Phase Space Invertible Asynchronous Cellular Automata

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While for synchronous deterministic cellular automata there is an accepted definition of reversibility, the situation is less clear for asynchronous cellular automata. We first discuss a few possibilities and then investigate what we call *phase space invertible* asynchronous cellular automata in more detail. We will show that for each Turing machine there is such a cellular automaton simulating it, and that it is decidable whether an asynchronous cellular automaton has this property or not, even in higher dimensions.

1 Introduction

For synchronous deterministic cellular automata the topic of reversibility has gained a good deal of attention. Reversibility is known to be decidable in one dimension [1] and undecidable in two and more dimensions [4]. Reversible synchronous cellular automata are also computationally universal (see, for example, [7] for the one-dimensional case).

In the present paper we take a first look at the analogous questions for *asynchronous* cellular automata. Hence the rest of this paper is organized as follows: In section 2 we introduce some notation used in this paper and two variants of asynchronicity. In section 3 we discuss several possibilities for the definition of "reversibility" of asynchronous cellular automata. In section 4 we show that each Turing machine can be simulated by a phase space invertible purely asynchronous cellular automaton, and in section 5 that the property of being phase space invertible is decidable for several variants of asynchronous cellular automata.

This paper is based on the diploma thesis of the first author [11].

2 Basics

2.1 General Notation

We write $\mathbf{N}$ for the set of natural numbers without 0, $\mathbf{Z}$ for the set of integers, $0 \in \mathbf{Z}^d$ for the d-tuple with each component equal to zero, B^A for the set of all total functions from A to B, and 2^M for the powerset of a set M. The cardinality of a set M is denoted as $|M|$. The restriction of a function $f : A \to M$ to a subset $B \subseteq A$ of its domain is written $f|_B$.

In this paper we are interested in d-dimensional cellular automata

($d \in \mathbf{N}$) and the set of *cells* is usually denoted as R, that is, $R = \mathbf{Z}^d$. If the set of *states* of one cell is denoted as Q, the set of all *(global) configurations* is Q^R. A *neighborhood* is a finite set $N = \{n_1, \dots, n_k\} \subseteq \mathbf{Z}^d$ of d-tuples of integers. A

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local configuration is a mapping $\ell : N \to Q$; thus Q^N is the set of all local configurations. The local configuration c_{i+N} observed by cell $i \in R$ in the global configuration $c \in Q^R$ is defined as

$$\begin{aligned} c_{i+N} &: N \to Q, \\ & n \mapsto c(i+n). \end{aligned}$$

The behavior of each single cell of a deterministic cellular automaton is described by the *local transition function* $\delta : Q^N \to Q$.

2.2 Asynchronous Updating Schemes

A local structure (R, N, Q, δ) of a deterministic cellular automaton together with a prescription how cells are updated induces a *global transition relation* $\mathscr{R} \subseteq Q^R \times Q^R$ describing the possible global steps which satisfies

$$c \,\mathscr{R}\, c' \Longrightarrow \forall i \in R : c'(i) \in \{\delta(c_{i+N}), c(i)\}.$$

We will use the same symbol for the related *global transition function*

$$\begin{aligned} \mathscr{R} &: Q^R \to 2^{Q^R}, \\ & c \mapsto \{c' \mid c \,\mathscr{R}\, c'\}. \end{aligned}$$

With this notation $c \,\mathscr{R}\, c'$ is equivalent to $c' \in \mathscr{R}(c)$ and both indicate that it is possible to reach global configuration $c' \in Q^R$ in one step from global configuration $c \in Q^R$.

In a global step each cell has two possibilities: to be *active* and make a state transition according to the rule or to be *passive* and maintain its state. Restrictions made by different updating schemes lead to different possible behaviors, that is, different relations/functions $\mathscr{R}$, of cellular automata.

Now, we will have a look at two different types of asynchronous updating.

Purely Asynchronous Updating. This version of asynchronous updating has been considered for many years now [8, 3]. In order to distinguish it from the other form mentioned below we call it *purely asynchronous* updating.

In each global step there are no restrictions on whether a cell may be active or passive. Thus in each step there is a subset $A \subseteq R$ of active cells which make a transition, while the cells in the complement $R \setminus A$ are passive and simply maintain their state.

Note that A is is allowed to be empty. The additional requirement $A \neq \emptyset$ might look irrelevant, but a closer look at the constructions and theorems reveals that it would render the main results wrong (see for example the remark after Lemma 6).

Given a deterministic cellular automaton and a set $A \subseteq R$ we define the function

$$\Delta_A : Q^R \to Q^R,$$

$$\forall i \in R: \qquad \Delta_A(c)(i) = \begin{cases} \delta(c_{i+N}) & \text{if } i \in A, \\ c(i) & \text{if } i \notin A. \end{cases}$$

Synchronous updating is described by Δ_R. The union of all Δ_A, interpreted as relations, is the general step relation for purely asynchronous updating, for which we will write

$$\Delta = \bigcup_{A \subseteq R} \Delta_A \,.$$

Note that for each global configuration $c \in Q^R$ holds

$$\Delta(c) = \{\Delta_A(c) \mid A \subseteq R\}.$$

Fully Asynchronous Updating. In the fully asynchronous updating scheme it is required that in each global step exactly one cell is active. Using the notation from above one may say that one only looks at the relations $\Delta_{\{i\}}$ where the set of active cells is a singleton. For the union of these relations we will write

$$\Delta_1 = \bigcup_{i \in R} \Delta_{\{i\}} .$$

Even for relatively simple deterministic cellular automata, for example, the elementary deterministic cellular automata or two-dimensional minority, the analysis of their behavior under fully asynchronous updating is surprisingly “non-simple” [2, 9, 6].

To distinguish the global step relations of two purely or fully asynchronous cellular automata C and G we will use Δ_A, Δ, and Δ_1 for the respective relations of C and Γ_A, Γ, and Γ_1 for the respective relations of G.

3 Which Definition of Invertibility?

For the global transition function Δ of a synchronous deterministic cellular automaton C the following conditions are equivalent:

R1 Each global configuration $c \in Q^R$ has exactly one predecessor under Δ.

R2 Each global configuration $c \in Q^R$ has at most one predecessor under Δ.

R3 The inverse of the transition graph of C, that is, the direction of each transition is reversed, is the transition graph of a(nother) synchronous deterministic cellular automaton C'.

For asynchronous cellular automata these are really different conditions. In that case, obviously, condition R1 still implies R2 as well as R3. But the reverse implications do not hold. In order to clarify this, we first show that there is essentially only one asynchronous cellular automaton satisfying R1 or R2: The identity.

Lemma 1. If a purely or fully asynchronous cellular automaton $C = (R, N, Q, \delta)$ has a local transition function δ which is non-trivial in the sense that $\delta(\ell) \neq \ell(0)$ holds for at least one local configuration ℓ, then there are two different global configurations $\hat{c}$ and $\check{c}$ and two singleton sets of active cells $\{\hat{a}\}$ and $\{\check{a}\}$ such that $\Delta_{\{\hat{a}\}}(\hat{c}) = \Delta_{\{\check{a}\}}(\check{c})$. (Without loss of generality, we assume that 0 is in the neighborhood.)

In other words, there is a global configuration which has two different predecessors.

Lemma 1 shows that the requirement of R1 or R2 leaves the trivial identity as the only asynchronous cellular automaton. Later, in section 4 and section 6, we will see, that there are non-trivial asynchronous cellular automata satisfying R3. Therefore, for asynchronous cellular automata, R3 does not imply R1 or R2.

Proof (of Lemma 1). Let $\ell \in Q^N$ be a local configuration with $q' = \delta(\ell) \neq \ell(0) = q$.

Let $a \in R$ be a cell large enough such that the neighborhoods $(-a) + N$ and $a + N$ are disjoint. We consider a global configuration $c \in Q^R$ in which cell $-a$ and cell a both observe ℓ in their neighborhoods, that is, $c_{(-a)+N} = \ell$ and $c_{a+N} = \ell$, in particular $c(-a) = q$ and $c(a) = q$. Define two global configurations $\hat{c}$ and $\check{c}$ which are identical to c with the only exceptions $\hat{c}(-a) = q'$ and $\check{c}(a) = q'$ respectively. Since $q' \neq q$ the global configurations $\hat{c}$ and $\check{c}$ are different.

