An Iterative Fixpoint Semantics for MKNF Hybrid Knowledge Bases with Function Symbols

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Hybrid Knowledge Bases based on Lifschitz's logic of Minimal Knowledge with Negation as Failure are a successful approach to combine the expressivity of Description Logics and Logic Programming in a single language. Their syntax, defined by Motik and Rosati, disallows function symbols. In order to define a well-founded semantics for MKNF HKBs, Knorr et al. define a partition of the modal atoms occurring in it, called the alternating fixpoint partition. In this paper, we propose an iterated fixpoint semantics for HKBs with function symbols. We prove that our semantics extends Knorr et al.'s, in that, for a function-free HKBs, it coincides with its alternating fixpoint partition. The proposed semantics lends itself well to a probabilistic extension with a distribution semantic approach, which is the subject of future work.

1 Introduction

When modelling complex domains it is of foremost importance to choose the logic that better fits with what must be represented. Therefore, many languages have been defined, based on First Order Logic such as Logic Programming (LP) or Description Logic (DL). These languages share many similarities but, on the other hand, they differ in the domain closure assumption they make: closed-world assumption for LP and open-world assumption for DLs.

Since many domains, such as legal reasoning [1], require different closure assumptions to coexist in the same model, combinations of LP and DL have been proposed by several authors. One of the most effective approaches is called Minimal Knowledge with Negation as Failure (MKNF) [7]. MKNF was then applied to define hybrid knowledge bases (HKBs) [10], which are defined as the combination of a logic program and a DL KB.

In the original HKB language, function symbols are not allowed. However, this is a feature that is useful in many domains. Consider, for example, the behaviour of a virus, which can mutate and spillover may happen due to each mutation. To trace the evolution of a virus, it is necessary to identify the sequence of spillover events starting from the initial version of the virus. We can represent the spillover count by Peano numbers, by means of a function symbol $s/1$ modelling that, e.g., $s(Y)$ represents the spillover event that follows the spillover identified by Y, which may have happened after another spillover, and so forth.

In this paper, we propose to extend the HKB syntax with function symbols, and we present an iterated fixpoint semantics for HKBs with Function Symbols (HKB^{FS}). We prove that our semantics coincides with that of [5] and [8] in the case of HKBs not including function symbols, and therefore can be considered an extension of that semantics to the case with function symbols.

The proposed semantics will also serve as the basis for a further (probabilistic) extension of the language, based on a distribution semantics approach, which is the subject of an ongoing effort.

We provide the necessary background notions in Section 2. We define the syntax and semantics of HKB^{FS}s in Section 3. In Section 4, we prove that our semantics extends Knorr et al's. We conclude the paper in Section 5.

2   Background

In this section, we provide the necessary background notions on the syntax and semantics of the language of MKNF Hybrid Knowledge Bases, which we extend with function symbols in Section 3. We start with Description Logics, which are a part of the language of HKBs.

2.1   Description Logics

Description Logics (DLs) are decidable fragments of First Order Logic used to model ontologies [3]. Usually their syntax is based on concepts and roles, corresponding to unary and binary predicates, respectively. In the following we briefly recall the DL $\mathscr{ALC}$; see [2] for a complete introduction to DLs.

$\mathscr{ALC}$'s alphabet is composed of a set $\mathbf{C}$ of *atomic concepts*, a set $\mathbf{R}$ of *atomic roles* and a set $\mathbf{I}$ of individuals. A *concept C* is defined by:

$$C ::= C_1 \mid \bot \mid \top \mid (C \sqcap C) \mid (C \sqcup C) \mid \neg C \mid \exists R.C \mid \forall R.C$$

where $C_1 \in \mathbf{C}$ and $R \in \mathbf{R}$.

A *TBox* $\mathscr{T}$ is a finite set of *concept inclusion axioms* $C \sqsubseteq D$, where C and D are concepts. An *ABox* $\mathscr{A}$ is a finite set of *concept membership axioms* $a : C$ and *role membership axioms* $(a,b) : R$, where C is a concept, $R \in \mathbf{R}$ and $a,b \in \mathbf{I}$. An $\mathscr{ALC}$ knowledge base $\mathscr{K} = (\mathscr{T}, \mathscr{A})$ consists of a TBox $\mathscr{T}$ and an ABox $\mathscr{A}$.

DL axioms can be mapped to FOL formulas by the transformation π shown in Table 1 for the $\mathscr{ALC}$ DL [15]. π is applied to concepts as follows:

$$
\begin{aligned}
\pi_x(A) &= A(x) \\
\pi_x(\neg C) &= \neg \pi_x(C) \\
\pi_x(C \sqcap D) &= \pi_x(C) \wedge \pi_x(D) \\
\pi_x(C \sqcup D) &= \pi_x(C) \wedge \pi_x(D) \\
\pi_x(\exists R.C) &= \exists y.R(x,y) \wedge \pi_y(C) \\
\pi_x(\forall R.C) &= \forall y.R(x,y) \rightarrow \pi_y(C)
\end{aligned}
$$

2.2   MKNF-based Hybrid Knowledge Bases

The logic of Minimal Knowledge with Negation as Failure (MKNF) was introduced in [7] to support epistemic queries on logic programs. MKNF was inspired by several works [6, 12] on epistemic query answering on non-monotonic databases, which is essential when databases contain incomplete information.

The syntax of MKNF is the syntax of FOL augmented with the modal operators $\mathbf{K}$ and $\mathbf{not}$.

Axiom	Translation
$C \sqsubseteq D$	$\forall x. \pi_x(C) \rightarrow \pi_x(D)$
$a : C$	$\pi_a(C)$
$(a,b) : R$	$R(a,b)$
$a = b$	$a = b$
$a \neq b$	$a \neq b$

Table 1: Translation of $\mathscr{ALC}$ axioms into FOL.

MKNF-based Hybrid Knowledge Bases [10] are combinations of DL axioms and LP rules that can be mapped to a MKNF formula, as follows. As shown in [10], MKNF-based HKBs exhibits desirable properties (faithfulness, i.e., preservation of the semantics of both formalisms when the other is absent; tightness, i.e., no layering of LP and DL; flexibility, i.e., the possibility to view each predicate under both open and closed world assumption; decidability), which each of the other existing approaches to LP and DL integration lacks at least partly.

Definition 1. *A Hybrid Knowledge Base (HKB) is a pair $\mathscr{K} = \langle \mathscr{O}, \mathscr{P} \rangle$ where $\mathscr{O}$ is a set of axioms in a description logic (Section 2.1) and $\mathscr{P}$ is a finite set of normal function-free logic programming rules.*

In the rest of the paper, with a slightly abuse of notation, we will say that a HKB $\mathscr{K}_1 = \langle \mathscr{O}_1, \mathscr{P}_1 \rangle$ is a subset of a HKB $\mathscr{K}_2 = \langle \mathscr{O}_2, \mathscr{P}_2 \rangle$, i.e., $\mathscr{K}_1 \subseteq \mathscr{K}_2$ iff $\mathscr{O}_1 \subseteq \mathscr{O}_2$ and $\mathscr{P}_1 \subseteq \mathscr{P}_2$. Given a HKB $\mathscr{K} = \langle \mathscr{O}, \mathscr{P} \rangle$, an atom in $\mathscr{P}$ is a *DL-atom* if its predicate occurs in $\mathscr{O}$, a non-DL-atom otherwise.

Definition 2 (DL-safety). *A rule is* DL-safe *if each of its variables occurs in at least one positive non-DL-atom in the body; a HKB is* DL-safe *if all its rules are DL-safe.*

In this paper, we assume that all HKBs are DL-safe.

An HKB $\mathscr{K} = \langle \mathscr{O}, \mathscr{P} \rangle$ can be mapped to an MKNF formula by extending the standard transformation π for DL axioms (Table 1) to support LP rules:

- if r is a rule of the form $h \leftarrow a_1, \ldots, a_n, \sim b_1, \ldots, \sim b_m$ where all a_i and b_j are atoms and $\mathbf{X}$ is the tuple of all variables in r, then $\pi(r) = \forall \mathbf{X}(\mathbf{K} a_1 \wedge \ldots \wedge \mathbf{K} a_n \wedge \mathbf{not} \, b_1 \wedge \ldots \wedge \mathbf{not} \, b_m \rightarrow \mathbf{K} h)$

- $\pi(\mathscr{P}) = \bigwedge_{r \in \mathscr{P}} \pi(r)$

- $\pi(\langle \mathscr{O}, \mathscr{P} \rangle) = \mathbf{K} \pi(\mathscr{O}) \wedge \pi(\mathscr{P})$

This transformation is a way to give a semantics to a HKB: MKNF formulas have been given two-valued [10] and three-valued [5] semantics, so the (two or three-valued) semantics of the resulting MKNF formula can be taken as the semantics of the original HKB. We refer the reader to those articles for an in-depth discussion of the semantics and their respective merits.

