Symmetry and Self-Duality in Categories of Probabilistic Models

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This note adds to the recent spate of derivations of the probabilistic apparatus of finite-dimensional quantum theory from various axiomatic packages. We offer two different axiomatic packages that lead easily to the Jordan algebraic structure of finite-dimensional quantum theory. The derivation relies on the Koecher-Vinberg Theorem, which sets up an equivalence between order-unit spaces having homogeneous, self-dual cones, and formally real Jordan algebras.

1 Introduction and Overview

The last several years have seen a spate of derivations of the probabilistic apparatus of finite-dimensional quantum theory from various axiomatic packages, many having an information-theoretic motivation [7, 8, 10, 12, 13]. This note (which in part echoes, but greatly improves upon [19]) adds to the flow. I offer two different, though overlapping, axiomatic packages, both stressing symmetry principles, that lead quickly and easily to the Jordan algebraic structure of finite-dimensional quantum theory. Quickly and easily, at any rate, if one is familiar with the Koecher-Vinberg Theorem [11, 15], which sets up an equivalence between order-unit spaces having homogeneous, self-dual cones, and formally real Jordan algebras.

A probabilistic system can be described, in a standard way, in terms of an order-unit space A, the positive elements of which are scalar multiples of "effects". The strategy, then, is to show that certain strong, but not unreasonable, assumptions force the positive cone A_+ to be homogeneous and self-dual, and hence, isomorphic to the cone of squares of such a Jordan algebra. In [3], several conditions are adduced that lead to a homogeneous and *weakly* self-dual cone — that is, a homogeneous cone that is *order-isomorphic* to its dual cone in A^*. However, proper self-duality is a much more stringent condition, requiring that the isomorphism be mediated by an inner product.

The line of attack here is to assume that systems individually have a great deal of symmetry, and collectively, can be organized into a symmetric monoidal category [1, 2, 14]. Here is a sketch. Further details can be found in the longer paper [20].

2 Probabilistic Models

A *test space* [16] is a pair $(X,\mathfrak{A})$ where X is a set of *outcomes* and $\mathfrak{A}$ is a covering of X by non-empty (for our purposes here, finite) subsets called *tests*, each understood as the set of possible outcomes of some measurement, experiment, etc. Two outcomes $x,y \in X$ are *distinguishable* iff they belong to a common test. In this case, I write $x \perp y$. Notice that there is, as yet, no linear structure in view, let alone an inner product; so this notation is promissory.

A *state* on a test space $(X,\mathfrak{A})$ is a function $\alpha : X \to [0,1]$, summing independently to unity on each test. A *symmetry* of $(X,\mathfrak{A})$ is a mapping $g : X \to X$ such that $g(E), g^{-1}(E) \in \mathfrak{A}$ for every $E \in \mathfrak{A}$. By a *probabilistic model*, I mean a structure $(X,\mathfrak{A},\Omega,G)$ where $(X,\mathfrak{A})$ is a test space, Ω is a compact convex set of states on $(X,\mathfrak{A})$, and G is a group acting on $(X,\mathfrak{A})$ by symmetries, and leaving Ω invariant.

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For illustration, if $\mathbf{H}$ is a finite-dimensional Hilbert space (real or Complex), the corresponding *quantum model* is $(X(\mathbf{H}), \mathfrak{A}(\mathbf{H}), \Omega(\mathbf{H}), U(\mathbf{H}))$, where $X = X(\mathbf{H})$ is the set of rank-one projection operators on $\mathbf{H}$, $\mathscr{A} = \mathfrak{A}(\mathbf{H})$ is set of (projective) *frames*, i.e., maximal pairwise orthogonal sets of projections, $\Omega(\mathbf{H})$ is the convex set of density operators on $\mathbf{H}$, and $U(\mathbf{H})$ is the group of unitary operators on $\mathbf{H}$, acting on $X(\mathbf{H})$ by conjugation.

Categories of Models. I will be interested in categories of models. A *morphism* from a model $(X, \mathfrak{A}, \Omega, G)$ to a model $(Y, \mathfrak{B}, \Gamma, H)$ is a pair (ϕ, ψ), where

(i) $\phi : X \to Y$ with $\phi(\mathfrak{A}) \subseteq \mathfrak{B}$, $\phi^*(\Gamma) \subseteq \Omega$

(ii) $\psi \in \mathrm{Hom}(G, H)$;

(iii) $\phi(gx) = \psi(g)\phi(x)$ for all $x \in X, g \in G$.