But the two global configurations $\hat{c}' = \Delta_{\{a\}}(\hat{c})$ and $\check{c}' = \Delta_{-a}(\check{c})$ are the same (see Figure 1).

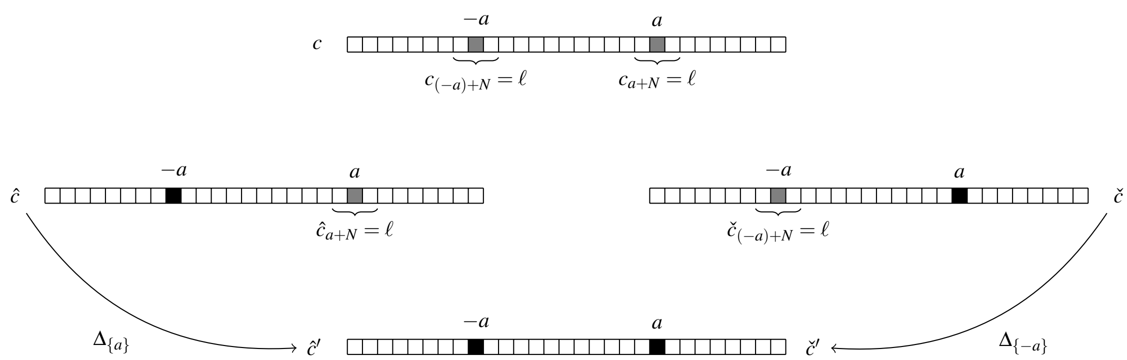


Figure 1: Displayed are the relevant parts of the global configurations c, $\hat{c}$, $\check{c}$, $\hat{c}'$ and $\check{c}'$ in the one-dimensional case, that is, $R = \mathbf{Z}$, with neighborhood $N = \{-1,0,1\}$. Grey colored cells are in state q and black colored cells in state q'. The state of white colored cells is not further specified. The solid directed lines indicate the two interesting global transitions.

- Each cell $i \notin \{-a,a\}$ is passive; therefore $\hat{c}'(i) = \hat{c}(i) = c(i) = \check{c}(i) = \check{c}'(i)$.
- $\hat{c}'(-a) = \hat{c}(-a) = q' = \delta(\ell) = \delta(\check{c}_{(-a)+N}) = \check{c}'(-a)$.
- $\hat{c}'(a) = \delta(\hat{c}_{a+N}) = \delta(\ell) = q' = \check{c}(a) = \check{c}'(a)$. ■

In a short presentation given at Automata 2011 Sarkar and Das [10] also consider some kind of "reversibility of 1-dimensional asynchronous cellular automata". We note that at least their setting is completely different from ours: They only look at finite configurations (with both, periodic and null, boundary conditions). And as far as we understand their Definitions 1 and 2 [10, p. 32], they call an asynchronous cellular automaton, restricted to configurations of a fixed length, reversible if each (finite) configuration has *at least one* predecessor. We do not pursue this line of thought.

Instead we use R3 as the guiding light. In order not to overload the word "reversible" with too many meanings we will speak of phase space invertibility, which we will often abbreviate as invertibility.

Definition 2. A purely asynchronous cellular automaton $C = (R,N,Q,\delta)$ is called *phase space invertible*, if there is a purely asynchronous cellular automaton $G = (R,M,Q,\gamma)$ such that for each pair of global configurations $c,c' \in Q^R$ holds

$$c' \in \Delta(c) \iff c \in \Gamma(c'),$$

which is equivalent to

$$\exists A \subseteq R : c' = \Delta_A(c) \iff \exists A' \subseteq R : c = \Gamma_{A'}(c').$$

We give a name to the set of cells on which two global configurations differ in

Definition 3. The *difference* $D_{c,c'}$ of two global configurations $c,c' \in Q^R$ is defined as

$$D_{c,c'} = \{i \in R \mid c_i \neq c'_i\}.$$

Note that $D_{c,c'} = D_{c',c}$.

Since for purely asynchronous cellular automata there are no restrictions on the set of active cells, cells which do not change their state when active, can be removed from the activity set without changing the outcome of a transition.

This suggests the characterization of invertibility given in

Lemma 4. Two purely asynchronous cellular automata $C = (R,N,Q,\delta)$ and $G = (R,M,Q,\gamma)$ are inverse to each other if, and only if, for each pair of global configurations $c,c' \in Q^R$ with $D_{c,c'} \neq \emptyset$ holds

$$c' = \Delta_{D_{c,c'}}(c) \iff c = \Gamma_{D_{c,c'}}(c').$$

Proof.

"only if": Let C and G be inverse to each other. Moreover let $c,c' \in Q^R$ be an arbitrary pair of global configurations with $D_{c,c'} \neq \emptyset$.

If $c' = \Delta_{D_{c,c'}}(c)$, then there exists a set of active cells A' such that $c = \Gamma_{A'}(c')$ by premise. In this case $D_{c,c'} \subseteq A'$, since $c_i = c'_i$ for each cell $i \notin A'$, and $c = \Gamma_{D_{c,c'}}(c')$, since $c_i = c'_i$ for each cell $i \in A' \setminus D_{c,c'}$. Therefore

$$c' = \Delta_{D_{c,c'}}(c) \Longrightarrow c = \Gamma_{D_{c,c'}}(c').$$

The other direction follows by symmetry.

"if": For each pair of global configurations $c,c' \in Q^R$ with $D_{c,c'} \neq \emptyset$ let

$$c' = \Delta_{D_{c,c'}}(c) \iff c = \Gamma_{D_{c,c'}}(c').$$

Moreover let $c,c' \in Q^R$ be an arbitrary pair of global configurations.

If there exists a set of active cells A such that $c' = \Delta_A(c)$, then $c' = \Delta_{D_{c,c'}}(c)$, since $D_{c,c'} \subseteq A$ and $c'_i = c_i$ for each cell $i \in A \setminus D_{c,c'}$. In this case we get $c = \Gamma_{D_{c,c'}}(c')$ by premise if $D_{c,c'} \neq \emptyset$ or by definition of $\Gamma_\emptyset$ and equality $c = c'$ if $D_{c,c'} = \emptyset$. Therefore

$$\exists A \subseteq R : c' = \Delta_A(c) \Longrightarrow c = \Gamma_{D_{c,c'}}(c').$$

The other direction follows by symmetry. ■

Analogously to definition 2 we define invertibility for fully asynchronous cellular automata in

Definition 5. A fully asynchronous cellular automaton $C = (R,N,Q,\delta)$ is called *phase space invertible*, if there is a fully asynchronous cellular automaton $G = (R,M,Q,\gamma)$ such that for each pair of global configurations $c,c' \in Q^R$ holds

$$c' \in \Delta_1(c) \iff c \in \Gamma_1(c'),$$

which is equivalent to

$$\exists a \in R : c' = \Delta_{\{a\}}(c) \iff \exists a' \in R : c = \Gamma_{\{a'\}}(c').$$

We characterize inversion for fully asynchronous cellular automata in

Lemma 6. Two fully asynchronous cellular automata $C = (R,N,Q,\delta)$ and $G = (R,M,Q,\gamma)$ are inverse to each other if, and only if, for each pair of global configurations $c,c' \in Q^R$ with $D_{c,c'} = \{a\}$ holds

$$c' = \Delta_{\{a\}}(c) \iff c = \Gamma_{\{a\}}(c')$$

and for each global configuration $c \in Q^R$ holds

$$\exists a \in R : c = \Delta_{\{a\}}(c) \iff \exists a' \in R : c = \Gamma_{\{a'\}}(c).$$

A comparison with the formulation in lemma 4 shows a complication. This is due to the fact that for fully asynchronous cellular automata one always needs at least one active cell.