In the following, we recall the three-valued MKNF semantics, which is more relevant to our work. For simplicity, we omit the signature Σ from the definitions.

Three-valued MKNF semantics [5]   The truth of an MKNF formula ψ is defined relatively to a *three-valued MKNF structure* $(I, \mathscr{M}, \mathscr{N})$, which consists of a first-order interpretation I over a universe Δ and two pairs $\mathscr{M} = (M, M_1)$ and $\mathscr{N} = (N, N_1)$ of sets of first-order interpretations over Δ where $M_1 \subseteq M$ and $N_1 \subseteq N$. $\mathbf{K} \psi$ is true (resp. false) with respect to (M, M_1) if and only if ψ is true in all elements of M (resp. not true in all elements of M_1). N and N_1 serve the same purpose for defining the truth value of $\mathbf{not} \, \psi$.

Satisfaction of a closed formula by a three-valued MKNF structure is defined as follows (where p is a predicate, ψ is a formula, the values `true`, `undefined` and `false` follow the order `false` $<$ `undefined` $<$ `true`, and ε^I represents the individual or relation in the domain of discourse assigned to ε by the interpretation I):

$$
\begin{aligned}
(I,\mathcal{M},\mathcal{N})(p(t_1,\ldots,t_n)) \quad & \texttt{true iff } (t_1^I,\ldots,t_n^I) \in p^I \\
& \texttt{false iff } (t_1^I,\ldots,t_n^I) \notin p^I \\
(I,\mathcal{M},\mathcal{N})(\neg\psi) \quad & \texttt{true iff } (I,\mathcal{M},\mathcal{N})(\psi) = \texttt{false}, \\
& \texttt{undefined iff } (I,\mathcal{M},\mathcal{N})(\psi) = \texttt{undefined}, \\
& \texttt{false iff } (I,\mathcal{M},\mathcal{N})(\psi) = \texttt{true} \\
(I,\mathcal{M},\mathcal{N})(\psi_1 \wedge \psi_2) \quad & min\{(I,\mathcal{M},\mathcal{N})(\psi_1),(I,\mathcal{M},\mathcal{N})(\psi_2)\} \\
(I,\mathcal{M},\mathcal{N})(\psi_1 \rightarrow \psi_2) \quad & \texttt{true iff } (I,\mathcal{M},\mathcal{N})(\psi_1) \leq (I,\mathcal{M},\mathcal{N})(\psi_2), \\
& \texttt{false otherwise} \\
(I,\mathcal{M},\mathcal{N})(\exists x : \psi) \quad & max\{(I,\mathcal{M},\mathcal{N})(\psi[\alpha/x])|\alpha \in \Delta\} \\
(I,\mathcal{M},\mathcal{N})(\mathbf{K}\,\psi) \quad & \texttt{true iff } (J,(M,M_1),\mathcal{N})(\psi) = \texttt{true for all } J \in M, \\
& \texttt{false iff } (J,(M,M_1),\mathcal{N})(\psi) = \texttt{false for some } J \in M_1, \\
& \texttt{undefined otherwise} \\
(I,\mathcal{M},\mathcal{N})(\mathbf{not}\,\psi) \quad & \texttt{true iff } (J,\mathcal{M},(N,N_1))(\psi) = \texttt{false for some } J \in N_1, \\
& \texttt{false iff } (J,\mathcal{M},(N,N_1))(\psi) = \texttt{true for all } J \in N, \\
& \texttt{undefined otherwise}
\end{aligned}
$$

An *MKNF interpretation* over a universe Δ is a non-empty set of first order interpretations over Δ. An *MKNF interpretation pair* (M,N) over a universe Δ consists of two MKNF interpretations M, N over Δ, with $\emptyset \subset N \subseteq M$. An MKNF interpretation pair (M,N) *satisfies* a closed MKNF formula ψ iff, for each $I \in M$, $(I,(M,N),(M,N))(\psi) = \texttt{true}$. If $M = N$, then the MKNF interpretation pair (M,N) is called *total*. If there exists an MKNF interpretation pair satisfying ψ, then ψ is *consistent*. An MKNF interpretation pair (M,N) over a universe Δ is a *three-valued MKNF model* for a given closed MKNF formula ψ if

- (M,N) satisfies ψ and

- for each MKNF interpretation pair (M',N') over Δ with $M \subseteq M'$ and $N \subseteq N'$, where at least one of the inclusions is proper and $M' = N'$ if $M = N$, there is $I' \in M'$ such that $(I',(M',N'),(M,N))(\psi) = \texttt{false}$. In other words, M and N cannot be extended while satisfying ψ; the semantics implements minimal knowledge by requiring as many possible worlds as possible.

2.3   Well Founded HKB Semantics

In [5], the well-founded model of an MKNF formula is defined as the three-valued MKNF model that, intuitively, leaves as much as possible undefined. Not all HKBs have a well-founded model; *MKNF-coherent* HKBs [8] have a unique well-founded model that is characterized by a partition of the atoms that occur in rules, called the alternating fixpoint partition and defined by [5].

The NoHR query answering system [4] is based on the well-founded semantics for HKBs.

We recall these definitions below.

An MKNF formula ψ is ground if ψ does not contain variables. Given a hybrid MKNF knowledge base $\mathcal{K} = \langle \mathcal{O},\mathcal{P} \rangle$, the ground instantiation of $\mathcal{K}$ is the KB $\mathcal{K}_g = \langle \mathcal{O},\mathcal{P}_g \rangle$ where $\mathcal{P}_g$ is obtained from $\mathcal{P}$ by replacing each rule r of $\mathcal{P}$ with a set of rules substituting each variable in r with constants from $\mathcal{K}$ in all possible ways. Let $\mathcal{K} = \langle \mathcal{O},\mathcal{P} \rangle$ be a ground HKB. Note that, if an HKB is DL-safe, it has the same two-valued [10] and three-valued [5] MKNF models of its grounding over the constants that occur in it, so it can be assumed, without loss of generality, that the HKB is ground. The set of *known atoms* of $\mathcal{K}$, $\mathsf{KA}(\mathcal{K})$, is the set of all (ground) atoms occurring in $\mathcal{P}$ [10].

Definition 3. *A* partition *of* $\mathsf{KA}(\mathscr{K})$ *is a pair* (P,N) *such that* $P \subseteq N \subseteq \mathsf{KA}(\mathscr{K})$; (P,N) *is* exact *if* $P = N$.

Given $S \subseteq \mathsf{KA}(\mathscr{K})$, the *objective knowledge of* $\mathscr{O}$ *with respect to* S is the set of first order formulas

$$\mathsf{OB}_{\mathscr{K},S} = \{\pi(\mathscr{O})\} \cup S \tag{1}$$

where π is the standard transformation π for DL axioms (Table 1).

The operators $R_{\mathscr{K}}$, $D_{\mathscr{K}}$ and $T_{\mathscr{K}}$ derive atoms that are consequences of a positive HKB $\mathscr{K}$ (i.e., one where no negative literals occur in rules) and a set S of atoms. $R_{\mathscr{K}}(S)$ is the set of immediate consequences due to rules, i.e., the heads of rules in $\mathscr{P}$ whose bodies are composed of atoms that are a subset of S; $D_{\mathscr{K}}(S)$ is the set of immediate consequences due to axioms, i.e., the atoms from $\mathsf{KA}(\mathscr{K})$ entailed by $\mathsf{OB}_{\mathscr{K},S}$; and $T_{\mathscr{K}}(S) = R_{\mathscr{K}}(S) \cup D_{\mathscr{K}}(S)$. Given an HKB $\mathscr{K}$ and a set of atoms $S \subseteq \mathsf{KA}(\mathscr{K})$, the following transformations, which yield positive knowledge bases, are defined: the *MKNF transformation* $\mathscr{K}/S$ is $\langle \mathscr{O}, \mathscr{P}/S \rangle$ where $\mathscr{P}/S$ is the set of rules $h \leftarrow a_1, \dots, a_m$ such that there exists in $\mathscr{P}$ a rule $h \leftarrow a_1, \dots, a_m, \sim b_1, \dots, \sim b_n$ with $\{b_1, \dots, b_n\} \cap S = \emptyset$, and the *MKNF-coherent transformation* $\mathscr{K}//S$ is $\langle \mathscr{O}, \mathscr{P}//S \rangle$ where $\mathscr{P}//S$ is the set of rules $h \leftarrow a_1, \dots, a_m$ such that there exists in $\mathscr{P}$ a rule $h \leftarrow a_1, \dots, a_m, \sim b_1, \dots, \sim b_n$ with $\{b_1, \dots, b_m\} \cap S = \emptyset$ and $\mathsf{OB}_{\mathscr{K},S} \not\models \neg h$.

