, we eventually arrive at a distribution @ as required by f.

To achieve this, we permute values in between applications of AS^*. In this (slow) way, we eventually get to the distribution of $\#(w)$ required by f.

The final permutation π_k maps the v with multiplicities $\#(v) > 0$ to the correct values $w \in Im(f)$.

Since we always have two words v_1, v_2 mapping to the same w under AS^*, and also two words w_1, w_2 outside $Im(\text{AS}^*)$, any $\pi_k \in S_{2^n} \backslash A_{2^n}$ can be extended by one of the transpositions (v_1, v_2) or (w_1, w_2) to an equivalent $\pi'_k \in A_{2^n}$.

Concerning the "only if" part, already the first application of AS^* would decrease the number of values below $|Im(f)|$. □

3.3 Examples

The superscript $^{(n)}$ indicates the torus size.

INC$^{(3)}$ For $n = 3$, let $w = v + 1 \mod 8$. This is an odd bijective function, and hence representable for this $n = 3$.

MUL-BY-3$^{(3)}$ For $n = 3$, let $w = 3 \cdot v \mod 8$. Same as with INC.

MUL-BY-2$^{(3)}$ For $n = 3$, let $w = 2 \cdot v \mod 8$. From $2 \cdot 0 = 2 \cdot 4 = 0 \mod 8$, this function is not bijective, and hence not representable by ECA-57 for $n = 3$.

INC$^{(4)}$ For $n = 4$, let $w = v + 1 \mod 16$. As with $n = 3$, this is an odd bijective function. Contrary to the case $n = 3$, a representation by ECA-57 is not possible for $n \geq 4$.

INC'$^{(4)}$ For $n = 4$, let $w = v + 1 \mod 15, 15 \mapsto 15$. This is an even bijective function, and thus representable.

MUL-2-BY-2$^{(4)}$ For $n = 4$, let $v = a|b$, $0 \leq a, b \leq 3$ and $w = a \cdot b$. The range is given by the multiset $\{0^7, 1, 2^2, 3^2, 4, 6^2, 9\}$, where superscripts show the number of occurrences. The sum $\sum_{w \in \mathbb{F}_2^4} \lfloor \#(w)/2 \rfloor = 6 \geq 2^{n-3} = 2$ is large enough (the range is sufficiently small thus) to allow shrinking by *e.g.* repeated application of the asynchronicity pattern "$<=>>$" and suitable permutations. Multiplication can thus be computed by ECA-57 through asynchronicity. *How* to do it exactly, is a more complicated case, see Open Problems.

MUL-k-BY-k$^{(2k)}$: Zero appears $2 \cdot 2^k - 1$ times, and $1 \leq a < b \leq 2 - 1$ yields $a \cdot b = b \cdot a$ that is at least $\binom{2^k - 1}{2}$ pairs. Hence, we have $\sum_{w \in \mathbb{F}_2^4} \lfloor \#(w)/2 \rfloor \geq 2^k - 1 + (2^k - 1) \cdot (2^k - 2)/2 > 2^{2k-3} = 2^{n-3}$ (with $k \geq 2$). All these multiplications can therefore be computed by ECA-57, using asynchronicity.

Boolean and arithmetic functions on k bits, $n = 2k$:

Let $v = a|b$ with $0 \le a, b < 2k$. Then $\underline{f_{\vee,1}}(v) = (0|a \vee b)$, $\underline{f_{\vee,2}}(v) = (a \vee b|a \vee b)$, $\underline{f_{\wedge,1}}(v) = (0|a \wedge b)$, $\underline{f_{\wedge,2}}(v) = (a \wedge b|a \wedge b)$, $\underline{f_{\oplus,1}}(v) = (0|a \oplus b)$, $\underline{f_{\oplus,2}}(v) = (a \oplus b|a \oplus b)$, $\underline{f_{-}}(v) = (a - b \mod 2^{2k})$, $\underline{f_{-'}}(v) = (0|(a-b) \mod 2^k)$, can all be computed by ECA-57 under asynchronicity, for any k that is any even n:

All these Boolean functions are commutative, $a \circ b = b \circ a$, and thus enough pairs (v_1, v_2) with $f(v_1) = f(v_2)$ exist to have $\sum \lfloor \#(w)/2 \rfloor \ge 2^{2k-3}$.

For $0 \le v < 2^n$, let $\underline{f_{\mathrm{NEG}}}(v) = (-v)$ (2's complement). This is an odd bijection with the two fixed points 0 and 2^{n-1}, and $2^{n-1} - 1$ transpositions, hence computable by ECA-57 only for $n = 3$.

$\underline{f_{\mathrm{COMP}}}(v) = (v \oplus 111 \cdots 111)$ (1's complement), on the other hand, is an even bijection, computable for all n.