Proof (of lemma 6). Let $c, c' \in Q^R$ be an arbitrary pair of global configurations. There are three cases to consider:

Case 1: $|D_{c,c'}| \geq 2$, that is, c and c' differ in more than one cell. Then $c' \notin \Delta_1(c)$ and $c \notin \Gamma_1(c')$, and hence

$$c' \in \Delta_1(c) \iff c \in \Gamma_1(c').$$

Case 2: $D_{c,c'} = \{a\}$, that is, c and c' differ in exactly one cell a. To reach c' in one step from c by C or c in one step from c' by G cell a must be active, thus

$$c' \in \Delta_1(c) \iff c' = \Delta_{\{a\}}(c) \text{ and}$$
$$c \in \Gamma_1(c') \iff c = \Gamma_{\{a\}}(c').$$

Case 3: $D_{c,c'} = \emptyset$, that is, $c = c'$. If c' can be reached from c by C, that is, $c' \in \Delta_1(c)$, c may not be reachable from c' by G or be reachable with a different active cell, and vice versa. Thus, a better characterization of this case is not as simple as in the other cases and is postponed until lemma 15.

Note, that for purely asynchronous cellular automata this case is trivial: Simply choose the empty activity set. ∎

Lemma 4 is only correct because we allow the activity set to be empty. If we would not allow this, we would additionally have to require for each global configuration $c \in Q^R$ that

$$\exists a \in R : c = \Delta_{\{a\}}(c) \iff \exists a' \in R : c = \Gamma_{\{a'\}}(c).$$

As we will see, this makes it much more difficult to prove completeness and decidability.

4 Turing Completeness

We will show that invertible purely asynchronous cellular automata are computationally universal.

It is known that reversible deterministic synchronous cellular automata are computationally universal. For one-dimensional cellular automata this can be shown by reversibly simulating reversible Turing machines which are computationally universal; for d-dimensional cellular automata the result holds too (more details can be found in [5]).

Hence any construction which transforms a reversible synchronous cellular automaton into an invertible asynchronous cellular automaton is sufficient to show the computational universality of the latter class. It turns out that Nakamura's method [8] of transforming the local transition function of any synchronous cellular automaton into one for an asynchronous cellular automaton while "basically preserving its global behavior" (irrespective of reversibility) is all that is needed for purely asynchronous updating.

In general the following transformations which maintain local synchronicity and guarantee invertibility are performed:

1. Each cell, additionally to its current state, remembers its previous state and manages a three-valued time stamp.

2. Each *active* cell only changes its state, if thereby no information that may be needed by neighboring cells is lost. This is the case, if neighboring cells have the same time stamp or are one step ahead.

3. Moreover, each active cell maintains its state, if the local configuration it observes is "illegal".

More specifically, let $C = (R, N, Q, \delta)$ and $G = (R, N, Q, \gamma)$ be two synchronous cellular automata with corresponding global transition functions Δ and Γ. The interesting case will be that they are inverse to each other, that is, $\Delta^{-1} = \Gamma$. Without loss of generality, we assume that the neighborhood includes 0 and is symmetric, that is, $N = \{-n \mid n \in N\}$.

We will now construct an asynchronous cellular automaton $\bar{C} = (R, N, \bar{Q}, \bar{\delta})$ from C. In order to save parentheses we will occasionally write ℓ_n instead of $\ell(n)$ ($\forall l \in Q^N, n \in N$) and c_i instead of $c(i)$ ($\forall c \in Q^R, i \in R$). The cellular automaton $\bar{C}$ is defined as follows:

1. The set of states is $\bar{Q} = Q \times Q \times \{0,1,2\}$. For $\bar{q} = (q_1, q_2, t) \in \bar{Q}$ we denote the first component as $curr(\bar{q})$, the second component as $old(\bar{q})$ and the third as $time(\bar{q})$.
2. Given a local configuration $\bar{\ell}$ we say, that cell 0 is *ahead* (of its neighbors) if, and only if, there is an $n \in N$ such that $time(\bar{\ell}_0) = time(\bar{\ell}_n) + 1 \pmod 3$.
3. If in a local configuration $\bar{\ell}$ of $\bar{C}$ cell 0 is *not* ahead define the *corresponding current local C-configuration* $curr(\bar{\ell})$ of C (not $\bar{C}$!) as

$$curr(\bar{\ell})_n = \begin{cases} curr(\bar{\ell}_n) & \text{if } time(\bar{\ell}_n) = time(\bar{\ell}_0), \\ old(\bar{\ell}_n) & \text{if } time(\bar{\ell}_n) = time(\bar{\ell}_0) + 1. \end{cases}$$

4. A local configuration $\bar{\ell}$ is *forward movable* if cell 0 is not ahead and $old(\bar{\ell}_0) = \gamma(curr(\bar{\ell}))$.
5. The local transition function $\bar{\delta}$ is then defined by

$$\bar{\delta}(\bar{\ell}) = \begin{cases} (\ \delta(curr(\bar{\ell})),\ curr(\bar{\ell}_0),\ time(\bar{\ell}_0) + 1\ ) & \text{if } \bar{\ell} \text{ is forward movable,} \\ \bar{\ell}_0 & \text{otherwise.} \end{cases}$$

Analogously, apply the following construction to G, resulting in $\bar{G} = (R, N, \bar{Q}, \bar{\gamma})$.

1. $\bar{Q} = Q \times Q \times \{0,1,2\}$, $curr(\bar{q})$, $old(\bar{q})$ and $time(\bar{q})$ are defined as above.
2. Given a local configuration $\bar{\ell}$ we say, that cell 0 is *behind* (of its neighbors) if, and only if, there is an $n \in N$ such that $time(\bar{\ell}_0) = time(\bar{\ell}_n) - 1 \pmod 3$.
3. If in a local configuration $\bar{\ell}$ of $\bar{C}$ cell 0 is *not* behind define the *corresponding old local C-configuration* $old(\bar{\ell})$ of G (not $\bar{G}$!) as

$$old(\bar{\ell})_n = \begin{cases} old(\bar{\ell}_n) & \text{if } time(\bar{\ell}_n) = time(\bar{\ell}_0), \\ curr(\bar{\ell}_n) & \text{if } time(\bar{\ell}_n) = time(\bar{\ell}_0) - 1. \end{cases}$$

4. A local configuration $\bar{\ell}$ is *backward movable* if cell 0 is not behind and $curr(\bar{\ell}_0) = \delta(old(\bar{\ell}))$.
5. The local transition function $\bar{\delta}$ is then defined by

$$\bar{\gamma}(\bar{\ell}) = \begin{cases} (\ old(\bar{\ell}_0),\ \gamma(old(\bar{\ell})),\ time(\bar{\ell}_0) - 1\ ) & \text{if } \bar{\ell} \text{ is backward movable,} \\ \bar{\ell}_0 & \text{otherwise.} \end{cases}$$

We now have

Theorem 7. *If C and G are synchronous cellular automata which are inverse to each other, then $\bar{C}$ and $\bar{G}$ are purely asynchronous inverses of each other.*

Proof. Let $c, c' \in \bar{Q}^R$ be two arbitrary global configurations and

$$D = D_{c,c'} = \{i \in R \mid c_i \neq c'_i\}$$

their difference. According to lemma 4 it is sufficient to prove

$$c' = \bar{\Delta}_D(c) \iff c = \bar{\Gamma}_D(c').$$

We will prove in detail that

$$c' = \bar{\Delta}_D(c) \Longrightarrow c = \bar{\Gamma}_D(c').$$

Because of the symmetry of the constructions is it not surprising that a proof of the inverse implication can be given analogously.

Now let $c' = \bar{\Delta}_D(c)$. Consider an arbitrary cell $i \in D$. Since $c_i \neq c'_i = \bar{\delta}(c_{i+N})$ the local configuration c_{i+N} is forward movable, because otherwise cell i would maintain its state by definition of $\bar{\delta}$. Therefore

$$\begin{aligned} \forall n \in N\colon\ \mathit{time}(c_{i+n}) &\in \{\mathit{time}(c_i), \mathit{time}(c_i)+1\}, \\ \mathit{old}(c_i) &= \gamma(\mathit{curr}(c_{i+N})), \\ \mathit{curr}(c'_i) &= \delta(\mathit{curr}(c_{i+N})), \\ \mathit{old}(c'_i) &= \mathit{curr}(c_i), \text{ and} \\ \mathit{time}(c'_i) &= \mathit{time}(c_i)+1. \end{aligned}$$

Now consider an arbitrary neighbor $n \in N$. Since we have assumed that N is symmetric, cell i is a neighbor of cell $i+n$.