Since, as shown in [5], $T_{\mathscr{K}}$ is monotonic if $\mathscr{K}$ is a ground positive HKB, the following transformations of sets of atoms are well defined: $\Gamma_{\mathscr{K}}(S) = \mathsf{lfp}(T_{\mathscr{K}/S})$ and $\Gamma'_{\mathscr{K}}(S) = \mathsf{lfp}(T_{\mathscr{K}//S})$. Using these transformations, it is possible to define a partition of $\mathscr{K}$'s known atoms as follows.

Definition 4. *For an HKB* $\mathscr{K}$, *the sequences of sets of atoms* $\mathbf{P}$ *and* $\mathbf{N}$ *are defined as follows:* $\mathbf{P}_0 = \emptyset$, $\mathbf{N}_0 = \mathsf{KA}(\mathscr{K})$, $\mathbf{P}_{n+1} = \Gamma_{\mathscr{K}}(\mathbf{N}_n)$ *and* $\mathbf{N}_{n+1} = \Gamma'_{\mathscr{K}}(\mathbf{P}_n)$, $\mathbf{P}_\omega = \bigcup \mathbf{P}_i$, $\mathbf{N}_\omega = \bigcap \mathbf{N}_i$.
The pair $(\mathbf{P}_\omega, \mathbf{N}_\omega)$ *is called* $\mathscr{K}$'s alternating fixpoint partition.

[8] identify the class of *MKNF-coherent* HKBs, i.e., those whose alternating fixpoint partition defines a well-founded model, as well as some sufficient conditions for a HKB to be MKNF-coherent.
We assume that the HKBs that we consider are MKNF-coherent.

Definition 5 (MKNF-coherent HKB (Def. 10 of [8])). *An HKB* $\mathscr{K}$ *is* MKNF-coherent *if* (I_P, I_N), *where* $I_P = \{I \mid I \models \mathsf{OB}_{\mathscr{K},\mathbf{P}_\omega}\}$ *and* $I_N = \{I \mid I \models \mathsf{OB}_{\mathscr{K},\mathbf{N}_\omega}\}$, *is a three-valued MKNF model of* $\mathscr{K}$.

For MKNF-coherent HKBs, the model determined by the alternating fixpoint partition as in Definition 5 is the unique well-founded model.

Proposition 1 (Proposition 2 of [8]). *If* $\mathscr{K}$ *is an MKNF-coherent HKB, then it has the unique well-founded model* $(\{I \mid I \models \mathsf{OB}_{\mathscr{K},\mathbf{P}_\omega}\}, \{I \mid I \models \mathsf{OB}_{\mathscr{K},\mathbf{N}_\omega}\})$

We report some sufficient conditions for MKNF-coherence from [8] in Appendix A.
Intuitively, the alternating fixpoint partition marks each known atom in $\mathsf{KA}(\mathscr{K})$ and induces the well founded model.

3 HKBs with function symbols

In this section, we extend the language of HKBs (Section 2.2) to allow function symbols. We define the syntax in Section 3.1 and the semantics in Section 3.2. We also provide a running example (Example 1) of the proposed syntax and semantics, which takes advantage of function symbols to model natural numbers.

3.1   Language

The syntax extension amounts to lifting the function-free limitation of the original HKB syntax.

Definition 6 (Hybrid Knowledge Base with Function Symbols)**.** *A Hybrid Knowledge Base with Function Symbols (HKB^{FS}) is a Hybrid Knowledge Base (Section 2.2) whose rules can contain function applications.*

The definition of DL-safety (Def. 2) also applies to HKB^{FS}s. In this paper, we assume that all HKB^{FS}s are DL-safe.

Example 1 (Spillover)**.** *Let $\mathscr{K} = \langle \mathscr{O}, \mathscr{P} \rangle$, where*

$$\mathscr{P} = safe(X) \leftarrow \sim spillover\_count(X, s(s(Y))).$$
$$spillover\_count(X, s(Y)) \leftarrow virus(X), mutated(X), spillover\_count(X, Y).$$
$$spillover\_count(X, 0) \leftarrow virus(X).$$
$$virus(t).$$
$$\mathscr{O} = \exists mutation.\top \sqsubseteq mutated$$
$$t : \exists mutation.\top$$

This HKB models that t is a virus and there is at least a mutation of the virus t. If there exists at least one mutation for virus t, it is mutated, and so, a spillover may have happened. Finally, we can model the series of spillover events by means of predicate spillover_count. Function $s(Y)$ represents the successor of Y. Finally, a virus is safe if the spillover count is less than two.

3.2   Iterated fixpoint HKB^{FS} semantics

In this section, we define the semantics of an HKB^{FS} as a partition of its known atoms.

We proceed in a bottom-up way, similarly to [11]. In particular, we define two inner operators (Def. 8) that, assuming sets of true and false atoms (a 3-valued interpretation for the HKB^{FS}, Def. 7) possibly derive new true and false atoms, respectively. These operators are monotonic in their argument (Proposition 2), so they have a least and a greatest fixpoint, which are used to define the outer operator (Def. 9) which updates the 3-value interpretation. The outer operator is itself monotonic (Proposition 4), so it has a least fixpoint, which we define (Definition 10) as the semantics of the HKB^{FS}.

Definition 7. *A 3-valued interpretation for an HKB^{FS} $\mathscr{K}$ is a pair $\langle I_T, I_F \rangle$ where I_T and I_F are disjoint sets of $\mathscr{K}$'s known atoms, i.e., $I_T \subseteq \mathsf{KA}(\mathscr{K})$, $I_F \subseteq \mathsf{KA}(\mathscr{K})$, $I_T \cap I_F = \emptyset$*

Given a 3-valued interpretation $\langle I_T, I_F \rangle$, an atom a is true in it if $a \in I_T$, false in it if $a \in I_F$, undefined in it otherwise.

We also define $\langle I_T, I_F \rangle \leq \langle I_T', I_F' \rangle$ iff $I_T \subseteq I_T'$ and $I_F \subseteq I_F'$.

We denote by $Int_3^{\mathscr{K}}$ the set of 3-valued interpretations for an HKB^{FS} $\mathscr{K}$.

Definition 8. *Given a ground HKB^{FS} $\mathscr{K} = \langle \mathscr{O}, \mathscr{P} \rangle$, and a 3-valued interpretation $\mathscr{I} = \langle I_T, I_F \rangle$ for $\mathscr{K}$, we define the operators $OpTrue_{\mathscr{I}}^{\mathscr{K}} : 2^{\mathsf{KA}(\mathscr{K})} \to 2^{\mathsf{KA}(\mathscr{K})}$ and $OpFalse_{\mathscr{I}}^{\mathscr{K}} : 2^{\mathsf{KA}(\mathscr{K})} \to 2^{\mathsf{KA}(\mathscr{K})}$ as*

- *$OpTrue_{\mathscr{I}}^{\mathscr{K}}(Tr) = \{a \in \mathsf{KA}(\mathscr{K}) \mid$ there is a clause $a \leftarrow a_1, ..., a_n, \sim b_1, \ldots, \sim b_r$ in the grounding of $\mathscr{P}$ such that for every i $(1 \leq i \leq n)$ a_i is true in $\mathscr{I}$ or $a_i \in Tr$, and for every j $(1 \leq j \leq r)$ b_j is false in $\mathscr{I}\} \cup \{a \in \mathsf{KA}(\mathscr{K}) | \mathsf{OB}_{\mathscr{K}, I_T \cup Tr} \models a\};$*

- $OpFalse_{\mathscr{I}}^{\mathscr{K}}(Fa) = \{a \in \mathsf{KA}(\mathscr{K}) \mid \mathsf{OB}_{\mathscr{K},I_T} \models \neg a$, *or, for every clause* $a \leftarrow a_1,...,a_n,\sim b_1,\ldots,\sim b_r$ *in the grounding of* $\mathscr{P}$, *there is some* i $(1 \leq i \leq n)$ *such that* a_i *is false in* $\mathscr{I}$ *or* $a_i \in Fa$, *or there is some* j $(1 \leq j \leq r)$ *such that* b_j *is true in* $\mathscr{I}\} \cap \{a \in \mathsf{KA}(\mathscr{K}) | \mathsf{OB}_{\mathscr{K},\mathsf{KA}(\mathscr{K})\setminus(I_F \cup Fa)} \not\models a\}$

In words, $OpTrue_{\mathscr{I}}^{\mathscr{K}}(Tr)$ represents the true atoms that can be derived from $\mathscr{K}$ knowing $\mathscr{I}$ and true atoms Tr, while $OpFalse_{\mathscr{I}}^{\mathscr{K}}(Fa)$ represents the false atoms that can be derived from $\mathscr{K}$ by knowing $\mathscr{I}$ and false atoms Fa.