In what follows, $\mathscr{C}$ is a symmetric monoidal category of probabilistic models $A = (X(A), \mathfrak{A}(A), \Omega(A), G(A))$, with morphisms as above. I shall make two further assumptions:

(1) For every $A \in \mathscr{C}$, $G(A) \subseteq \mathscr{C}(A, A)$.

(2) The model $A \otimes B \in \mathscr{C}$ is a *composite* of the models $A, B \in \mathscr{C}$, in the sense of [5]. This means, in particular, that there are canonical injections $\otimes : X(A) \times X(B) \to X(A \otimes B)$ and $\otimes : \Omega(A) \times \Omega(B) \to \Omega(A \otimes B)$, with
$$E \otimes F = \{x \otimes y | x \in E, y \in F\} \in \mathfrak{A}(A \otimes B)$$
for every $E \in \mathfrak{A}(A), F \in \mathfrak{A}(B)$, and
$$(\alpha \otimes \beta)(x \otimes y) = \alpha(x)\beta(y)$$
for all $\alpha \in \Omega(A), \beta \in \Omega(B), x \in X(A)$ and $y \in X(B)$. A *bipartite state* between $A, B \in \mathscr{C}$ is a state ω in $\Omega(A \otimes B)$. It is also part of the definition that the *un-normalized conditional state* $\hat{\omega}(x) := \omega(x, \cdot)$ belong to $\Omega(B)$ for every $x \in X$, and similarly with A and B reversed.

Models Linearized. Every model $A = (X(A), \mathfrak{A}(A), \Omega(A), G(A)) \in \mathscr{C}$ generates, in a standard and quite canonical way, an order-unit space $\mathbf{E}(A)$. To be precise, $\mathbf{E}(A)$ is the span in $\mathbb{R}^{\Omega}$ of the evaluation functionals associated with measurement outcomes $x \in X$.) In the case of a quantum model $A(\mathbf{H}) = (X(\mathbf{H}), \mathfrak{A}(\mathbf{H}), \Omega(\mathbf{H}), U(\mathbf{H}))$, one has $\mathbf{E}(A) \simeq \mathscr{L}(\mathbf{H})$, the space of Hermitian operators on $\mathbf{H}$, with the usual ordering and $u_A = \mathbf{1}_{\mathbf{H}}$.

The construction $A \mapsto \mathbf{E}(A)$ is functorial, so we obtain from $\mathscr{C}$ a category $\mathbf{E}(\mathscr{C})$ of order-unit spaces and positive linear mappings. It is natural to enlarge this to a category $\mathscr{E}$ in which each hom-set $\mathscr{E}(A, B)$ is an ordered linear space, and in which, e.g., $\mathscr{E}(I, A) \simeq \mathbf{E}(A)$. In what follows, I assume that such a "linearized" category $\mathscr{E}$ has been fixed.

3 Bi-Symmetric Models

To tighten this structure further, I now ask that every $A \in \mathscr{C}$ enjoy a property I call *bi-symmetry*.

Definition. A model $A \in \mathscr{C}$ is *bi-symmetric* iff

(i) $G(A)$ acts transitively on the pure states (that is, extreme points) of $\Omega(A)$,

(ii) $G(A)$ acts transitively on $\mathfrak{A}$, and on pairs (x, y) of outcomes with $x \perp y$.

If, in place of (ii), we require that arbitrary bijections $f : E \to F$, $E, F \in \mathfrak{A}$, extend to elements of G, then A is *fully bi-symmetric*.

If A is bi-symmetric, then G acts transitively. Clearly, the quantum model discussed above is fully bi-symmetric. Bi-symmetry and full bi-symmetry, are very natural conditions. (See [17] for further discussion and motivation of the latter.)