Example 5 A detailed description of the calculation of $\mathrm{INC}^{(3)}$ and MUL-BY-$3^{(3)}$. The left column indicates the temporal rule and partition of $\mathbb{Z}/3\mathbb{Z}$

$\mathrm{INC}^{(3)}$

	000	001	010	011	100	101	110	111
$<><(1\mid 0\mid 2)$	011	101	100	110	111	010	001	000
$<>>(1\mid 2\mid 0)$	010	110	011	000	001	101	100	111
$<>\equiv(1\mid 0,2)$	101	000	110	011	100	010	111	001
$\equiv><(0,1\mid 2)$	010	111	100	110	011	001	000	101
$>><(0\mid 1\mid 2)$	001	010	011	100	101	110	111	000

MUL-BY-$3^{(3)}$

	000	001	010	011	100	101	110	111
$<<>(2\mid 1\mid 0)$	101	010	111	100	110	011	001	000
$><>(2\mid 0\mid 1)$	001	101	100	010	011	000	110	111
$\equiv<>(2\mid 0,1)$	110	011	010	111	000	101	001	100
$>\equiv<(0\mid 1,2)$	101	100	001	010	110	000	111	011
$>><(0\mid 1\mid 2)$	000	011	110	001	100	111	010	101

Table 5: $\mathrm{INC}^{(3)}$ and MUL-BY-$3^{(3)}$ in detail

Further Research and Open Problems

1. Give an algorithm to calculate the temporal sequence $(\mathrm{AS}_1, \mathrm{AS}_2, \ldots, \mathrm{AS}_k)$ for a function on $\mathbb{F}_2^n$ directly from the function values, given *e.g.* as permutation on $\{0, 1, \ldots, 2^n - 1\}$, instead of searching through the full tree AS_n^k.

2. Consider temporal sequences that do not depend on the position, but on the rule to be applied, *e.g.* first update at all corresponding sites $000 \mapsto p_0$, then $110 \mapsto p_6$, then $010 \mapsto p_2$ etc. There are 8! = 40320 such temporal rules, independent of n.

How do we treat actions that already had their turn, but whose neighborhood only turns up later? Update immediately upon creation, never in this round, ...?

This could mimic chemical reactions, *e.g.* in cell biology, DNA expression, where some reactions are faster than others, depending on their reaction rate constant k.

3. As in 2., but associate a latency time with each temporal rule: As soon as the corresponding neighborhood pattern is created, wait for its latency time and then update according to the temporal rule.

4. What can we say about the alphabet $\{0,1,2\}$ instead of $\{0,1\}$?

There are now $3^{3^3} \approx 2^{43}$ ECA's to be considered. Since there are $3^n - 2^{n+1} + 2 = \Theta(3^n)$ asynchronicities (Definition 1) and exactly 3^n configurations on $\mathbb{Z}/n\mathbb{Z}$, an analogue of properties (o) to (iv) is now impossible due to lack of temporal rules. However, we may ask, how the number of configurations actually reached grows with n. Do we ever obtain the full diversity of $3^n - 2^{n+1} + 2$ results?

Conclusion

We have introduced temporal order via temporal rules as a means to diversify the behaviour of elementary cellular automata.

In particular, ECA's with update rules 19, 41, and 57 are pattern universal for $n \leq 12$, achieving any desired pattern transduction $v \mapsto w$, applying iteratedly a single temporal rule. We conjecture that they are indeed uniformly pattern-universal.

ECA-57 produces any even (as permutation) bijective function on $\{0,1\}^n$, for $n \leq 10$, and all non-bijective ones that join at least 2^{n-3} pairs of argument values.

Temporal order is thus a third way to encode information and algorithms, after programs (ECA's) and data (initial configurations).

This may have farreaching consequences, *e.g.* for modeling gene expression, since physico-biological processes seldomly achieve exact synchronicity.

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Appendix A – ECA Families

Each family (equivalence class under the symmetries 0/1 and L/R ($c_{i-1} \leftrightarrow c_{i+1}$)) consists in up to 4 ECAs with numbers ECA=$abcdefgh_2$,
ECA$\stackrel{L/R}{\longleftrightarrow}$ $aecgbfdh_2$, ECA $\stackrel{0/1}{\longleftrightarrow}$ $\overline{hgfedcba}_2$, and ECA$\stackrel{L/R,0/1}{\longleftrightarrow}$ $\overline{hdfbgcea}_2$.

0 (255),	1 (127),	2 (191 16 247),	3 (63 17 119),
4 (223),	5 (95),	6 (159 20 215),	7 (31 21 87)
8 (239 64 253),	9 (111 65 125),	10 (175 80 245),	11 (47 81 117),
12 (207 68 221),	13 (79 69 93),	14 (143 84 213),	15 (85),
18 (183),	19 (55),	22 (151),	23,
24 (231 66 189),	25 (103 67 61),	26 (167 82 181),	27 (39 83 53),
28 (199 70 157),	29 (71),	30 (135 86 149),	32 (251),
33 (123),	34 (187 48 243),	35 (59 49 115),	36 (219),
37 (91),	38 (155 52 211),	40 (235 96 249),	41 (107 97 121),
42 (171 112 241),	43 (113),	44 (203 100 217),	45 (75 101 89),
46 (139 116 209),	50 (179),	51,	54 (147),
56 (227 98 185),	57 (99),	58 (163 114 177),	60 (195 102 153),
62 (131 118 145),	72 (237),	73 (109),	74 (173 88 229),
76 (205 76 205),	77,	78 (141 92 197),	90 (165 90 165),
94 (133),	104 (233),	105,	106 (169 120 225),
108 (201),	110 (137 124 193),	122 (161),	126 (129),
128 (254),	130 (190 144 246),	132 (222),	134 (158 148 214),
136 (238 192 252),	138 (174 208 244),	140 (206 196 220),	142 (212),
146 (182),	150,	152 (230 194 188),	154 (166 210 180),
156 (198),	160 (250),	162 (186 176 242),	164 (218),
168 (234 224 248),	170 (240),	172 (202 228 216),	178,
184 (226),	200 (236),	204,	232