Proposition 2. *Given an HKB^{FS} $\mathscr{K}$ and a 3-valued interpretation $\mathscr{I}$ for $\mathscr{K}$, $OpTrue_{\mathscr{I}}^{\mathscr{K}}$ and $OpFalse_{\mathscr{I}}^{\mathscr{K}}$ are both monotonic in their argument.*

Proof. Monotonicity of $OpTrue_{\mathscr{I}}^{\mathscr{K}}$ means that if $Tr \subseteq Tr'$, then $OpTrue_{\mathscr{I}}^{\mathscr{K}}(Tr) \subseteq OpTrue_{\mathscr{I}}^{\mathscr{K}}(Tr')$. Analogously, monotonicity of $OpFalse_{\mathscr{I}}^{\mathscr{K}}$ means that if $Fa \subseteq Fa'$, then $OpFalse_{\mathscr{I}}^{\mathscr{K}}(Fa) \subseteq OpFalse_{\mathscr{I}}^{\mathscr{K}}(Fa')$.

Regarding $OpTrue_{\mathscr{I}}^{\mathscr{K}}$, if $a \in OpTrue_{\mathscr{I}}^{\mathscr{K}}(Tr)$, Definition 8 ensures that either there is a clause $a \leftarrow a_1,...,a_n,\sim b_1,\ldots,\sim b_r$ in $\mathscr{P}$'s grounding such that for each $1 \leq i \leq n$ a_i is true in $\mathscr{I}$ or $a_i \in Tr$ and for each $1 \leq j \leq r$ b_j is false in $\mathscr{I}$, or $\mathsf{OB}_{\mathscr{K},I_T \cup Tr} \models a$, i.e., $\pi(\mathscr{O}) \cup I_T \cup Tr \models a$. Since $Tr \subseteq Tr'$, if $a_i \in Tr$, then also $a_i \in Tr'$, and if $\pi(\mathscr{O}) \cup I_T \cup Tr \models a$, then also $\pi(\mathscr{O}) \cup I_T \cup Tr' \models a$ by the monotonicity of first order logic. So $a \in OpTrue_{\mathscr{I}}^{\mathscr{K}}(Tr')$.

Regarding $OpFalse_{\mathscr{I}}^{\mathscr{K}}$, if $a \in OpFalse_{\mathscr{I}}^{\mathscr{K}}(Fa)$, then either

- $\mathsf{OB}_{\mathscr{K},I_T} \models \neg a$, or

- for each clause $a \leftarrow a_1,...,a_n,\sim b_1,\ldots,\sim b_r$ in $\mathscr{P}$ there is some i $(1 \leq i \leq n)$ such that either a_i is false in $\mathscr{I}$ or $a_i \in Fa$ (and since $Fa \subseteq Fa'$, $a_i \in Fa'$), or some j $(1 \leq j \leq r)$ such that b_j is true in $\mathscr{I}$.

Also, if $\mathsf{OB}_{\mathscr{K},\mathsf{KA}(\mathscr{K})\setminus(I_F \cup Fa)} \not\models a$, then $\mathsf{OB}_{\mathscr{K},\mathsf{KA}(\mathscr{K})\setminus(I_F \cup Fa')} \not\models a$ by the monotonicity of first order logic. So $a \in OpFalse_{\mathscr{I}}^{\mathscr{K}}(Fa')$. $\square$

Proposition 3. *Given an HKB^{FS} $\mathscr{K}$, $OpTrue_{\mathscr{I}}^{\mathscr{K}}$ and $OpFalse_{\mathscr{I}}^{\mathscr{K}}$ are monotonic in $\mathscr{I}$, i.e., if $\mathscr{I}$ and $\mathscr{I}'$ are three-valued interpretations for $\mathscr{K}$ such that $\mathscr{I} \leq \mathscr{I}'$, then*

  1. for each $Tr \subseteq \mathsf{KA}(\mathscr{K})$, $OpTrue_{\mathscr{I}}^{\mathscr{K}}(Tr) \subseteq OpTrue_{\mathscr{I}'}^{\mathscr{K}}(Tr)$

  2. for each $Fa \subseteq \mathsf{KA}(\mathscr{K})$, and $OpFalse_{\mathscr{I}}^{\mathscr{K}}(Fa) \subseteq OpFalse_{\mathscr{I}'}^{\mathscr{K}}(Fa)$.

Proof. 1. If $a \in OpTrue_{\mathscr{I}}^{\mathscr{K}}(Tr)$, then
  - either there is a clause $a \leftarrow a_1,...,a_n,\sim b_1,\ldots,\sim b_r$ in $\mathscr{P}$'s grounding such that for each i $(1 \leq i \leq n)$ a_i is true in $\mathscr{I}$ (and then it is true in $\mathscr{I}'$) or $a_i \in Tr$ and for each j $(1 \leq j \leq r)$ b_j is false in $\mathscr{I}$ (and then it is also false in $\mathscr{I}'$); which would ensure $a \in OpTrue_{\mathscr{I}'}^{\mathscr{K}}(Tr)$
  - or $\mathsf{OB}_{\mathscr{K},I_T \cup Tr} \models a$, i.e., $\pi(\mathscr{O}) \cup I_T \cup Tr \models a$. Since $I_T \subseteq I'_T$, then also $\pi(\mathscr{O}) \cup I'_T \cup Tr' \models a$ by the monotonicity of first order logic; which, also, would ensure $a \in OpTrue_{\mathscr{I}'}^{\mathscr{K}}(Tr)$

  So $a \in OpTrue_{\mathscr{I}'}^{\mathscr{K}}(Tr)$.

  2. If $a \in OpFalse_{\mathscr{I}}^{\mathscr{K}}(Fa)$, then
  - $\mathsf{OB}_{\mathscr{K},\mathsf{KA}(\mathscr{K})\setminus(I_F \cup Fa)} \not\models a$, so also $\mathsf{OB}_{\mathscr{K},\mathsf{KA}(\mathscr{K})\setminus(I'_F \cup Fa)u} \not\models a$ because $I'_F \supseteq I_F$ and by the monotonicity of first order logic;
  - and
    - either $\mathsf{OB}_{\mathscr{K},I_T} \models a$, so also $\mathsf{OB}_{\mathscr{K},I'_T} \models a$ by the monotonicity of first order logic;
    - or for all clauses $a \leftarrow a_1,...,a_n,\sim b_1,\ldots,\sim b_r$ in $\mathscr{P}$'s grounding either there exists an $a_i \in I_F \cup Fa$, so $a_i \in I'_F \cup Fa$, or a $b_j \in I_T$, so $b_j \in I'_T$.

In conclusion, $a \in OpFalse_{\mathscr{I}'}^{\mathscr{K}}(Fa)$.

$\square$

Given an HKB^{FS} $\mathscr{K}$ and a 3-valued interpretation $\mathscr{I}$, since $OpTrue_{\mathscr{I}}^{\mathscr{K}}$ and $OpFalse_{\mathscr{I}}^{\mathscr{K}}$ are monotonic in their argument, they both have least and greatest fixpoints.

So it is possible to define the following iterative operator on a 3-valued interpretation $\mathscr{I}$.