There are two possible cases:

Case 1: $\mathit{time}(c_{i+n}) = \mathit{time}(c_i)$.

Case 1.1: $i+n \in D$: Then $c_{i+n} \neq c'_{i+n}$ and therefore

$$\begin{aligned} \mathit{time}(c'_{i+n}) &= \mathit{time}(c_{i+n})+1 = \mathit{time}(c_i)+1 = \mathit{time}(c'_i) \text{ and} \\ \mathit{curr}(c_{i+N})(n) &= \mathit{curr}(c_{i+n}) = \mathit{old}(c'_{i+n}) = \mathit{old}(c'_{i+N})(n). \end{aligned}$$

Case 1.2: $i+n \notin D$: Then $c_{i+n} = c'_{i+n}$ and therefore

$$\begin{aligned} \mathit{time}(c'_{i+n}) &= \mathit{time}(c_{i+n}) = \mathit{time}(c_i) = \mathit{time}(c'_i)-1 \text{ and} \\ \mathit{curr}(c_{i+N})(n) &= \mathit{curr}(c_{i+n}) = \mathit{curr}(c'_{i+n}) = \mathit{old}(c'_{i+N})(n). \end{aligned}$$

Case 2: $\mathit{time}(c_{i+n}) = \mathit{time}(c_i)+1$. Since i is a neighbor of $i+n$ and $\mathit{time}(c_i) = \mathit{time}(c_{i+n})-1$ we have $c'_{i+n} = c_{i+n}$. Therefore

$$\begin{aligned} \mathit{time}(c'_{i+n}) &= \mathit{time}(c_{i+n}) = \mathit{time}(c_i)+1 = \mathit{time}(c'_i) \text{ and} \\ \mathit{curr}(c_{i+N})(n) &= \mathit{old}(c_{i+n}) = \mathit{old}(c'_{i+n}) = \mathit{old}(c'_{i+N})(n). \end{aligned}$$

Taken together we always have

$$\begin{aligned} \mathit{time}(c'_{i+n}) &\in \{\mathit{time}(c'_i), \mathit{time}(c'_i)-1\} \text{ and} \\ \mathit{curr}(c_{i+N})(n) &= \mathit{old}(c'_{i+N})(n), \end{aligned}$$

and therefore $curr(c'_i) = \delta(curr(c_{i+N})) = \delta(old(c'_{i+N}))$. As a consequence c'_{i+N} is backward movable and hence

$$\begin{aligned} curr(c_i) &= old(c'_i) = curr(\Gamma_D(c')_i), \\ old(c_i) &= \gamma(curr(c_{i+N})) = \gamma(old(c'_{i+N})) = old(\Gamma_D(c')_i), \text{ and} \\ time(c_i) &= time(c'_i) - 1 = time(\Gamma_D(c')_i), \end{aligned}$$

which is a long-winded way of saying $c_i = \Gamma_D(c')_i$. ∎

The above proof is incorrect if we restrict purely asynchronous cellular automata to non-empty sets of active cells. The problematic case happens when the minimal difference D is empty. In this case we cannot use it as the set of active cells. But nevertheless $c' = c$ may be reachable from c by $\bar{C}$ in one step with a non-empty set of active cells and we need to prove that the same holds for $\bar{G}$. We do not know whether this is always the case.

One part of the problem is that even if c is a global configuration in which the registers old, current and time do not conform we need to show the property. This could be solved by adding more restrictions that do not corrupt the simulation as in the second part of the definitions of backward and forward movable.

5 Decidability

We will show that phase space invertibility is decidable for arbitrary-dimensional purely asynchronous cellular automata and one-dimensional fully asynchronous cellular automata by presenting two algorithms which always terminate and for any given automaton find an inverse if one exists.

For any given asynchronous cellular automaton C these algorithms only look for inverses among the finitely many automata with the same neighborhood as C. This is justified in section 5.1 where we prove that if C is invertible there is an inverse with the same neighborhood.

To decide whether two asynchronous cellular automata C and G are inverse to each other the algorithms only verify that C and G are inverse to each other on a subspace of the phase space that is restricted to global transitions in which only cells of a fixed finite subset of cells are active. In sections 5.2 and 5.3 we prove that this is sufficient for arbitrary-dimensional purely and one-dimensional fully asynchronous cellular automata respectively.

5.1 Inverse Neighborhood and Translation Invariance

Consider any set of all global configurations that agree on cell 0 and all its neighbors. For each of these global configurations make the global transition step where only cell 0 is active. All resulting global configurations again agree on cell 0 and all its neighbors.

An inverse can undo all these transitions, whereby at most cell 0 is active. Regardless of the states of cells besides cell 0 and its neighborhood, the inverse does the same and can thus not benefit from a larger neighborhood.

Definition 8. We say that a cellular automaton $C = (R, N, Q, \delta)$ has *minimal neighborhood* if it has no dummy neighbors, meaning for each neighbor $n \in N$ there exist local configurations $\ell, \ell' \in Q^N$ with $\ell|_{N\setminus\{n\}} = \ell'|_{N\setminus\{n\}}$ and $\ell_n \neq \ell'_n$ such that $\delta(\ell) \neq \delta(\ell')$.

With this term we can rigorously state and prove

Lemma 9. Two inverse purely or fully asynchronous cellular automata $C = (R, N, Q, \delta)$ and $G = (R, M, Q, \gamma)$ with minimal neighborhoods have the same neighborhood.

Proof (by contradiction). Assume $N \neq M$. Without loss of generality, let $N \setminus M \neq \emptyset$. Then there exists a neighbor $\mathring{n} \in N \setminus M$. Because the neighborhoods are minimal there also exist local configurations $\hat{\ell}, \check{\ell} \in Q^N$ with $\hat{\ell}|_{N\setminus\{\mathring{n}\}} = \check{\ell}|_{N\setminus\{\mathring{n}\}}$ and $\hat{\ell}_{\mathring{n}} \neq \check{\ell}_{\mathring{n}}$ such that $\delta(\hat{\ell}) \neq \delta(\check{\ell})$.

Choose global configurations $\hat{c}, \check{c} \in Q^R$ with

$$\hat{c}_{\mathring{n}} = \hat{\ell}_{\mathring{n}} \text{ as well as } \check{c}_{\mathring{n}} = \check{\ell}_{\mathring{n}},$$
$$\hat{c}|_{N\setminus\{\mathring{n}\}} = \hat{\ell}|_{N\setminus\{\mathring{n}\}} = \check{\ell}|_{N\setminus\{\mathring{n}\}} = \check{c}|_{N\setminus\{\mathring{n}\}}, \text{ and}$$
$$\hat{c}|_{R\setminus N} = \check{c}|_{R\setminus N}.$$

For the global configurations $\hat{d} = \Delta_{\{0\}}(\hat{c})$ and $\check{d} = \Delta_{\{0\}}(\check{c})$ we obtain

$$\hat{d}_0 = \delta(\hat{\ell}) \neq \delta(\check{\ell}) = \check{d}_0,$$
$$\hat{d}_{\mathring{n}} = \hat{\ell}_{\mathring{n}} \neq \check{\ell}_{\mathring{n}} = \check{d}_{\mathring{n}},$$
$$\hat{d}|_{R\setminus\{0,\mathring{n}\}} = \check{d}|_{R\setminus\{0,\mathring{n}\}}, \text{ and}$$
$$\hat{d}|_{R\setminus\{0\}} = \hat{c}|_{R\setminus\{0\}} \text{ as well as } \check{d}|_{R\setminus\{0\}} = \check{c}|_{R\setminus\{0\}}.$$

Because C and G are inverse to each other we furthermore have

$$\hat{c} = \Gamma_{D_{\hat{c},\hat{d}}}(\hat{d}) \text{ as well as } \check{c} = \Gamma_{D_{\check{c},\check{d}}}(\check{d}).$$

Note that $D_{\hat{c},\hat{d}}, D_{\check{c},\check{d}} \subseteq \{0\}$. See Figure 2 for a graphical representation of the situation.