Definition. A *SPIN form*[1] for the model A is a real bilinear form B on $\mathbf{E}(A)$ that is *symmetric*, *positive* in the sense that $B(a,b) \geq 0$ for all $a, b \in \mathbf{E}(A)_+$, *invariant*, in the sense that $B(ga, gb) = B(a,b)$ for all $g \in G(A)$, and *normalized*, in the sense that $B(u_A, u_A) = 1$. A SPIN form is *orthogonalizing* iff $B(x,y) = 0$ for all distinguishable measurement outcomes $x, y \in X(A)$.

An example is the usual tracial inner product on $\mathscr{L}(\mathbf{H})$. Call $\mathbf{E}(A)$ *irreducible* iff (with respect to any SPIN form B), the subspace $u^\perp = \{a \in \mathbf{E}(A) | B(a,u) = 0\}$ (this is independent of B) is irreducible under the group $G(A)$. Quantum models are irreducible in this sense.

Theorem 1. *If $\mathbf{E}(A)$ is irreducible, it supports at most one orthogonalizing SPIN form, which — if it exists — is an inner product.*

4 Conjugates and Daggers

At this point, the aim is to find sufficient conditions for the existence of an orthogonalizing SPIN form on $\mathbf{E}(A)$. I shall provide two.

Definition. By a *conjugate* for a model A, I mean a structure $(\overline{A}, \gamma_A, \eta_A)$, where $\gamma_A : A \mapsto \overline{A}$ is an isomorphism of models, and η_A is a bipartite state on $A \times \overline{A}$ such that $\eta(x, \gamma_A(x)) = 1/n$ (where n is the rank of A) for every $x \in X(A)$.

In the case of a quantum-mechanical model $A = A(\mathbf{H})$ associated with a Hilbert space $\mathbf{H}$, the obvious conjugate model is just that associated with the conjugate Hilbert space $\overline{\mathbf{H}}$, with γ_A taking the rank-one projection x to the corresponding projection $\bar{x}$ on $\overline{\mathbf{H}}$, and with η_A the pure state associated with the unit vector $\frac{1}{\sqrt{n}} \sum_i e_i \otimes \overline{e}_i$, $\{e_i\}$ any basis for $\mathbf{H}$ (note that this is basis-independent).

Returning to the general case, note that by averaging over $G(A)$, we can choose η_A to be invariant, in the sense that $\eta_A(gx, \gamma_A(gy)) = \eta_A(gx, gy)$ for all $g \in G(A)$. This gives us an invariant SPIN form on $\mathbf{E}(A)$, defined on outcomes by $B(x,y) := \eta(x, \gamma_A(y))$. Applying Theorem 1, we have

Theorem 2. *Let $\mathbf{E}(A)$ be irreducible, and suppose A has a conjugate. Then $\mathbf{E}(A)$ carries a canonical orthogonalizing inner product.*

Under some mild auxiliary hypotheses, the existence of a conjugate for every $A \in \mathscr{C}$ (with γ_A and η_A appropriately belonging to $\mathscr{C}$) can be used to construct a dagger on the category $\mathscr{E} \supseteq \mathbf{E}(\mathscr{C})$ discussed above. In fact, however, the mere existence of a reasonable dagger-monoidal structure on $\mathscr{E}$ is enough to obtain much the same result.

Theorem 3. *Suppose $\mathscr{E}$ supports a dagger-monoidal structure, such that for every $g \in G(A)$, $g^\dagger = g^{-1}$ (i.e., $g \in G(A)$ is "unitary"). If $\mathbf{E}(A)$ is irreducible, then it carries an orthogonalizing inner product.*

In order to obtain the self-duality of $\mathbf{E}(A)_+$ for an irreducible model A, it now suffices to assume either of two simple further conditions:

Theorem 4. *Suppose that either*

[1] This is probably not the best choice of terminology.

(a) In the context of Theorem 2, A has a conjugate such that the state η_A is an isomorphism state *or*

(b) In the context of Theorem 3, A is sharp*, meaning that every outcome has probability one in a unique state on* $\mathbf{E}(A)$.

Then $\mathbf{E}(A)_+$ *is self-dual.*

The homogeneity of $\mathbf{E}(A)_+$ can now be enforced by any of several conditions discussed in [4, 19]. Applying the Koecher-Vinberg Theorem, we can conclude that $\mathbf{E}(A)$ carries a unique Jordan product making it a formally real Jordan algebra.