Definition 9 (Iterated Fixed Point). *For an HKB^{FS} $\mathscr{K}$, we define $IFP^{\mathscr{K}} : Int_3^{\mathscr{K}} \to Int_3^{\mathscr{K}}$ as*
$$IFP^{\mathscr{K}}(\mathscr{I}) = \langle \mathsf{lfp}(OpTrue_{\mathscr{I}}^{\mathscr{K}}), \mathsf{gfp}(OpFalse_{\mathscr{I}}^{\mathscr{K}}) \rangle.$$

Proposition 4. *For each HKB^{FS} $\mathscr{K}$, $IFP^{\mathscr{K}}$ is monotonic w.r.t. the order relation among 3-valued interpretations defined in Definition 7.*

Proof. Let $\mathscr{I}$ and $\mathscr{I}'$ be two three-valued interpretations of $\mathscr{K}$ such that $\mathscr{I} \leq \mathscr{I}'$. By propositions 2 and 3,

1. $OpTrue_{\mathscr{I}}^{\mathscr{K}} \uparrow n \subseteq OpTrue_{\mathscr{I}'}^{\mathscr{K}} \uparrow n$ for all n
2. $OpFalse_{\mathscr{I}}^{\mathscr{K}} \downarrow n \subseteq OpFalse_{\mathscr{I}'}^{\mathscr{K}} \downarrow n$ for all n

Thus,

1. $\mathsf{lfp}(OpTrue_{\mathscr{I}}^{\mathscr{K}}) \subseteq \mathsf{lfp}(OpTrue_{\mathscr{I}'}^{\mathscr{K}})$
2. $\mathsf{gfp}(OpFalse_{\mathscr{I}}^{\mathscr{K}}) \subseteq \mathsf{gfp}(OpFalse_{\mathscr{I}'}^{\mathscr{K}})$

i.e., $IFP^{\mathscr{K}}(\mathscr{I}) \leq IFP^{\mathscr{K}}(\mathscr{I}')$.

$\square$

By virtue of being monotonic, $IFP^{\mathscr{K}}$ admits a least fixpoint for each HKB^{FS} $\mathscr{K}$, which we define as the semantics of the HKB^{FS}.

Definition 10 (Iterated fixpoint semantics). *Given an HKB^{FS} $\mathscr{K}$, its iterated fixpoint semantics is $\mathsf{lfp}(IFP^{\mathscr{K}})$.*

Example 2 (Spillover cont.). *Consider the $\mathscr{K} = \langle \mathscr{O}, \mathscr{P} \rangle$ of Example 1. Figure 1 shows the computation of the iterated fixpoint semantics for the HKB $\mathscr{K}$. Given the presence of the function symbol $s(\cdot)$, the model is infinite because there are countably many substitutions for spillover_count.*

$$
\begin{array}{llll}
I_{T0} & = & \emptyset & \\
I_{T1} & = & \{virus(t), & \\
       &   & mutated(t), & \\
       &   & spillover\_count(t,0), & \\
       &   & spillover\_count(t,s(0)), & \\
       &   & spillover\_count(t,s(s(0))), & \\
       &   & \dots\} & \\
I_{T2} & = & I_{T1} & \\
I_{T3} & = & I_{T2} & \\
\end{array}
\qquad
\begin{array}{lll}
I_{F0} & = & \emptyset \\
I_{F1} & = & \mathsf{KA}(\mathscr{K}) \setminus I_{T1} \setminus \{safe(t)\} \\
       &   & \\
       &   & \\
       &   & \\
       &   & \\
       &   & \\
I_{F2} & = & \mathsf{KA}(\mathscr{K}) \setminus I_{T1} \\
I_{F3} & = & I_{F2} \\
\end{array}
$$

Figure 1: Iterations of the $IFP^{\mathscr{K}}$ operator for Example 1.

Each $\mathscr{I}_m$, for $m = 1, 2, 3$ is determined by the fixpoints of $OpTrue_{\mathscr{I}_{m-1}}^{\mathscr{K}}$ and $OpFalse_{\mathscr{I}_{m-1}}^{\mathscr{K}}$ as follows.

- $OpTrue_{\mathscr{I}_0}^{\mathscr{K}} \uparrow 0 = \emptyset$,
- $OpTrue_{\mathscr{I}_0}^{\mathscr{K}} \uparrow 1 = OpTrue_{\mathscr{I}_0}^{\mathscr{K}} \uparrow 0 \cup \{virus(t), mutated(t)\}$,

- $OpTrue_{\mathscr{I}_0}^{\mathscr{K}} \uparrow 2 = OpTrue_{\mathscr{I}_0}^{\mathscr{K}} \uparrow 1 \cup \{spillover\_count(t,0)\}$,

- $OpTrue_{\mathscr{I}_0}^{\mathscr{K}} \uparrow 3 = OpTrue_{\mathscr{I}_0}^{\mathscr{K}} \uparrow 2 \cup \{spillover\_count(t,s(0))\}$,

- $OpTrue_{\mathscr{I}_0}^{\mathscr{K}} \uparrow 4 = OpTrue_{\mathscr{I}_0}^{\mathscr{K}} \uparrow 3 \cup \{spillover\_count(t,s(s(0)))\}$,

and so on to the least fixpoint I_{T1}.

- $OpFalse_{\mathscr{I}_0}^{\mathscr{K}} \downarrow 0 = \mathsf{KA}(\mathscr{K})$

- $OpFalse_{\mathscr{I}_0}^{\mathscr{K}} \downarrow 1 = OpFalse_{\mathscr{I}_0}^{\mathscr{K}} \downarrow 0 \setminus \{virus(t), mutated(t), safe(t)\}$

- $OpFalse_{\mathscr{I}_0}^{\mathscr{K}} \downarrow 2 = OpFalse_{\mathscr{I}_0}^{\mathscr{K}} \downarrow 1 \setminus \{spillover\_count(t,0)\}$

- $OpFalse_{\mathscr{I}_0}^{\mathscr{K}} \downarrow 3 = OpFalse_{\mathscr{I}_0}^{\mathscr{K}} \downarrow 2 \setminus \{spillover\_count(t,s(0))\}$

- $OpFalse_{\mathscr{I}_0}^{\mathscr{K}} \downarrow 4 = OpFalse_{\mathscr{I}_0}^{\mathscr{K}} \downarrow 3 \setminus \{spillover\_count(t,s(s(0)))\}$

and so on to the greatest fixpoint I_{F1}.

- $OpTrue_{\mathscr{I}_1}^{\mathscr{K}} \uparrow 0 = \emptyset$

- $OpTrue_{\mathscr{I}_1}^{\mathscr{K}} \uparrow 1 = I_{T1}$

  which is the least fixpoint.

- $OpFalse_{\mathscr{I}_1}^{\mathscr{K}} \downarrow 0 = \mathsf{KA}(\mathscr{K})$

- $OpFalse_{\mathscr{I}_1}^{\mathscr{K}} \downarrow 1 = OpFalse_{\mathscr{I}_1}^{\mathscr{K}} \downarrow 0 \setminus \{virus(t), mutated(t)\}$. *In this case, safe(t) is kept because spillover_count(s(s(0))) is false in $\mathscr{I}_1$.*

- $OpFalse_{\mathscr{I}_1}^{\mathscr{K}} \downarrow 2 = OpFalse_{\mathscr{I}_1}^{\mathscr{K}} \downarrow 1 \setminus \{spillover\_count(t,0)\}$

- $OpFalse_{\mathscr{I}_1}^{\mathscr{K}} \downarrow 3 = OpFalse_{\mathscr{I}_1}^{\mathscr{K}} \downarrow 2 \setminus \{spillover\_count(t,s(0))\}$

- $OpFalse_{\mathscr{I}_1}^{\mathscr{K}} \downarrow 4 = OpFalse_{\mathscr{I}_1}^{\mathscr{K}} \downarrow 3 \setminus \{spillover\_count(t,s(s(0)))\}$

to the greatest fixpoint $I_{F2} = \mathsf{KA}(\mathscr{K}) \setminus I_{T1} \cup \{safe(t)\}$.

For all m, $OpTrue_{\mathscr{I}_2}^{\mathscr{K}} \uparrow m = OpTrue_{\mathscr{I}_1}^{\mathscr{K}} \uparrow m$ and $OpFalse_{\mathscr{I}_2}^{\mathscr{K}} \downarrow m = OpFalse_{\mathscr{I}_1}^{\mathscr{K}} \downarrow m$, so $\mathscr{I}_2 = \mathscr{I}_3 = \mathsf{lfp}(IFP^{\mathscr{K}})$.

4 Properties

In this section, we prove that, for function-free HKB^{FS}s, which are also HKBs, Knorr et al.'s alternating fixpoint partition (def. 4) and our iterated fixpoint (def. 10) coincide, modulo a set complement operation.