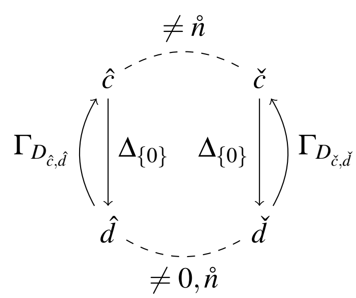


Figure 2: The labels on the dashed lines indicate on which cells the connected global configurations differ, the solid lines show possible global transitions.

Case 1: $\mathring{n} \neq 0$. Then $\hat{c}_0 = \check{c}_0$. With $\hat{d}_0 \neq \check{d}_0$ it follows that $\hat{d}_0 \neq \hat{c}_0$ or $\check{d}_0 \neq \check{c}_0$. Without loss of generality, let $\hat{d}_0 \neq \hat{c}_0$. Then $\hat{c} = \Gamma_{\{0\}}(\hat{d})$ and thus $\hat{c}_0 = \gamma(\hat{d}_{0+M})$.

Consider global configuration $\mathring{d} \in Q^R$ with $\mathring{d}|_{R\setminus\{0\}} = \check{d}|_{R\setminus\{0\}}$ and $\mathring{d}_0 = \hat{d}_0$. From $\mathring{n} \notin M$ we have $\mathring{d}_{0+M} = \hat{d}_{0+M}$ and therefore

$$\Gamma_{\{0\}}(\mathring{d})_0 = \gamma(\mathring{d}_{0+M}) = \gamma(\hat{d}_{0+M}) = \hat{c}_0 = \check{c}_0 \text{ and}$$
$$\Gamma_{\{0\}}(\mathring{d})|_{R\setminus\{0\}} = \mathring{d}|_{R\setminus\{0\}} = \check{d}|_{R\setminus\{0\}} = \check{c}|_{R\setminus\{0\}},$$

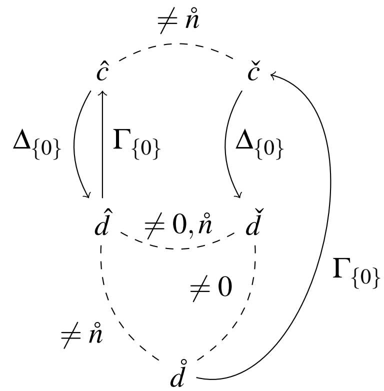


Figure 3: The labels on the dashed lines indicate on which cells the connected global configurations differ, the solid lines show possible global transitions.

which is a long-winded way of saying $\check{c} = \Gamma_{\{0\}}(\mathring{d})$.

From $\mathring{d}_0 = \hat{d}_0 \neq \hat{c}_0 = \check{c}_0$ we conclude $\mathring{d} = \Delta_{\{0\}}(\check{c}) = \check{d}$ and $\check{d}_0 = \mathring{d}_0 = \hat{d}_0$, which contradicts $\hat{d}_0 \neq \check{d}_0$; see Figure 3.

Case 2: $\mathring{n} = 0$. Then $0 \notin M$ and hence $\hat{d}_{0+M} = \check{d}_{0+M}$ as well as $\hat{c}_{0+M} = \check{c}_{0+M}$.

Case 2.1: $\hat{c}_0 \neq \hat{d}_0$ or $\check{c}_0 \neq \check{d}_0$. Without loss of generality, let $\hat{c}_0 \neq \hat{d}_0$. Then $\hat{c} = \Gamma_{\{0\}}(\hat{d})$ and therefore $\hat{c}_0 = \gamma(\hat{d}_{0+M})$. With $\hat{d}_{0+M} = \check{d}_{0+M}$ it follows that $\hat{c}_0 = \gamma(\check{d}_{0+M})$ and therefore $\check{c} \neq \hat{c} = \Gamma_{\{0\}}(\check{d})$. Hence $\check{c} = \Gamma_{\emptyset}(\check{d}) = \check{d}$ and $\hat{c} = \Gamma_{\{0\}}(\check{c})$.

Finally, from $\hat{c}_0 \neq \check{c}_0$ we have $\check{c} = \Delta_{\{0\}}(\hat{c})$ and conclude $\check{d} = \hat{d}$, which contradicts $\hat{d}_0 \neq \check{d}_0$; see Figure 4.

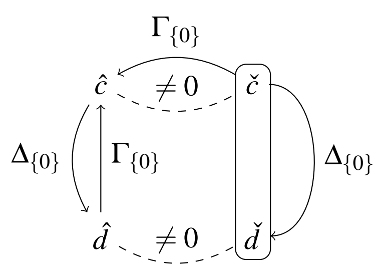


Figure 4: The labels on the dashed lines indicate on which cells the connected global configurations differ, the solid lines show possible global transitions, and the rounded rectangle signifies that the enclosed global configurations are found to be identical during the proof.

Case 2.2: $\hat{c}_0 = \hat{d}_0$ and $\check{c}_0 = \check{d}_0$. Then $\hat{c} = \hat{d}$ and $\check{c} = \check{d}$. Consider global configuration $b = \Gamma_{\{0\}}(\hat{c}) = \Gamma_{\{0\}}(\check{c})$. From $\hat{c}_0 \neq \check{c}_0$ it follows that $b_0 \neq \hat{c}_0$ or $b_0 \neq \check{c}_0$.

Without loss of generality, let $b_0 \neq \hat{c}_0$. Then $\hat{c} = \Delta_{\{0\}}(b)$ and with $\check{c} \neq \hat{c}$ also $\check{c} = \Delta_{\emptyset}(b) = b$. Therefore $\hat{c} = \Delta_{\{0\}}(\check{c}) = \check{d} = \check{c}$, which contradicts $\hat{c}_0 \neq \check{c}_0$; see Figure 5.

Because every possible case led to a contradiction, the assumption must be false, meaning $N = M$. ■

Since dummy neighbors can be added and removed without affecting the phase space we get

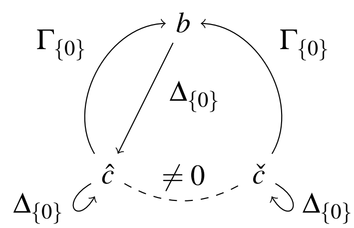


Figure 5: The labels on the dashed lines indicate on which cells the connected global configurations differ, the solid lines show possible global transitions.

Corollary 10. For each invertible asynchronous cellular automaton exists an inverse with the same neighborhood. ■

Let us now briefly consider translation invariance.

Definition 11. For each *translation vector* $j \in R$ the map $\tau_j : Q^R \to Q^R$ with

$$\forall i \in R\colon\ \tau_j(c)_i = c_{i+j}$$

is called *(j-)translation*.

Similar to synchronous, asynchronous cellular automata are also translation invariant – we only need to translate the set of active cells as well.

Lemma 12. For each translation vector $j \in R$ and set of active cells $A \subseteq R$ holds

$$\tau_j \circ \Delta_A = \Delta_{A-j} \circ \tau_j,$$

where $A - j = \{a - j \mid a \in A\}$. ■

5.2 Purely Asynchronous Cellular Automata

If every transition of a purely asynchronous cellular automaton, in which only cell 0 is active as well as arbitrary neighbors of cell 0, can be inverted by another automaton and vice versa, then these automata are inverse to each other.

This follows from locality properties and translation invariance of purely asynchronous cellular automata and is rigorously stated and proved in

Lemma 13. Two purely asynchronous cellular automata $C = (R,N,Q,\delta)$ and $G = (R,N,Q,\gamma)$ are inverse to each other if, and only if, for each pair of global configurations $c, c' \in Q^R$ with $0 \in D_{c,c'} \subseteq \{0\} \cup N$ holds

$$c' = \Delta_{D_{c,c'}}(c) \iff c = \Gamma_{D_{c,c'}}(c').$$

Proof. The forward direction follows directly from lemma 4. For the backward direction consider any global configurations $c, c' \in Q^R$.