One of these conditions is so simple it's worth pausing to describe it. Any bipartite state ω between $A, B \in \mathscr{C}$ gives rise to a natural positive linear mapping $\hat{\omega} : \mathbf{E}(A) \to \mathbf{E}(B)^*$, uniquely defined by $\hat{\omega}(x)(y) = \omega(x,y)$. Where $\hat{\omega}$ is an *order-isomorphism* — that is, where $\hat{\omega}$ is an order-isomorphism (that is, invertible and having a positive inverse), we call ω an *isomorphism state*. A basic observation from [4], translated into the present context, is that if every state in the interior of $\Omega(B)$ is the marginal of an isomorphism state, then the cone in $\mathbf{E}(B)$ generated by $\Omega(B)$ is homogeneous.

5 Image-Closure

In order to extend these results to possibly reducible systems, I impose one further constraint on $\mathscr{C}$. Call a morphism $(\phi, \psi) : (X, \mathfrak{A}, \Omega, G) \to (Y, \mathfrak{B}, \Gamma, H)$ is *surjective* iff $\phi(X) = Y$, $\mathfrak{B} \subseteq \phi(\mathfrak{A})$, $H = \psi(G)$, and $\Gamma = \{\beta \in \Omega(Y, \mathfrak{B}) | \phi^*(\beta) \in \Omega\}$. In this case, we call $(Y, \mathfrak{B}, \Gamma, H)$ the *image* of $(X, \mathfrak{A}, \Omega, G)$ under (ϕ, ψ). Call $\mathscr{C}$ *image-closed* iff, for any $A \in \mathscr{C}$ and any surjective morphism $(\phi, \psi) : (X_A, \mathfrak{A}_A, \Omega_A, G_A) \to (Y, \mathfrak{B}, \Gamma, H)$, (i) the model $B := (Y, \mathfrak{B}, \Gamma, H)$ belongs to $\mathscr{C}$, and (ii) $(\phi, \psi) \in \mathscr{C}(A, B)$. in $\mathscr{C}$, again belong to $\mathscr{C}$.

Theorem 5. *Let $\mathscr{C}$ be an image-closed category of bi-symmetric probabilistic models, and let $\mathscr{E}$ be the corresponding linearized category as discussed in Section 1. If either*

(a) every $A \in \mathscr{C}$ has a conjugate $\overline{A} \in \mathscr{C}$, with η_A an isomorphism state, or

(b) $\mathscr{E}$ has a dagger-monoidal structure making every $g \in G(A)$ unitary for all $A \in \mathscr{C}$, and every $A \in \mathscr{C}$ is sharp, then for every $A \in \mathscr{C}$, $\mathbf{E}(A)_+$ is self-dual.

Again, adding any of the sufficient conditions for homogeneity from [4, 17] — or simply assuming it outright — will yield a category of formally real Jordan algebras.

Operationally, it is reasonable to suppose that any image $\phi(A)$ of a model $A \in \mathscr{C}$ can be *simulated* by means of the model A. Hence, if we wish to think of $\mathscr{C}$ as closed under operationally reasonable constructions, it is not far-fetched that $\phi(A)$ should belong to $\mathscr{C}$. In fact, the image of a bi-symmetric model is 2-symmetric, so one can simply "close up" $\mathscr{C}$ without sacrificing this assumption. (To suppose that this closure continues to support, e.g., a symmetric-monoidal structure, or conjugate systems, is a sharper constraint, of course.) Categories of finite-dimensional quantum models turn out to be image-closed for the simple reason that a quantum model *has no* non-trivial images.

6 Conclusion

These results raise any number of interesting questions. For one thing, it is possible that the assumptions are stronger than advertised, singling out a narrower class than formally real Jordan algebras. It is noteworthy that I have not had to assume that $\mathscr{C}$'s monoidal product is locally tomographic. In fact, in a forthcoming paper with Howard Barnum [6], we show (using a result of Hanche-Olsen) that if $\mathscr{C}$ is a dagger-monoidal category of finite-dimensional order-unit spaces with homogeneous self-dual cones, then local tomography, plus the existence in $\mathscr{C}$ of a system having the structure of a qubit, implies that every $A \in \mathscr{C}$ is isomorphic to the Hermitian part of a finite-dimensional complex C^* algebra.

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