Theorem 1. *Given a function-free HKB^{FS} $\mathscr{K} = \langle \mathscr{O}, \mathscr{P} \rangle$, let $\mathsf{lfp}(IFP^{\mathscr{K}}) = \langle I_T, I_F \rangle$. Then $\langle I_T, \mathsf{KA}(\mathscr{K}) \setminus I_F \rangle$ is $\mathscr{K}$'s alternating fixpoint partition.*

Proof. We prove the claim by double induction. Since $\mathscr{P}$ is function-free, its grounding is finite so all fixpoints occur at finite ordinals and it is not necessary to consider limit ordinals.

We show by induction that $IFP^{\mathscr{K}} \uparrow n = \langle \mathbf{P}_n; \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}_n \rangle$.

For $n = 0$ (base case), $IFP^{\mathscr{K}} \uparrow 0 = \langle \emptyset, \emptyset \rangle$, while $\mathbf{P}_0 = \emptyset$ and $\mathbf{N}_0 = \mathsf{KA}(\mathscr{K})$, thus $\langle \mathbf{P}_0, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}_0 \rangle = \langle \emptyset, \emptyset \rangle = IFP^{\mathscr{K}} \uparrow 0$.

For the inductive case, assume $IFP^{\mathscr{K}} \uparrow n = \langle \mathbf{P}_n, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}_n \rangle = \mathscr{I} = \langle I_T, I_F \rangle$.

We now prove that (1) $\mathsf{lfp}(OpTrue_{\mathscr{G}}^{\mathscr{K}}) = \mathsf{lfp}(T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F})$ and that (2) $\mathsf{KA}(\mathscr{K}) \setminus (\mathsf{gfp}(OpFalse_{\mathscr{G}}^{\mathscr{K}})) = \mathsf{lfp}(T_{\mathscr{K}//I_T})$.

To prove (1), we first show by induction that $T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F} \uparrow m \subseteq OpTrue_{\mathscr{G}}^{\mathscr{K}} \uparrow m$.

For $m = 0$ $T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F} \uparrow 0 = \emptyset$ so $T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F} \uparrow 0 \subseteq OpTrue_{\mathscr{G}}^{\mathscr{K}} \uparrow 0$.

For $m + 1$, let Tr be $T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F} \uparrow m$, and assume that $Tr \subseteq OpTrue_{\mathscr{G}}^{\mathscr{K}} \uparrow m$.

If $a \in T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F}(Tr)$, suppose $a \in R_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F}(Tr)$. Then there exists a rule $a \leftarrow l_1, ..., l_n$ in $\mathscr{P}_g/\mathsf{KA}(\mathscr{K}) \setminus I_F$, where $\mathscr{P}_g$ is the grounding of $\mathscr{P}$, with each $l_i \in Tr$. This means that $\mathscr{P}_g$ contains a rule $a \leftarrow l_1, ..., l_n, \sim b_1, ..., \sim b_r$ with $b_1, ..., b_r$ in I_F. So $a \in OpTrue_{\mathscr{G}}^{\mathscr{K}} \uparrow m + 1$ by the definition of $OpTrue_{\mathscr{G}}^{\mathscr{K}}$. If $a \in D_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F}(Tr)$ then $\mathsf{OB}_{\mathscr{K}, Tr} \models a$ so $a \in OpTrue_{\mathscr{G}}^{\mathscr{K}}(Tr)$ by the definition of $OpTrue_{\mathscr{G}}^{\mathscr{K}}$.

Since $T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F} \uparrow m \subseteq OpTrue_{\mathscr{G}}^{\mathscr{K}} \uparrow m$, for all m, $\mathsf{lfp}(T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F}) \subseteq \mathsf{lfp}(OpTrue_{\mathscr{G}}^{\mathscr{K}})$, so to prove (1) it is sufficient to show that $\mathsf{lfp}(OpTrue_{\mathscr{G}}^{\mathscr{K}}) \subseteq \mathsf{lfp}(T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F})$.

To this end, consider the sequence S of sets defined by $S_0 = I_T$, $S_{m+1} = T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F}(S_m)$. Note that $S_m \subseteq S_{m+1}$ for all m, which can be proved by induction. For $m = 0$, $I_T = T_{\mathscr{K}'}(I_T)$ where $\mathscr{K}'$ is a subset of $\mathscr{K}/\mathsf{KA}(\mathscr{K}) \setminus I_F$, so if $a \in I_T$, then $a \in T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F}(I_T) = S_1$. For the inductive case, assume $S_{m-1} \subseteq S_m$: by the monotonicity of $T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F}$ (because of Proposition 4 of [5], $\mathscr{K}/\mathsf{KA}(\mathscr{K}) \setminus I_F$ being positive), $S_m = T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F}(S_{m-1}) \subseteq T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F}(S_m) = S_{m+1}$.

We now prove by induction that $OpTrue_{\mathscr{G}}^{\mathscr{K}} \uparrow m \subseteq S_m$. For $m = 0$, $\emptyset \subseteq I_T$. Assuming the inclusion holds for a generic m, then if $a \in OpTrue_{\mathscr{G}}^{\mathscr{K}} \uparrow (m+1)$, either there is a rule $a \leftarrow a_1, ..., a_n, \sim b_1, ..., \sim b_r$ in $\mathscr{P}_g$ with $\{a_1, ..., a_n\} \subseteq (OpTrue_{\mathscr{G}}^{\mathscr{K}} \uparrow m \cup I_T) \subseteq (S_m \cup S_0) \subseteq S_m$ and $\{b_1, ..., b_r\} \subseteq I_F$, so $R_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F}$ can be applied to derive a; or $\mathsf{OB}_{\mathscr{K}, I_T \cup OpTrue_{\mathscr{G}}^{\mathscr{K}} \uparrow m}$ entails a, but then so does $\mathsf{OB}_{\mathscr{K}, S_m}$, by the inductive hypothesis and because $I_T \subseteq S_m$, so $D_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F}$ applies.

Also, note that $I_T \subseteq \mathsf{lfp}(T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F})$ because $I_T = \mathbf{P}_n$, $\mathbf{P}_n$ is the least fixpoint of a $T_{\mathscr{K}'}$ operator where $\mathscr{K}' \subseteq \mathscr{K}/\mathsf{KA}(\mathscr{K}) \setminus I_F$, and $T_{\mathscr{K}}$ is monotonic in its (positive) HKB argument. In fact $T_{\mathscr{K}'}(S) \subseteq T_{\mathscr{K}}(S)$ if $\mathscr{K}' \subseteq \mathscr{K}$ because, if a is the head of a program rule of $\mathscr{K}'$ whose body is true in S, that rule is also in $\mathscr{K}$, and if $\mathsf{OB}_{\mathscr{K}', S} \models a$, then $\mathsf{OB}_{\mathscr{K}, S} \models a$ by the monotonicity of first order logic.

Moreover, $S_m \subseteq \mathsf{lfp}(T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F})$ for all m. By induction: $S_0 = I_T \subseteq \mathsf{lfp}(T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F})$. Suppose $S_m \subseteq \mathsf{lfp}(T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F})$. Then $a \in S_{m+1}$ is the head of a rule of $\mathscr{K}/\mathsf{KA}(\mathscr{K}) \setminus I_F$ whose body is true in S_m. By the inductive hypothesis, it is also true in $\mathsf{lfp}(T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F})$ so $a \in \mathsf{lfp}(T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F})$.

Thus, $\mathsf{lfp}(OpTrue_{\mathscr{G}}^{\mathscr{K}}) \subseteq \mathsf{lfp}(T_{\mathscr{K}/\mathsf{KA}(\mathscr{K})\setminus I_F})$, which concludes the proof of (1).

We prove (2) by proving that, for all m, $T_{\mathscr{K}//I_T} \uparrow m = \mathsf{KA}(\mathscr{K}) \setminus (OpFalse_{\mathscr{G}}^{\mathscr{K}} \downarrow m)$.

For the base case of $m = 0$, $T_{\mathscr{K}//I_T} \uparrow 0 = \emptyset$ and $OpFalse_{\mathscr{G}}^{\mathscr{K}} \downarrow 0 = \mathsf{KA}(\mathscr{K})$, so $T_{\mathscr{K}//I_T} \uparrow 0 = \mathsf{KA}(\mathscr{K}) \setminus (OpFalse_{\mathscr{G}}^{\mathscr{K}} \downarrow 0)$.