First, let $c' = \Delta_{D_{c,c'}}(c)$. Choose any cell $i \in D_{c,c'}$ that changes its state during the transition from c to c'. Because C and G are translation invariant it follows that

$$\begin{aligned}
\Gamma_{D_{c,c'}}(c')_i &= \tau_i(\Gamma_{D_{c,c'}}(c'))_0 \\
&= \Gamma_{D_{c,c'}-i}(\tau_i(c'))_0 \\
&= \Gamma_{D_{c,c'}-i}(\tau_i(\Delta_{D_{c,c'}}(c)))_0 \\
&= \Gamma_{D_{c,c'}-i}(\Delta_{D_{c,c'}-i}(\tau_i(c)))_0.
\end{aligned}$$

With $0 \in D_{c,c'} - i$ we further get

$$\Gamma_{D_{c,c'}}(c')_i = \Gamma_{\{0\}}(\Delta_{D_{c,c'}-i}(\tau_i(c)))_0.$$

Because the transition state of cell 0 only depends on its neighborhood we farther deduce

$$\Gamma_{D_{c,c'}}(c')_i = \Gamma_{\{0\}}(\underbrace{\Delta_{(D_{c,c'}-i)\cap(\{0\}\cup N)}(\overbrace{\tau_i(c)}^{d})}_{d'})_0.$$

We can make cells, which do not change state if active, inactive without disturbing the transition from d to d':

$$d' = \Delta_{D_{d,d'}}(d).$$

The difference $D_{d,d'}$ of d and d' is contained in $\{0\} \cup N$. From $0 \in (D_{c,c'} - i) \cap (\{0\} \cup N)$ we furthermore get

$$d_0 = \tau_i(c)_0 = c_i \neq c'_i = \delta(c_{i+N}) = \delta(\tau_i(c)_{0+N}) = d'_0,$$

meaning $0 \in D_{d,d'}$. Thus, by the premise of the backward direction,

$$d = \Gamma_{D_{d,d'}}(d')$$

and therefore

$$\Gamma_{D_{c,c'}}(c')_i = \Gamma_{\{0\}}(d')_0 = \Gamma_{D_{d,d'}}(d')_0 = d_0 = \tau_i(c)_0 = c_i.$$

Because cell i was arbitrarily chosen, $c = \Gamma_{D_{c,c'}}(c')$ and we have proved the implication

$$c' = \Delta_{D_{c,c'}}(c) \Longrightarrow c = \Gamma_{D_{c,c'}}(c').$$

By symmetry we conclude that the implication

$$c' = \Delta_{D_{c,c'}}(c) \Longleftarrow c = \Gamma_{D_{c,c'}}(c')$$

holds as well. Because the global configurations c and c' were arbitrarily chosen C and G are inverse to each other according to lemma 4. ■

Thus only cell 0, the neighbors of cell 0, and the neighbors of each neighbor of cell 0 have to be considered when testing whether two purely asynchronous cellular automata are inverse to each other, because the transition state of cell 0 or an arbitrary neighbor of cell 0 only depends in its observed local configuration.

Theorem 14. *Phase space invertibility is decidable for purely asynchronous cellular automata.*

Proof. Let $C = (R, N, Q, \delta)$ be an arbitrary purely asynchronous cellular automaton. Consider each of the $|Q|^{|Q|^{|N|}}$ purely asynchronous cellular automata $G = (R, N, Q, \gamma)$ in turn. Test whether for each pair $c, c' \in Q^{\{0\} \cup N \cup (N+N)}$ of finite configurations with $0 \in D_{c,c'} \subseteq \{0\} \cup N$ the equivalence

$$c' = \Delta_{D_{c,c'}}(c) \iff c = \Gamma_{D_{c,c'}}(c')$$

holds. If this is the case for one automaton G, then C and G are inverse to each other, and C is invertible. Otherwise C is not invertible.

Note that $N + N = \{m + n \mid m, n \in N\}$. ■

Remark. It is in fact not necessary to consider all purely asynchronous cellular automata with the same neighborhood. One can show that if C is invertible, the local transition function of its inverse can be constructed easily from the one of C. Thus only one candidate needs to be considered.

For further details see [11, section 3.5].

The above proofs are incorrect if we restrict purely asynchronous cellular automata to non-empty sets of active cells. But at least in the one-dimensional case invertibility is still decidable (see [11, section 6.2.3] for further details). This is due to the fact that a purely asynchronous cellular automaton that is restricted to non-empty sets of active cells is invertible if, and only if, it is invertible without the restriction and invertible as fully asynchronous cellular automaton (see [11, corollary 3.15]). And, as we shall see, the latter is decidable in the one-dimensional case.

5.3 Fully Asynchronous Cellular Automata

In each transition of fully asynchronous cellular automata exactly one cell is active, whereas in each transition of purely asynchronous cellular automata arbitrary cells are active. One may naively think that deciding invertibility is thus easier for fully than for purely asynchronous cellular automata. But the added difficulty is that in every transition one cell *must* be active.

To decide whether two fully asynchronous cellular automata C and G are inverse to each other or not, we have to show that if C transits a global configuration c into c' when cell a is active, that G can transit c' into c when a cell a' is active or that no such cell exists, and vice versa.

If $c_a \neq c'_a$ then a and a' must be the same cell or C and G are not inverse. But if $c_a = c'_a$ then a and a' are in general different cells. So how or where can we either find such a cell a' or decide that no such cell exists?

We answer this question in the one-dimensional case in

Lemma 15. Let $C = (\mathbf{Z}, N, Q, \delta)$ and $G = (\mathbf{Z}, N, Q, \gamma)$ be two fully asynchronous one-dimensional cellular automata. Define the maximal distance of a neighbor

$$m = \begin{cases} \max_{n \in N} |n| & \text{if } N \neq \emptyset, \\ -\infty & \text{if } N = \emptyset, \end{cases}$$

and a finite set of active candidate cells

$$\mathscr{A} = \begin{cases} \{-|Q|^{2m+1}, \ldots, -1, 0, 1, \ldots, |Q|^{2m+1}\} & \text{if } m \neq -\infty, \\ \{0\} & \text{if } m = -\infty. \end{cases}$$

Then C and G are inverse to each other if, and only if, for each pair of global configurations $c, c' \in Q^{\mathbf{Z}}$ with $D_{c,c'} = \{0\}$

$$c' = \Delta_{\{0\}}(c) \iff c = \Gamma_{\{0\}}(c') \tag{1}$$

and for each pair of global configurations $c, c' \in Q^{\mathbf{Z}}$ with $c = \Delta_{\{0\}}(c)$ and $c' = \Gamma_{\{0\}}(c')$

$$\exists a \in \mathscr{A} : c = \Gamma_{\{a\}}(c) \text{ and } \exists a' \in \mathscr{A} : c' = \Delta_{\{a'\}}(c'). \tag{2}$$

In the proof of this lemma we need segments of global configurations: For each global configuration $c \in Q^{\mathbf{Z}}$ and cells $i, j \in \mathbf{Z}$ with $i \leq j$ the (i, j)-*segment (of c)* $c[i, j]$ is the $(j - i + 1)$-tuple with $c[i, j] = (c_i, c_{i+1}, \ldots, c_j)$.

Proof. If there is only one state, meaning $|Q| = 1$, there is nothing to show. We thus assume $|Q| \geq 2$.

"⇒" Let C and G be inverse to each other. According to lemma 6: Equality (1) holds. To show equality (2) we choose an arbitrary global configuration $c \in Q^{\mathbf{Z}}$ with $c = \Delta_{\{0\}}(c)$ or $c = \Gamma_{\{0\}}(c)$.

At first, let $N = \emptyset$. Then $\mathscr{A} = \{0\}$. Let $c = \Delta_{\{0\}}(c)$. Assume $c \neq \Gamma_{\{0\}}(c)$. Consider the global configuration $c' \in Q^{\mathbf{Z}}$ with $c'_i = c_0$ for each cell $i \in \mathbf{Z}$. Then $c' = \Delta_{\{0\}}(c')$ and from

$$c'_a = c_0 \neq \Gamma_{\{0\}}(c)_0 = \Gamma_{\{a\}}(c')_a$$

we obtain $c' \neq \Gamma_{\{a\}}(c')$ for each cell $a \in \mathbf{Z}$ – in contradiction to the premise that C and G are inverse to each other. Thus our assumption is false, meaning $c = \Gamma_{\{0\}}(c)$. One can analogously show that $c = \Gamma_{\{0\}}(c)$ implies $c = \Delta_{\{0\}}(c)$.