For the inductive case, $m + 1$, let S be $T_{\mathscr{K}//I_T} \uparrow m$ and let Fa be $OpFalse_{\mathscr{G}}^{\mathscr{K}} \downarrow m$. Note that, for all m, $I_F \subseteq OpFalse_{\mathscr{G}}^{\mathscr{K}} \downarrow m$, because by Proposition 4 $I_F \subseteq \mathsf{gfp}(OpFalse_{\mathscr{G}}^{\mathscr{K}})$; thus, $I_F \cup Fa = Fa$.

By the inductive hypothesis, $S = \mathsf{KA}(\mathscr{K}) \setminus (I_F \cup Fa)$. We now show that, for all $a \in \mathsf{KA}(\mathscr{K})$, $a \in T_{\mathscr{K}//I_T}(S)$ if and only if $a \notin OpFalse_{\mathscr{G}}^{\mathscr{K}}(Fa)$.

Assume $a \in T_{\mathscr{K}//I_T}(S)$: if $\mathsf{OB}_{\mathscr{K}, S} \models a$, then $\mathsf{OB}_{\mathscr{K}, \mathsf{KA}(\mathscr{K})\setminus(I_F \cup Fa)} \models a$, so $a \notin OpFalse_{\mathscr{G}}^{\mathscr{K}}(Fa)$; otherwise, $\mathsf{OB}_{\mathscr{K}, I_T} \not\models \neg a$ and there exists a rule $a \leftarrow a_1, ..., a_m, \sim b_1, ..., \sim b_n$ in $\mathscr{P}_g$ such that $\{a_1, ..., a_m\} \subseteq S$ and $\{b_1, ..., b_n\} \cap I_T = \emptyset$, which, by De Morgan's laws and because $S = \mathsf{KA}(\mathscr{K}) \setminus (I_F \cup Fa)$, is the negation of the fact that $\mathsf{OB}_{\mathscr{K}, I_T} \models \neg a$ or, for each rule $a \leftarrow a_1, ..., a_m, \sim b_1, ..., \sim b_n$ in $\mathscr{P}_g$, $\{a_1, ..., a_m\} \cap (I_F \cup Fa) \neq \emptyset$ or $\{b_1, ..., b_n\} \cap I_T \neq \emptyset$; so again $a \notin OpFalse_{\mathscr{G}}^{\mathscr{K}}(Fa)$.

On the other hand, if $a \notin OpFalse_{\mathscr{G}}^{\mathscr{K}}(Fa)$, then either (i) $\mathsf{OB}_{\mathscr{K}, \mathsf{KA}(\mathscr{K})\setminus(I_F \cup Fa)} \models a$ (and, since $\mathsf{KA}(\mathscr{K}) \setminus (I_F \cup Fa) = S$, $\mathsf{OB}_{\mathscr{K}, S} \models a$, so $a \in T_{\mathscr{K}//I_T}(S)$), or (ii) $\mathsf{OB}_{\mathscr{K}, I_T} \not\models \neg a$ and for a rule $a \leftarrow a_1, ..., a_m, b_1, ..., b_n$ in $\mathscr{P}$'s grounding $\{a_1, ..., a_m\} \cap (I_F \cup Fa) = \emptyset$ (i.e., $\{a_1, ..., a_m\} \subseteq S$) and $\{b_1, ..., b_n\} \cap I_T = \emptyset$, so again

$a \in T_{\mathscr{K}//I_{\mathrm{T}}}(S).$ □

5 Conclusions and future work

In this paper we proposed an extension of the language of MKNF-based Hybrid Knowledge Bases to support function symbols in rules. We extended the syntax and proposed an iterative fixpoint semantics for the extended language. We showed that the proposed semantics coincides with the one proposed by [5] in the case of HKBs without function symbols, so it is an extension of it.

The proposed iterative fixpoint semantics also opens the way to the introduction of probabilities in HKBs. We are currently working on a probabilistic extension of HKBs with function symbols, inspired by Sato's distribution semantics [14], which will be based on the iterated fixpoint operator defined in this paper. The probabilistic language of probabilistic HKB^{FS} will be also equipped with a query answering system, in the style of what we did in TRILL [17, 16] and PITA [13], comparing our system with that of Knorr and colleagues [9].

Acknowledgements

References

[1] Marco Alberti, Ana Sofia Gomes, Ricardo Gonçalves, João Leite & Martin Slota (2011): *Normative Systems Represented as Hybrid Knowledge Bases*. In João Leite, Paolo Torroni, Thomas Ågotnes, Guido Boella & Leon van der Torre, editors: *Computational Logic in Multi-Agent Systems - 12th International Workshop, CLIMA XII, Barcelona, Spain, Proceedings, Lecture Notes in Artificial Intelligence* 6814, Springer, Berlin, pp. 330–346. Available at `http://dx.doi.org/10.1007/978-3-642-22359-4_23`.

[2] Franz Baader, Diego Calvanese, Deborah L. McGuinness, Daniele Nardi & Peter F. Patel-Schneider, editors (2003): *The Description Logic Handbook: Theory, Implementation, and Applications*. Cambridge University Press, New York, NY, USA, doi:10.1017/CBO9780511711787.

[3] Alex Borgida (1996): *On the relative expressiveness of description logics and predicate logics*. Artif. Intell. 82(1-2), pp. 353–367, doi:10.1016/0004-3702(96)00004-5.

[4] Vedran Kasalica, Matthias Knorr, João Leite & Carlos Lopes (2020): *NoHR: An Overview*. Künstliche Intell. 34(4), pp. 509–515, doi:10.1007/s13218-020-00650-1.

[5] Matthias Knorr, José Júlio Alferes & Pascal Hitzler (2011): *Local closed world reasoning with description logics under the well-founded semantics*. Artif. Intell. 175(9-10), pp. 1528 – 1554, doi:10.1016/j.artint.2011.01.007.

[6] Hector J. Levesque (1984): *Foundations of a Functional Approach to Knowledge Representation*. Artif. Intell. 23(2), pp. 155–212, doi:10.1016/0004-3702(84)90009-2.

[7] Vladimir Lifschitz (1991): *Nonmonotonic Databases and Epistemic Queries*. In John Mylopoulos & Raymond Reiter, editors: *12th International Joint Conference on Artificial Intelligence (IJCAI 1991)*, Morgan Kaufmann, San Francisco, CA, USA, pp. 381–386.

[8] Fangfang Liu & Jia-Huai You (2017): *Three-valued semantics for hybrid MKNF knowledge bases revisited*. Artificial Intelligence 252(Supplement C), pp. 123 – 138, doi:10.1016/j.artint.2017.08.003. Available at `http://www.sciencedirect.com/science/article/pii/S0004370217300917`.

[9] Carlos Lopes, Matthias Knorr & João Leite (2017): *NoHR: Integrating XSB Prolog with the OWL 2 Profiles and Beyond*. In Marcello Balduccini & Tomi Janhunen, editors: *14th International Conference on Logic Programming and Non-monotonic Reasoning (LPNMR 2017)*, Lecture Notes in Computer Science 10377, Springer, pp. 236–249, doi:10.1007/978-3-319-61660-5_22.

[10] Boris Motik & Riccardo Rosati (2010): *Reconciling Description Logics and Rules*. J. ACM 57(5), pp. 30:1–30:62, doi:10.1145/1754399.1754403.

[11] Teodor C. Przymusinski (1989): *Every Logic Program Has a Natural Stratification And an Iterated Least Fixed Point Model*. In: *Proceedings of the 8th ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems (PODS-1989)*, ACM Press, pp. 11–21. Available at `http://doi.acm.org/10.1145/73721.73723`.

[12] Raymond Reiter (1990): *On Asking What a Database Knows*. In John W. Lloyd, editor: *Computational Logic: Symposium Proceedings, Brussels, November 13/14, 1990*, Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 96–113, doi:10.1007/978-3-642-76274-1_4.

[13] Fabrizio Riguzzi & Terrance Swift (2011): *The PITA System: Tabling and Answer Subsumption for Reasoning under Uncertainty*. Theor. Pract. Log. Prog. 11(4–5), pp. 433–449, doi:10.1017/S147106841100010X.

[14] Taisuke Sato (1995): *A Statistical Learning Method for Logic Programs with Distribution Semantics*. In Leon Sterling, editor: *Logic Programming, Proceedings of the Twelfth International Conference on Logic Programming, Tokyo, Japan, June 13-16, 1995*, MIT Press, pp. 715–729, doi:10.7551/mitpress/4298.003.0069.

[15] U. Sattler, D. Calvanese & R. Molitor (2003): *Relationships with other Formalisms*. In: *Description Logic Handbook*, pp. 137–177, doi:10.5555/885746.885751.

[16] Riccardo Zese, Elena Bellodi, Fabrizio Riguzzi, Giuseppe Cota & Evelina Lamma (2018): *Tableau Reasoning for Description Logics and its Extension to Probabilities*. Ann. Math. Artif. Intell. 82(1–3), pp. 101–130, doi:10.1007/s10472-016-9529-3.

[17] Riccardo Zese, Giuseppe Cota, Evelina Lamma, Elena Bellodi & Fabrizio Riguzzi (2019): *Probabilistic DL Reasoning with Pinpointing Formulas: A Prolog-based Approach*. Theor. Pract. Log. Prog. 19(3), pp. 449–476, doi:10.1017/S1471068418000480.

A MKNF-coherent HKBs

We recall that the HKBs such that the alternating fixpoint partition defines a three-valued MKNF model are called *MKNF-coherent* [8].

From Definition 5[1] An HKB $\mathscr{K}$ is *MKNF-coherent* if (I_P, I_N), where $I_P = \{I \mid I \models \mathsf{OB}_{\mathscr{K}, \mathbf{P}_\omega}\}$ and $I_N = \{I \mid I \models \mathsf{OB}_{\mathscr{K}, \mathbf{N}_\omega}\}$, is a three-valued MKNF model of $\mathscr{K}$.

For MKNF-coherent HKBs, the model determined by the alternating fixpoint partition as in Definition 5 is the unique well-founded model.

From Proposition 1[2] If $\mathscr{K}$ is an MKNF-coherent HKB, then it has the unique well-founded model $(\{I \mid I \models \mathsf{OB}_{\mathscr{K}, \mathbf{P}_\omega}\}, \{I \mid I \models \mathsf{OB}_{\mathscr{K}, \mathbf{N}_\omega}\})$

For an MKNF-coherent HKB $\mathscr{K}$ with alternating fixpoint partition $(\mathbf{P}_\omega, \mathbf{N}_\omega)$ and $a \in \mathsf{KA}(\mathscr{K})$, we write $\mathsf{WFM}(\mathscr{K}) \models a$ if $a \in \mathbf{P}_\omega$ and $\mathsf{WFM}(\mathscr{K}) \models \neg a$ if $a \in \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}_\omega$.

[8] show a bijection between the three-valued MKNF models of an HKB $\mathscr{K}$ and certain partitions of $\mathsf{KA}(\mathscr{K})$, called *stable partitions*. In the following, we report the definition of stable partition and two results on stable partitions.

The definition of stable partition depends on the following evaluation scheme of rules and logic programs w.r.t. partitions of the set of known atoms of an HKB.

In the following, let $\mathscr{K} = \langle \mathscr{O}, \mathscr{P} \rangle$ be an HKB, and T and F two subsets of $\mathsf{KA}(\mathscr{K})$ such that $T \cap F = \emptyset$.

- A rule r in $\mathscr{P}$ is evaluated to a new rule as follows:
    - $r[\mathbf{K}, T, F]$ denotes the rule obtained by replacing each positive literal a in r with `true` if $a \in T$, with `false` if $a \in F$, and with `undefined` otherwise;
    - $r[\mathbf{not}, T, F]$ denotes the rule obtained by replacing each negative literal $\sim a$ in r with `true` if $a \in F$, with `false` if $a \in T$, and with `undefined` otherwise;
    - $r[T, F]$ denotes $r[\mathbf{K}, T, F][\mathbf{not}, T, F]$.

- an evaluated rule is simplified as follows:
    - if the value of the head atom in a rule is equal to or greater than the value of its body, the rule is replaced by `true ←`;
    - if the value of the head atom in a rule is less than the value of its body, then the rule is replaced by `false ←`.

- A logic program is evaluated as follows:
    - $\mathscr{P}[\mathbf{K}, T, F]$, $\mathscr{P}[\mathbf{not}, T, F]$, $\mathscr{P}[T, F]$ denote the logic programs obtained by replacing each rule r in $\mathscr{P}$ with $r[\mathbf{K}, T, F]$, $r[\mathbf{not}, T, F]$, $r[T, F]$, respectively;
    - $\mathscr{P}[\mathbf{K}, T, F]$, $\mathscr{P}[\mathbf{not}, T, F]$, $\mathscr{P}[T, F]$ evaluate to `true` if they are empty or if all of their rules are of the form `true ←`; they evaluate to `false` if at least one rule is of the form `false ←`.

Definition 11 (Stable partition – Def. 11 of [8]). *Let $\mathscr{K}$ be an HKB and $P \subseteq N \subseteq \mathsf{KA}(\mathscr{K})$. (P, N) is a stable partition of $\mathscr{K}$ if*

1. $\mathsf{OB}_{\mathscr{K}, N}$ *is satisfiable;*

2. $\forall a \in \mathsf{KA}(\mathscr{K})$, *if* $\mathsf{OB}_{\mathscr{K}, P} \models a$ *then* $a \in P$ *and if* $\mathsf{OB}_{\mathscr{K}, N} \models a$ *then* $a \in N$; *and* $\mathscr{P}[P, \mathsf{KA}(\mathscr{K}) \setminus N] =$ `true`

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[1] MKNF-coherent HKB (Def. 10 of [8])

[2] Proposition 2 of [8]

3. *for any other partition (P',N') with $P' \subseteq P$ and $N' \subseteq N$ where at least one of the inclusions is proper, $\exists a \in \mathsf{KA}(\mathscr{K}) \setminus P' \mid \mathsf{OB}_{\mathscr{K},P'} \models a$, or $\exists a \in \mathsf{KA}(\mathscr{K}) \setminus N' \mid \mathsf{OB}_{\mathscr{K},N'} \models a$, or $\mathscr{P}[\mathbf{not}, P, \mathsf{KA}(\mathscr{K}) \setminus N][\mathbf{K}, P', \mathsf{KA}(\mathscr{K}) \setminus N'] = \mathtt{false}$*

Definition 12 (Induced partition (Def. 7 and Lemma 1 of [8])). *Let $S \subseteq \mathsf{KA}(\mathscr{K})$. An MKNF interpretation pair (M,N) induces the partition (T,P) of S by placing each atom $a \in S$ as follows:*

- *$a \in T$ if and only if $\forall I \in M, (I,(M,N),(M,N))(a) = \mathtt{true}$*

- *$a \notin P$ if and only if $\forall I \in M, (I,(M,N),(M,N))(a) = \mathtt{false}$*

- *$a \in P \setminus T$ if and only if $\forall I \in M, (I,(M,N),(M,N))(a) = \mathtt{undefined}$*

The following result establishes the correspondence between an HKB's three-valued models and the stable partitions of its known atoms.

Theorem 2 (Theorem 1 of [8]). *Let $\mathscr{K} = (\mathscr{O}, \mathscr{P})$ be a hybrid MKNF KB.*

- *If an MKNF interpretation pair (M,N) is a three-valued MKNF model of $\mathscr{K}$, then the partition (T,P) of $\mathsf{KA}(\mathscr{K})$ induced by (M,N) is a stable partition of $\mathscr{K}$.*

- *If a partition (T,P) is a stable partition of $\mathscr{K}$, then the interpretation pair (M,N), where $(M,N) = (\{I \mid I \models \mathsf{OB}_{\mathscr{K},T}\}, \{I \mid I \models \mathsf{OB}_{\mathscr{K},P}\})$, is a three-valued MKNF model of $\mathscr{K}$.*

The following theorem shows that, for certain HKBs, the alternating fixpoint partition is stable, so it defines a three-valued model which, by Theorem 2, is the HKB's unique well-founded model.

Theorem 3 (Theorem 3 of [8]). *Let $\mathscr{K} = (\mathscr{O}, \mathscr{P})$ be a hybrid MKNF KB.*

- *Assume $\pi(\mathscr{O})$ is satisfiable. Then, for any $E \subseteq \mathsf{KA}(\mathscr{K})$, (E,E) is a stable partition of $\mathsf{KA}(\mathscr{K})$ iff $E = \Gamma_{\mathscr{K}}(E) = \Gamma'_{\mathscr{K}}(E)$.*

- *Assume $\mathscr{K}$ is MKNF-coherent. Then, for any partition (T,P) of $\mathsf{KA}(\mathscr{K})$, (T,P) is a stable partition of $\mathscr{K}$ iff $T = \Gamma_{\mathscr{K}}(P)$ and $P = \Gamma'_{\mathscr{K}}(T)$.*