Now let $N \neq \emptyset$. The idea is to construct a global configuration $c' \in Q^{\mathbf{Z}}$ for which

$$c' \in \Delta_1(c') \iff c' \in \Gamma_1(c')$$

does *not* hold if there does *not* exist a cell $a \in \mathscr{A}$ such that $c = \Gamma_{\{a\}}(c)$, if $c = \Delta_{\{0\}}(c)$, or $c = \Delta_{\{a\}}(c)$, if $c = \Gamma_{\{0\}}(c)$, contradicting that C and G are inverse to each other.

Consider the $|Q|^{2m+1} + 1$ segments

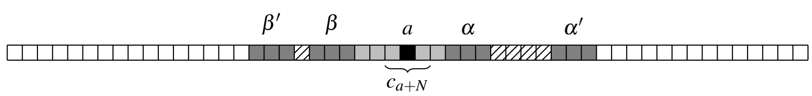
$$c[\alpha - m, \alpha + m]$$

for cells $\alpha \in \{0, 1, \ldots, |Q|^{2m+1}\}$. Because at most $|Q|^{2m+1}$ of these segments are different, there exist two cells $\alpha, \alpha' \in \{0, 1, \ldots, |Q|^{2m+1}\}$ with $\alpha < \alpha'$ such that

$$c[\alpha - m, \alpha + m] = c[\alpha' - m, \alpha' + m].$$

Analogously there exist two cells $\beta', \beta \in \{-|Q|^{2m+1}, \ldots, 1, 0\}$ with $\beta' < \beta$ such that

$$c[\beta' - m, \beta' + m] = c[\beta - m, \beta + m].$$

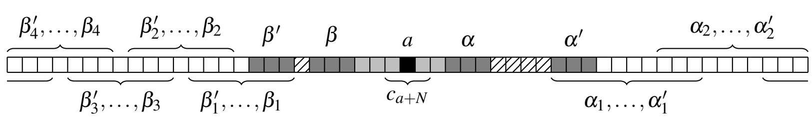


Define for each index $k \in \mathbf{N}$

$$\alpha_k = \alpha' + (k-1)(\alpha' - \alpha) - m,$$
$$\alpha'_k = \alpha' + k(\alpha' - \alpha) + m,$$

and

$$\beta_k = \beta' - (k-1)(\beta - \beta') + m,$$
$$\beta'_k = \beta' - k(\beta - \beta') - m.$$



Because $0 \leq \alpha < \alpha'$ and $\beta' < \beta \leq 0$ we have

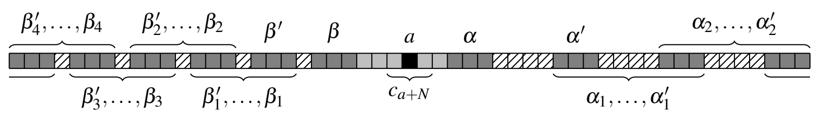
$$-m < \alpha' - m = \alpha_1 < \alpha_2 < \alpha_3 < \cdots,$$
$$m < \alpha' + m < \alpha' + (\alpha' - \alpha) + m = \alpha'_1 < \alpha'_2 < \alpha'_3 < \cdots,$$

and

$$\cdots < \beta_3 < \beta_2 < \beta_1 = \beta' + m < m,$$
$$\cdots < \beta'_3 < \beta'_2 < \beta'_1 = \beta' - (\beta - \beta') - m < \beta' - m < -m.$$

Consider the global configuration $c' \in Q^{\mathbf{Z}}$ with

$$c'[\beta' - m, \alpha' + m] = c[\beta' - m, \alpha' + m],$$
$$c'[\alpha_k, \alpha'_k] = c[\alpha - m, \alpha' + m] \ (\forall k \in \mathbf{N}), \text{ and}$$
$$c'[\beta'_k, \beta_k] = c[\beta' - m, \beta + m] \ (\forall k \in \mathbf{N}).$$



This global configuration exists, because the rear and front parts in which the definitions of two consecutive indices overlap are precisely the identical segments $c[\alpha - m, \alpha + m] = c[\alpha' - m, \alpha' + m]$ and $c[\beta' - m, \beta' + m] = c[\beta - m, \beta + m]$. In fact, for each index $k \in \mathbf{N}$ it holds that

$$\{\beta' - m, \ldots, \alpha' + m\} \cap \{\alpha_1, \ldots, \alpha'_1\} = \{\alpha' - m, \ldots, \alpha' + m\},$$
$$\{\beta' - m, \ldots, \alpha' + m\} \cap \{\beta'_1, \ldots, \beta_1\} = \{\beta' - m, \ldots, \beta' + m\},$$
$$\begin{aligned}\{\alpha_k, \ldots, \alpha'_k\} \cap \{\alpha_{k+1}, \ldots, \alpha'_{k+1}\} &= \{\alpha_{k+1}, \ldots, \alpha'_k\} \\ &= \{\alpha'_k - 2m, \ldots, \alpha'_k\} \\ &= \{\alpha_{k+1}, \ldots, \alpha_{k+1} + 2m\},\end{aligned}$$
$$\begin{aligned}\{\beta'_k, \ldots, \beta_k\} \cap \{\beta'_{k+1}, \ldots, \beta_{k+1}\} &= \{\beta'_k, \ldots, \beta_{k+1}\} \\ &= \{\beta'_k, \ldots, \beta'_k + 2m\} \\ &= \{\beta_{k+1} - 2m, \ldots, \beta_{k+1}\},\end{aligned}$$

and

$$\begin{aligned}c'[\alpha' - m, \alpha' + m] &= c[\alpha' - m, \alpha' + m] \\ &= c[\alpha - m, \alpha + m] = c'[\alpha_1, \alpha_1 + 2m], \\ c'[\beta' - m, \beta' + m] &= c[\beta' - m, \beta' + m] \\ &= c[\beta - m, \beta + m] = c'[\beta_1 - 2m, \beta_1], \\ c'[\alpha'_k - 2m, \alpha'_k] &= c[\alpha' - m, \alpha' + m] \\ &= c[\alpha - m, \alpha + m] = c'[\alpha_{k+1}, \alpha_{k+1} + 2m], \\ c'[\beta'_k, \beta'_k + 2m] &= c[\beta' - m, \beta' + m] \\ &= c[\beta - m, \beta + m] = c'[\beta_{k+1} - 2m, \beta_{k+1}].\end{aligned}$$

We will now show that for the global configuration c' any cell whose neighborhood is not contained in $\mathscr{A}$ behaves the same as at least one cell whose neighborhood is contained in $\mathscr{A}$ if active. Note that $N \subseteq \{-m, \dots, m\}$.

Assertion. For each cell $a' \in \mathbf{Z}$ exists a cell $a \in \{\beta', \dots, \alpha'\}$ such that

$$c'[a'-m, a'+m] = c[a-m, a+m].$$

Proof. Let $a' \in \mathbf{Z}$ be an arbitrary cell. If $a' \in \{\beta', \dots, \alpha'\}$, choose $a = a'$. From now on let $a' \notin \{\beta', \dots, \alpha'\}$. We consider only the case that $a' \geq \alpha' + 1$. The case $a' \leq \beta' - 1$ can be shown analogously. Thus let $a' \geq \alpha' + 1$.

Because $a' \geq \alpha_1 + m$ and each of the intersections $\{\alpha_k, \dots, \alpha'_k\} \cap \{\alpha_{k+1}, \dots, \alpha'_{k+1}\}$ ($\forall k \in \mathbf{N}$) and $\{a'-m, \dots, a+m\}$ contains $2m+1$ elements, there exists an index $k \in \mathbf{N}$ such that

$$\{a'-m, \dots, a'+m\} \subseteq \{\alpha_k, \dots, \alpha'_k\}.$$

Choose $a = a' - k(\alpha' - \alpha) \in \{\alpha, \dots, \alpha'\}$. Then

$$c'[a'-m, a'+m] = c[a-m, a+m]. \qquad \blacksquare$$

With this setup we can conclude the proof:

1. Let $c = \Delta_{\{0\}}(c)$. From
$$c'[-m, m] = c[-m, m]$$
we have $c' = \Delta_{\{0\}}(c')$. Assume that for each cell $a \in \mathscr{A}$ it holds that $c \neq \Gamma_{\{a\}}(c)$. Choose an arbitrary cell $a' \in \mathbf{Z}$. Then there is a cell $a \in \{\beta', \dots, \alpha'\}$ such that
$$c'[a'-m, a'+m] = c[a-m, a+m].$$
Because cell a is contained in $\mathscr{A}$ it holds that
$$c'_{a'} = c_a \neq \Gamma_{\{a\}}(c)_a = \Gamma_{\{a'\}}(c')_{a'}$$
and thus $c' \neq \Gamma_{\{a'\}}(c')$. Because $a' \in \mathbf{Z}$ was arbitrarily chosen, this contradicts the premise that C and G are inverse to each other. Thus our assumption is false, meaning there exists a cell $a \in \mathscr{A}$ with $c = \Gamma_{\{a\}}(c)$.
2. Let $c = \Gamma_{\{0\}}(c)$. Like above one shows that there exists a cell $a' \in \mathscr{A}$ with $c = \Delta_{\{a'\}}(c)$.

Therefore equality (2) holds as well.

"$\Leftarrow$" For each pair of global configurations $c, c' \in Q^R$ with $D_{c,c'} = \{a\}$ holds

$$\begin{aligned} c' = \Delta_{\{a\}}(c) &\iff \tau_a(c') = \tau_a(\Delta_{\{a\}}(c)) = \Delta_{\{0\}}(\tau_a(c)) \\ &\overset{eq.\ (1)}{\iff} \tau_a(c) = \Gamma_{\{0\}}(\tau_a(c')) = \tau_a(\Gamma_{\{a\}}(c')) \\ &\iff c = \Gamma_{\{a\}}(c'), \end{aligned}$$

and for each global configuration $c \in Q^R$ holds

$$\begin{aligned} &\exists a \in R : c = \Delta_{\{a\}}(c) \\ &\quad \Longrightarrow \tau_a(c) = \tau_a(\Delta_{\{a\}}(c)) = \Delta_{\{0\}}(\tau_a(c)) \\ &\quad \overset{eq.\ (2)}{\Longrightarrow} \exists a' \in R : \tau_a(c) = \Gamma_{\{a'\}}(\tau_a(c)) = \tau_a(\Gamma_{\{a'+a\}}(c)) \\ &\quad \Longrightarrow c = \Gamma_{\{a'+a\}}(c), \end{aligned}$$

and analogously

$$\exists a \in R : c = \Gamma_{\{a\}}(c) \Longrightarrow \exists a' \in R : c = \Delta_{\{a'+a\}}(c).$$

Hence, by lemma 6, the cellular automata C and G are inverse to each other. ■

Thus only cell 0, the neighbors of cell 0, the cells in $\mathscr{A}$ and the neighbors of each cell in $\mathscr{A}$ have to be considered when testing whether two fully asynchronous one-dimensional cellular automata are inverse to each other.

Theorem 16. *Phase space invertibility is decidable for fully asynchronous one-dimensional cellular automata.*

Proof. Let $C = (\mathbf{Z}, N, Q, \delta)$ be an arbitrary fully asynchronous one-dimensional cellular automaton and $T = \{0\} \cup N \cup \mathscr{A} \cup (\mathscr{A} + N)$. Consider each of the $|Q|^{|Q|^{|N|}}$ fully asynchronous one-dimensional cellular automata $G = (\mathbf{Z}, N, Q, \gamma)$ in turn. Test whether for each pair $c, c' \in Q^T$ of finite configurations with $D_{c,c'} = \{0\}$ the equivalence

$$c' = \Delta_{\{0\}}(c) \iff c = \Gamma_{\{0\}}(c')$$

holds and for each finite configuration $c \in Q^T$ the implications

$$c = \Delta_{\{0\}}(c) \Longrightarrow \exists a \in \mathscr{A} : c = \Gamma_{\{a\}}(c) \text{ and}$$
$$c = \Gamma_{\{0\}}(c) \Longrightarrow \exists a \in \mathscr{A} : c = \Delta_{\{a\}}(c)$$

hold. If this is the case for one automaton G, then C and G are inverse to each other, and C is invertible. Otherwise C is not invertible.

Note that $\mathscr{A} + N = \{a + n \mid a \in \mathscr{A}, n \in N\}$. ■

Remark. It is in fact not necessary to consider all fully asynchronous one-dimensional cellular automata with the same neighborhood and states. One can show that if C is invertible, the local transition function of its inverse can be constructed easily from the one of C. Thus only one candidate needs to be considered.

For further details see [11, section 3.5].

6 Elementary Cellular Automata

Using the decision algorithms it can easily be shown that exactly the purely asynchronous elementary cellular automata with Wolfram numbers 0, 35, 43, 49, 51, 59, 113, 115, 204, and 255 are invertible and that exactly the fully asynchronous elementary cellular automata with Wolfram numbers 33, 35, 38, 41, 43, 46, 49, 51, 52, 54, 57, 59, 60, 62, 97, 99, 102, 105, 107, 108, 113, 115, 116, 118, 121, 123, 131, 139, 145, 147, 150, 153, 155, 156, 195, 198, 201, 204, 209, and 211 are invertible.

For further details see [11, chapter 7].

7 Summary and Outlook

We have introduced a definition of invertibility for asynchronous cellular automata, namely phase space invertibility, and shown that invertible purely asynchronous cellular automata are computationally universal, that invertibility can be decided for arbitrary-dimensional purely and one-dimensional fully asynchronous cellular automata.

It remains open whether invertible fully asynchronous cellular automata are computationally universal and whether invertibility for higher-dimensional fully asynchronous cellular automata is decidable. If the latter can be shown to be true then invertibility of purely asynchronous cellular automata restricted to non-empty sets of active cells would also be decidable in higher dimensions.

References

[1] Serafino Amoroso and Yale N. Patt. Decision procedures for surjectivity and injectivity of parallel maps for tessellation structures. *Journal of Computer and System Sciences*, 6(5):448–464, 1972, doi:10.1016/S0022-0000(72)80013-8.

[2] Nazim Fatès and Lucas Gerin. Examples of fast and slow convergence of 2D asynchronous cellular systems. In Hiroshi Umeo et al., editor, *Proceedings ACRI 2008*, pages 184–191, 2008, doi:10.1007/978-3-540-79992-4_24.

[3] Ulrich Golze. (A-)synchronous (non-)deterministic cell spaces simulating each other. *Journal of Computer and System Sciences*, 17(2):176–193, 1978, doi:10.1016/0022-0000(78)90003-X.

[4] Jarkko Kari. Reversibility and surjectivity problems of cellular automata. *Journal of Computer and System Sciences*, 48(1):149–182, 1994, doi:10.1016/S0022-0000(05)80025-X.

[5] Jarkko Kari. Reversible cellular automata. In Clelia de Felice and Antonio Restivo, editors, *Developments in Language Theory*, volume 3572 of *Lecture Notes in Computer Science*, pages 57–68, 2005, doi:10.1007/11505877_5.

[6] Jia Lee, Susumi Adachi, Ferdinand Peper, and Kenichi Morita. Asynchronous game of life. *Physica D*, 194(3-4):369–384, 2004, doi:10.1016/j.physd.2004.03.007.

[7] Kenichi Morita and Masateru Harao. Computation universality of one-dimensional reversible (injective) cellular automata. *IEICE Transactions*, E72:758–762, 1989.

[8] Katsuo Nakamura. Asynchronous cellular automata and their computational ability. *Systems, Computers, Control*, 5(5):58–66, 1974.

[9] Damien Regnault, Nicolas Schabanel, and Eric Thierry. Progresses in the analysis of stochastic 2D cellular automata: A study of asynchronous 2D minority. *Theoretical Computer Science*, 410(47-49):4844–4855, 2009, doi:10.1016/j.tcs.2009.06.024.

[10] Anindita Sarkar and Sukanta Das. On the reversibility of 1-dimensional asynchronous cellular automata. In Nazim Fatès et al., editor, *Local Proceedings of Automata 2011*, pages 29–40, 2011.

[11] Simon Wacker. Reversibilität asynchroner Zellularautomaten. Diploma thesis, Karlsruhe Institute of Technology, Department of Informatics, 2012.